PLATE AND PANEL STRUCTURES OF ISOTROPIC, COMPOSITE AND PIEZOELECTRIC MATERIALS, INCLUDING SANDWICH CONSTRUCTION
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Plate and Panel Structures of Isotropic, Composite and Piezoelectric Materials, Including Sandwich Construction

by

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This textbook is dedicated to my beautiful wife Midge, who through her encouragement and nurturing over these last two decades, has made the writing of this book possible.
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PREFACE

Plates and panels are primary structural components in many structures from space vehicles, aircraft, automobiles, buildings and homes, bridges decks, ships, and submarines. The ability to design, analyze, optimize and select the proper materials and architecture for plates and panels is a necessity for all structural designers and analysts, whether the adjective in front of the “engineer” on their degree reads aerospace, civil, materials or mechanical.

This text is broken into four parts. The first part deals with the behavior of isotropic plates. Most metals and pure polymeric materials used in structures are isotropic, hence this part covers plates and panels using metallic and polymeric materials.

The second part involves plates and panels of composite materials. Because these fiber reinforced matrix materials can be designed for the particular geometry and loading, they are very often anisotropic with the properties being functions of how the fibers are aligned, their volume fraction, and of course the fiber and matrix materials used. In general, plate and panel structures involving composite materials will weigh less than a plate or panel of metallic material with the same loads and boundary conditions, as well as being more corrosion resistant. Hence, modern structural engineers must be knowledgeable in the more complicated anisotropic material usage for composite plates and panels.

Sandwich plates and panels offer spectacular advantages over the monocoque constructions treated above. By having suitable face and core materials, isotropic or anisotropic, sandwich plates and panels subjected to bending loads can be 300 times as stiff in bending, with face stresses 1/30 of those using a monocoque construction of a thickness equal to the two faces of the sandwich. Thus, for only the additional weight of the light core material, the spectacular advantages of sandwich construction can be attained. In Part 3, the analyses, design and optimization of isotropic and anisotropic sandwich plates and panels are presented.

In Part 4, the use of piezoelectric materials in beams, plates and panels are treated. Piezoelectric materials are those that when an electrical voltage is applied, the effects are tensile, compressive or shear strains in the material. Conversely, with piezoelectric materials, when loads cause tensile, compressive or shear strains, an electrical voltage is generated. Thus, piezoelectric materials can be used as damage sensors, used to achieve a planned structural response due to an electrical signal, or to increase damping. Piezoelectric materials are often referred to as smart or intelligent materials. The means to describe this behavior and incorporate this behavior into beam, plate and panel construction is the theme of Part 4.

This book is intended for three purposes: as an undergraduate textbook for those students who have taken a mechanics of material course, as a graduate textbook, and as a reference for practicing engineers. It therefore provides the fundamentals of plate and panel behavior. It does not include all of the latest research information nor the complications associated with numerous complex structures – but those structures can be studied and analyzed better using the information provided herein.
Several hundred problems are given at the end of Chapters. Most if not all of these problems are homework and exam problems used by the author over several decades of teaching this material. Appreciation is expressed to Alejandro Rivera, who as the first student to take the course using this text, worked most of the problems at the end of the chapters. These solutions will be the basis of a solutions manual which will be available to professors using this text who contact me.

Special thanks is given to James T. Arters, Research Assistant, who has typed this entire manuscript including all of its many changes and enhancements. Finally, many thanks are given to Dr. Moti Leibowitz who reviewed and offered significant suggestions toward improving Chapter 18, 19 and 20.
CHAPTER 1

EQUATIONS OF LINEAR ELASTICITY IN CARTESIAN COORDINATES

References [1.1-1.6] derive in detail the formulation of the governing differential equations of elasticity. Those derivations will not be repeated here, but rather the equations are presented and then utilized to systematically make certain assumptions in the process of deriving the governing equations for rectangular plates and beams.

