Foliations and Geometric Structures
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by

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Preface

The theory of foliations of manifolds was created in the forties of the last century by Ch. Ehresmann and G. Reeb [ER44]. Since then, the subject has enjoyed a rapid development and thousands of papers investigating foliations have appeared. A list of papers and preprints on foliations up to 1995 can be found in Tondeur [Ton97].

Due to the great interest of topologists and geometers in this rapidly evolving theory, many books on foliations have also been published one after the other. We mention, for example, the books written by: I. Tamura [Tam76], G. Hector and U. Hirsch [HH83], B. Reinhart [Rei83], C. Camacho and A.L. Neto [CN85], H. Kitahara [Kit86], P. Molino [Mol88], Ph. Tondeur [Ton88], [Ton97], V. Rovenskii [Rov98], A. Candel and L. Conlon [CC03]. Also, the survey written by H.B. Lawson, Jr. [Law74] had a great impact on the development of the theory of foliations.

So it is natural to ask: why write yet another book on foliations? The answer is very simple. Our areas of interest and investigation are different. The main theme of this book is to investigate the interrelations between foliations of a manifold on one hand, and the many geometric structures that the manifold may admit on the other hand. Among these structures we mention: affine, Riemannian, semi–Riemannian, Finsler, symplectic, and contact structures. We also mention that, for the first time in the literature, we present in a book form results on degenerate (null, light–like) foliations of semi–Riemannian manifolds. Using these structures one obtains very interesting classes of foliations whose geometry is worth investigating. There are still many aspects of this geometry that can be promising areas for more research. We hope that the body of geometry and techniques developed in this book will show the richness of the subjects waiting to be studied further, and will present the means and tools needed for such investigations. Another point that makes our book different from the others, is that we use only two (adapted) linear connections which have been considered first by G. Vrânceanu [VG31], [VG57], and J.A. Schouten and E.R. Van Kampen [SVK30] for studying the geometry of non–holonomic spaces. Thus our study appears as a continuation of the study of
non–holonomic spaces (non–integrable distributions) to foliations (integrable distributions). Furthermore, the book shows how the scientific material developed for foliations can be used in some applications to physics.

We hope that the audience of this book will include graduate students who want to be introduced to the geometry of foliations, researchers interested in foliations and geometric structures, and physicists interested in gauge theory and its generalizations.

The first chapter is devoted to the geometry of distributions. We present here a modern approach to the geometry of non–holonomic manifolds, stressing the importance of the role of the Schouten–Van Kampen connection and the Vrânceanu connection for understanding this geometry.

The theory of foliations is introduced in Chapter 2. We give the different approaches to this theory with examples showing that foliations on manifolds appear in many natural ways. A tensor calculus is then built on foliated manifolds to enable us to study the geometry of both the foliations and the ambient manifolds.

Foliations on semi–Riemannian manifolds are studied in Chapter 3. Important classes of such foliations are investigated. These include foliations with bundle–like metrics, totally geodesic, totally umbilical, minimal, symmetric and transversally symmetric foliations.

Chapter 4 deals with parallelism of foliations on semi–Riemannian manifolds. Here we study both the degenerate and non–degenerate foliations on semi–Riemannian manifolds. The situation of parallel partially–null foliations is still very far from being fully understood. We hope that our exposition stimulates further investigations trying to tackle the remaining unsolved problems.

More geometric structures on foliated manifolds are displayed in the fifth chapter. These include Lagrange foliations on symplectic manifolds, Legendre foliations on contact manifolds, foliations on the tangent bundles of Finsler manifolds, and foliations on $CR$–submanifolds. It is interesting to note that in Section 5.3 we develop a new method for studying the geometry of a Finsler manifold. This is mainly based on the Vrânceanu connection whose local coefficients determine all classical Finsler connections.

The last chapter is dedicated to applications. Since any vector bundle admits a natural foliation by fibers, we use the theory of foliations to develop a gauge theory on the total space of a vector bundle. We investigate the invariance of Lagrangians and obtain the equations of motion and conservation laws for the full Lagrangian. Finally, we derive the Bianchi identities for the strength fields of the gauge fields.

The preparation of the manuscript took longer than originally planned. We would like to thank both Kluwer and Springer publishers for their patience, cooperation and understanding.

We are also grateful to all the authors of books and articles whose work on foliations has been used by us in preparing the book. Many thanks go to the staff of the library ”Seminarul Matematic Al. Myller” from Iaşi (Romania),
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1

GEOMETRY OF DISTRIBUTIONS
ON A MANIFOLD

In the third decade of the last century, Vrânceanu [VG26a] and Horak [Hor27] introduced independently the notion of non–holonomic manifold as a need for a geometric interpretation of non–holonomic mechanical systems. We present here a modern approach to the geometry of non–holonomic manifolds as manifolds endowed with non–integrable distributions, and extend this study to almost product manifolds. Our approach is mainly based on adapted linear connections