1.1 Stresses

Consider an elastic body of any general shape. Consider the material to be a continuum, ignoring its crystalline structure and its grain boundaries. Also consider the continuum to be homogeneous, i.e., no variation of material properties with respect to the spatial coordinates. Then, consider a material point anywhere in the interior of the elastic body. If one assigns a Cartesian reference frame with axes $x$, $y$ and $z$, shown in Figure 1.1, it is then convenient to assign a rectangular parallelepiped shape to the material point, and label it a control element of dimensions $dx$, $dy$ and $dz$. The control element is defined to be infinitesimally small compared to the size of the elastic body, yet infinitely large compared to elements of the molecular structure, in order that the material can be considered a continuum.

On the surfaces of the control element there can exist both normal stresses (those perpendicular to the plane of the face) and shear stresses (those parallel to the plane of the face). On any one face these three stress components comprise a vector, called a surface traction.

It is important to note the sign convention and the meaning of the subscripts of these surfaces stresses. For a stress component on a positive face, that is, a face whose outer normal is in the direction of a positive axis, that stress component is positive when it is directed in the direction of that positive axis. Conversely, when a stress is on a negative face of the control element, it is positive when it is directed in the negative axis direction. This procedure is followed in Figure 1.1. Also, the first subscript of any stress component on any face signifies the axis to which the outer normal of that face is parallel. The second subscript refers to the axis to which that stress component is parallel. In the case of normal stresses the subscripts are seen to be repeated and often the two subscripts are shortened to one, i.e. $\sigma_i = \sigma_j$ where $i = x$, $y$ or $z$.

* Numbers in brackets refer to references given at the end of chapters.
1.2 Displacements

The displacements \( u, v \) and \( w \) are parallel to the \( x, y \) and \( z \) axes respectively and are positive when in the positive axis direction.

1.3 Strains

Strains in an elastic body are also of two types, extensional and shear. Extensional strains, where \( i = x, y \) or \( z \), are directed parallel to each of the axes respectively and are a measure of the change in dimension of the control volume in the subscripted direction due to the normal stresses acting on all surfaces of the control volume. Looking at Figure 1.2, one can define shear strains.

The shear strain \( \gamma_{ij} \) (where \( i \) and \( j = x, y \) or \( z \), and \( i \neq j \)) is a change of angle. As an example shown in Figure 1.2, in the \( x-y \) plane, defining \( \gamma_{xy} \) to be

\[
\gamma_{xy} = \frac{\alpha}{2} \phi \text{ (in radians),} \tag{1.1}
\]

then,

\[
\varepsilon_{xy} = \frac{1}{2} \gamma_{xy} \text{.} \tag{1.2}
\]
It is important to define the shear strain $\gamma_{xy}$ to be one half the angle $\gamma_{xy}$ in order to use tensor notation. However, in many texts and papers the shear strain is defined as $\gamma_{xy}$. Care must be taken to insure awareness of which definition is used when reading or utilizing a text or research paper, to obtain correct results in subsequent analysis. Sometimes $\sigma_\gamma$ is termed tensor strain, and $\gamma_{xy}$ is referred to as engineering shear strain (not a tensor quantity).

The rules regarding subscripts of strains are identical to those of stresses presented earlier.

### 1.4 Isotropy and Its Elastic Constants

An isotropic material is one in which the mechanical and physical properties do not vary with orientation. In mathematically modeling an isotropic material, the constant of proportionality between a normal stress and the resulting extensional strain, in the sense of tensile tests is called the modulus of elasticity, $E$.

Similarly, from mechanics of materials, the proportionality between shear stress and the resulting angle $\gamma_{xy}$ described earlier, in a state of pure shear, is called the shear modulus, $G$.

One final quantity must be defined – the Poisson’s ratio, denoted by $\nu$. It is defined as the ratio of the negative of the strain in the $j$ direction to the strain in the $i$ direction caused by a stress in the $i$ direction, $\sigma_{ii}$. With this definition it is a positive quantity of magnitude $0 \leq \nu \leq 0.5$, for all isotropic materials.
The well known relationship between the modulus of elasticity, the shear modulus and Poisson’s ratio for an isotropic material should be remembered:

\[ G = \frac{E}{2(1 + \nu)}. \]  

It must also be remembered that (1.3) can only be used for isotropic materials.

The basic equations of elasticity for a control element of an elastic body in a Cartesian reference frame can now be written. They are written in detail in the following sections and the compact Einsteinian notation of tensor calculus is also provided.

### 1.5 Equilibrium Equations

A material point within an elastic body can be acted on by two types of forces: 

- **Body forces** \(( F_i )\) and surface tractions. The former are forces which are proportional to the mass, such as magnetic forces. Because the material is homogeneous, the body forces can be considered to be proportional to the volume. The latter involve stresses caused by neighboring control elements.

![Figure 1.3. Control element showing variation of stresses.](image)
Figure 1.1 is repeated above, but in Figure 1.3, the provision for stresses varying with respect to space is provided. Thus on the back face the stress \( \sigma_z \) is shown, while on the front face that stress value differs because \( \sigma_z \) is a function of \( x \); hence, its value is \( \sigma_z + (\partial \sigma_z / \partial x) dx \). Also shown are the appropriate expressions for the shear stresses.

The body forces per unit volume, \( F_i \) (\( i = x, y, z \)) are proportional to mass and, as stated before, because the body is homogeneous, are proportional to volume.

The summation of forces in the \( x \) direction can be written as

\[
\left( \sigma_x + \frac{\partial \sigma_z}{\partial x} dx \right) dy dz + \left( \sigma_{yx} + \frac{\partial \sigma_{zz}}{\partial y} dy \right) dx dz \\
+ \left( \sigma_{xz} + \frac{\partial \sigma_{yz}}{\partial z} dz \right) dx dy - \sigma_{xx} dx dy dz - \sigma_{yy} dy dz - \sigma_{zz} dz dx \\
- \sigma_{yx} dx dy + F_x dx dy dz = 0. 
\] (1.4)

After cancellations, every term is multiplied by the volume, which upon division by the volume, results in

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + F_x = 0. 
\] (1.5)

Likewise, in the \( y \) and \( z \) direction, the equilibrium equations are:

\[
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + F_y = 0 
\] (1.6)

\[
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + F_z = 0. 
\] (1.7)

In the compact Einsteinian notation, the above three equilibrium equations are written as

\[
\sigma_{ik} + F_k = 0 \quad (i, k = x, y, z) 
\] (1.8)

where this is the \( n \)th equation, and the repeated subscripts \( k \) refer to each term being repeated in \( x, y \) and \( z \), and where the comma means partial differentiation with respect to the subsequent subscript.
1.6 Stress-Strain Relations

The relationship between the stresses and strains at a material point in a three dimensional body mathematically describe the way the elastic material behaves. They are often referred to as the constitutive equations and are given below without derivation, because easy reference to many texts on elasticity can be made, such as [1.1 - 1.7].

\[ \varepsilon_x = \frac{1}{E} \left[ \sigma_x - v(\sigma_y + \sigma_z) \right], \quad \varepsilon_y = \frac{1}{E} \left[ \sigma_y - v(\sigma_x + \sigma_z) \right] \] (1.9), (1.10)

\[ \varepsilon_z = \frac{1}{E} \left[ \sigma_z - v(\sigma_y + \sigma_x) \right], \quad \varepsilon_y = \frac{1}{2G} \gamma_{yz} \] (1.11), (1.12)

\[ \varepsilon_x = \frac{1}{2G} \gamma_{xz}, \quad \varepsilon_y = \frac{1}{2G} \gamma_{xy} \] (1.13), (1.14)

From (1.9) the proportionality between the strain \( \varepsilon_x \) and the stress \( \sigma_x \) is clearly seen. It is also seen that stresses \( \sigma_y \) and \( \sigma_z \) affect the strain \( \varepsilon_x \), due to the Poisson’s ratio effect.

Similarly, in (1.12) the proportionality between the shear strain \( \varepsilon_{yz} \) and the shear stress \( \sigma_{yz} \) is clearly seen, the number ‘two’ being present due to the definition of \( \varepsilon_{yz} \) given in (1.2).

In the compact Einsteinian notation, the above six equations can be written as

\[ \varepsilon_{ij} = \sigma_{ij} \alpha_{kj} \] (1.15)

where \( \alpha_{ij} \) is the generalized compliance tensor.

1.7 Linear Strain-Displacement Relations

The strain-displacement relations are the kinematic equations relating the displacements that result from an elastic body being strained due to applied loads, or the strains that occur in the material when an elastic body is physically displaced.

\[ \varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y} \] (1.16), (1.17)

\[ \varepsilon_z = \frac{\partial w}{\partial z}, \quad \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \] (1.18), (1.19)
In compact Einsteinian notation, these six equations are written as:

\[
\nu_y = \frac{1}{2} \left( u_{i,j} + u_{j,i} \right) \quad (i, j = x, y, z)
\]  

(1.22)

### 1.8 Compatibility Equations

The purpose of the compatibility equations is to insure that the displacements of an elastic body are single-valued and continuous. They can be written as:

\[
\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial z} \left( -\frac{\partial \varepsilon_{xy}}{\partial x} + \frac{\partial \varepsilon_{yx}}{\partial y} + \frac{\partial \varepsilon_{yz}}{\partial z} \right)
\]  

(1.23)

\[
\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} = \frac{\partial}{\partial z} \left( -\frac{\partial \varepsilon_{yx}}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_{xz}}{\partial z} \right)
\]  

(1.24)

\[
\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial y} \left( -\frac{\partial \varepsilon_{xz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial y} + \frac{\partial \varepsilon_{yz}}{\partial y} \right)
\]  

(1.25)

\[
2 \frac{\partial^2 \varepsilon_{xx}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_{xy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{yx}}{\partial y^2}, \quad 2 \frac{\partial^2 \varepsilon_{yy}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_{yz}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zy}}{\partial y^2}
\]  

(1.26), (1.27)

\[
2 \frac{\partial^2 \varepsilon_{zz}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_{xz}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zx}}{\partial x^2}
\]  

(1.28)

In compact Einsteinian notation, the compatibility equations are written as follows:

\[
\varepsilon_{\alpha \beta} + \varepsilon_{\beta \alpha} - 2 \varepsilon_{\alpha \beta} - \varepsilon_{\beta \alpha} = 0 \quad (i, j, k, l = x, y, z).
\]  

(1.29)

However, in all of what follows herein, namely treating plates and beams, invariably the governing differential equations are placed in terms of displacements, and if the solutions are functions which are single-valued and continuous, it is not necessary to utilize the compatibility equations.
1.9 Summary

It can be shown that both the stress and strain tensor quantities are symmetric, i.e.,
\[ \sigma_{ij} = \sigma_{ji} \quad \text{and} \quad \varepsilon_{ij} = \varepsilon_{ji} \quad (i,j = x,y,z). \] (1.30)

Therefore, for the elastic solid there are fifteen independent variables; six stress components, six strain components and three displacements. In the case where compatibility is satisfied, there are fifteen equations: three equilibrium equations, six constitutive relations and six strain-displacement equations.

For a rather complete discussion [1.7] of the equations of elasticity for anisotropic materials, see Chapter 10 of this text.

1.10 References


1.11 Problems

1.1. Prove that the stresses are symmetric, i.e., \( \sigma_{ij} = \sigma_{ji} \).
   (Suggestion: take moments about the \( x, y \) and \( z \) axes.)

1.2. When \( v = 0.5 \) a material is called ‘incompressible’. Prove that for \( v = 0.5 \), under any set of stresses, the control volume of Figure 1.1 will not change volume when subjected to applied stresses.

1.3. An elastic body has the following strain field:
   \[ \varepsilon_{xx} = 2x^2 + 3xy + 4y^2 \quad \varepsilon_{xy} = 0 \]
   \[ \varepsilon_{yy} = x^2 - 2y^2 + z^2 \quad \varepsilon_{yz} = 2y^2 - 3z^2 \]
   \[ \varepsilon_{zz} = 2y^2 - z^2 \quad \varepsilon_{xz} = 3z^2 - 2y^2 \]
Does this strain field satisfy compatibility? Note: compatibility is not satisfied if any one or more of the compatibility equations is violated.
CHAPTER 2

DERIVATION OF THE GOVERNING EQUATIONS FOR ISOTROPIC RECTANGULAR PLATES

This approach in this chapter is to systematically derive the governing equations for an isotropic classical, thin elastic isotropic rectangular plate. Analogous derivations are given in [2.1 - 2.8].

2.1 Assumptions of Plate Theory

In classical, linear thin plate theory, there are a number of assumptions that are necessary in order to reduce the three dimensional equations of elasticity to a two dimensional set that can be solved. Consider an elastic body shown in Figure 2.1, comprising the region \(0 \leq x \leq a, \ 0 \leq y \leq b\) and \(-h/2 \leq z \leq h/2\), such that \(h \ll a\) and \(h \ll b\). This is called a plate.

The following assumptions are made.

1. A lineal element of the plate extending through the plate thickness, normal to the mid surface, \(x-y\) plane, in the unstressed state, upon the application of load:
   a. undergoes at most a translation and a rotation with respect to the original coordinate system;
   b. remains normal to the deformed middle surface.

2. A plate resists lateral and in-plane loads by bending, transverse shear stresses, and in-plane action, not through block like compression or tension in the plate in the
thickness direction. This assumption results from the fact that $h/a \ll 1$ and $h/b \ll 1$.

From 1a the following is implied:

3. A lineal element through the thickness does not elongate or contract.
4. The lineal element remains straight upon load application.

In addition,


It is seen from 1a that the most general form for the two in-plane displacements is:

$$u(x, y, z) = u_0(x, y) + z \alpha(x, y)$$

(2.1)

$$v(x, y, z) = v_0(x, y) + z \beta(x, y)$$

(2.2)

where $u_0$ and $v_0$ are the in-plane middle surface displacements ($z = 0$), and $\alpha$ and $\beta$ are rotations as yet undefined. Assumption 3 requires that $z_\alpha = 0$, which in turn means that the lateral deflection $w$ is at most (from Equation 1.18)

$$w = w(x, y).$$

(2.3)

Also, Equations (1.11) is ignored.

Assumption 4 requires that for any $z$, both $\epsilon_{xz} = \text{constant}$ and $\epsilon_{yz} = \text{constant}$ at any specific location $(x, y)$ on the plate middle surface for all $z$. Assumption 1b requires that the constant is zero, hence

$$\epsilon_{xz} = \epsilon_{yz} = 0.$$ 

Assumption 2 means that $\sigma_z = 0$ in the stress strain relations.

Incidentally, the assumptions above are identical to those of thin classical beam, ring and shell theory.

2.2 Derivation of the Equilibrium Equations for a Rectangular Plate

Figure 2.2 shows the positive directions of stress quantities to be defined when the plate is subjected to lateral and in-plane loads.

The stress couples are defined as follows:
Figure 2.2. Positive directions of stress resultants and couples.

\[ M_x = \int_{-b/2}^{+b/2} \sigma_{x} z \, dx \]  \hspace{1cm} (2.4)

\[ M_y = \int_{-h/2}^{+h/2} \sigma_{y} z \, dz \]  \hspace{1cm} (2.5)

\[ M_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} z \, dz \]  \hspace{1cm} (2.6)

\[ M_{yz} = \int_{-h/2}^{+h/2} \sigma_{yz} z \, dz = M_{yx} \]  \hspace{1cm} (2.7)

Physically, it is seen that the stress couple is the summation of the moment about the middle surface of all the stresses shown acting on all of the infinitesimal control elements through the plate thickness at a location \((x, y)\). In the limit the summation is replaced by the integration.

Similarly, the shear resultants are defined as,

\[ Q_x = \int_{-b/2}^{+b/2} \sigma_{xy} \, dx \]  \hspace{1cm} (2.8)
\[ Q_y = \int_{-h/2}^{+h/2} \sigma_{yx} \, dz \]  

(2.9)

Again the shear resultant is physically the summation of all the shear stresses in the thickness direction acting on all of the infinitesimal control elements across the thickness of the plate at the location \((x, y)\).

Finally, the stress resultants are defined to be:

\[ N_x = \int_{-h/2}^{+h/2} \sigma_{xx} \, dx \]  

(2.10)

\[ N_y = \int_{-h/2}^{+h/2} \sigma_{yy} \, dz \]  

(2.11)

\[ N_{xy} = \int_{-h/2}^{+h/2} \sigma_{xy} \, dz \]  

(2.12)

\[ N_{yx} = \int_{-h/2}^{+h/2} \sigma_{yx} \, dz = N_{xy} \]  

(2.13)

These then are the sum of all the in-plane stresses acting on all of the infinitesimal control elements across the thickness of the plate at \(x, y\).

Thus, in plate theory, the details of each control element under consideration are disregarded when one integrates the stress quantities across the thickness \(h\). Instead of considering stresses at each material point one really deals with the integrated stress quantities defined above. The procedure to obtain the governing equations for plates from the equations of elasticity is to perform certain integrations on them.

Proceeding, multiply Equation (1.5) by \(z \, dz\) and integrate between \(-h/2\) and \(+h/2\), as follows:

\[ \frac{\partial}{\partial x} \left( \int_{-h/2}^{+h/2} \sigma_{xx} z \, dz \right) + \frac{\partial}{\partial y} \left( \int_{-h/2}^{+h/2} \sigma_{xy} z \, dz \right) + \int_{-h/2}^{+h/2} \frac{\partial \sigma_{xy}}{\partial z} \, dz = 0 \]

\[ \frac{\partial}{\partial x} \left( \int_{-h/2}^{+h/2} \sigma_{yx} z \, dz \right) + \frac{\partial}{\partial y} \left( \int_{-h/2}^{+h/2} \sigma_{yy} z \, dz \right) + \int_{-h/2}^{+h/2} \frac{\partial \sigma_{yx}}{\partial z} \, dz = 0 \]

\[ \int_{-h/2}^{+h/2} \frac{\partial M_x}{\partial y} + \frac{\partial M_y}{\partial x} + z \mu_{xx} \left|_{-h/2}^{+h/2} \right. \right. - \int_{-h/2}^{+h/2} \sigma_{xx} \, dz = 0. \]
In the above, the order of differentiation and integration can be reversed because \( x \) and \( z \) are orthogonal one to the other. Looking at the third term, \( \sigma_{x} = \sigma_{z} = 0 \) when there are no shear loads on the upper or lower plate surface. If there are surface shear stresses then defining \( \tau_{1z} = \sigma_{x}(+h/2) \) and \( \tau_{z2} = \sigma_{x}(-h/2) \), the results are shown below in Equation (2.14). It should also be noted that for plates supported on an edge, \( \sigma_{x} \) may not go to zero at \( \pm h/2 \), and so the theory is not accurate at that edge, but due to St. Venant’s Principle, the solutions are satisfactory away from the edge supports.

\[
\frac{\partial M_{z}}{\partial x} + \frac{\partial M_{x}}{\partial y} + \frac{h}{2}(\tau_{1z} + \tau_{z2}) - Q_{y} = 0. \tag{2.14}
\]

Likewise Equation (1.6) becomes

\[
\frac{\partial M_{y}}{\partial x} + \frac{\partial M_{x}}{\partial y} + \frac{h}{2}(\tau_{1y} + \tau_{y2}) - Q_{y} = 0 \tag{2.15}
\]

where

\[\tau_{1y} = \sigma_{x}(+h/2) \quad \text{and} \quad \tau_{y2} = \sigma_{z}(-h/2).\]

These two equations describe the moment equilibrium of a plate element. Looking now at Equations (1.7), multiplying it by \( \partial z \), and integrating between \(-h/2\) and \(+h/2\), results in

\[
\int_{-h/2}^{+h/2} \left( \frac{\partial \sigma_{x}}{\partial x} + \frac{\partial \sigma_{y}}{\partial y} + \frac{\partial \sigma_{z}}{\partial z} \right) \partial z = 0
\]

\[
\frac{\partial Q_{x}}{\partial x} + \frac{\partial Q_{y}}{\partial y} + \int_{-h/2}^{+h/2} \sigma_{z} \partial z = 0
\]

\[
\frac{\partial Q_{x}}{\partial x} + \frac{\partial Q_{y}}{\partial y} + p_{1}(x,y) - p_{2}(x,y) = 0 \tag{2.16}
\]

where \( p_{1}(x,y) = \sigma_{x}(+h/2) \), \( p_{2}(x,y) = \sigma_{z}(-h/2) \).

One could also derive (2.16) by considering vertical equilibrium of a plate element shown in Figure 2.3.
One may ask why use is made of $\sigma_z$ in this equation and not in the stress-strain relation? The foregoing is not really inconsistent, since $\sigma_z$ does not appear explicitly in Equation (2.16) and once away from the surface the normal surface traction is absorbed by shear and in-plane stresses rather than by $\sigma_z$ in the plate interior, as stated previously in Assumption 2.

Similarly, multiplying Equations (1.5) and (1.6) by $dz$ and integrating across the plate thickness results in the plate equilibrium equations in the $x$ and $y$ directions respectively, in terms of the in-plane stress resultants and the surface shear stresses.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_y}{\partial y} + (\varepsilon_{xy} - \varepsilon_{yx}) = 0$$

(2.17)

$$\frac{\partial N_y}{\partial x} + \frac{\partial N_x}{\partial y} + (\varepsilon_{xy} - \varepsilon_{yx}) = 0.$$  (2.18)

### 2.3 Derivation of Plate Moment-Curvature Relations and Integrated Stress Resultant-Displacement Relations

Now, the plate equations must be derived corresponding to the elastic stress strain relations. The strains $\varepsilon_x$, $\varepsilon_y$, and $\varepsilon_{xy}$ will not be used explicitly since the stresses have been averaged by integrating through the thickness. Hence, displacements are utilized. Thus, combining (1.9) through (1.21) gives the following, remembering that $\sigma_z$ has been assumed zero in the interior of the plate and excluding Equation (1.11) for reasons given previously.

$$\frac{\partial u}{\partial x} = \frac{1}{E} \left[ \nu_x - v \nu_y \right]$$

(2.19)
\[
\frac{\partial v}{\partial y} = \frac{1}{E} [\sigma_y - v\sigma_z]
\]  
(2.20)

\[
\frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = \frac{1}{2G} \sigma_{xy}
\]  
(2.21)

\[
\frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = \frac{1}{2G} \sigma_{yz}
\]  
(2.22)

\[
\frac{1}{2} \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right) = \frac{1}{2G} \sigma_{xz}
\]  
(2.23)

Next, recall the form of the admissible displacements resulting from the plate theory assumptions, given in (2.1) through (2.3):

\[
u = v_0(x, y) + z\sigma(x, y)
\]  
(2.24)

\[
v = v_0(x, y) + z\beta(x, y)
\]  
(2.25)

\[
w = w(x, y)
\]  
only.  
(2.26)

In plate theory it is remembered that a lineal element through the plate will experience translations, rotations, but no extensions or contractions. For these assumptions to be valid, the lateral deflections are restricted to being small compared to the plate thickness. It is noted that if a plate is very thin, lateral loads can cause lateral deflections many times the thickness and the plate then behaves largely as a membrane because it has little or no bending resistance, i.e., \( D \to 0 \).

The assumptions of classical plate theory require that transverse shear deformation be zero. If \( \sigma_{yy} = \sigma_{zz} = 0 \) then from Equations (1.20) and (1.21)

\[
\frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = 0 \quad \text{or} \quad \frac{\partial u}{\partial z} = -\frac{\partial w}{\partial x}, \quad \text{likewise}
\]

\[
\frac{\partial v}{\partial z} = \frac{\partial w}{\partial y}.
\]

Hence, from Equations (2.24) through (2.26) and the above, it is seen that the rotations are

\[
\frac{\partial w}{\partial x} = -\frac{\partial w}{\partial x}
\]  
(2.27)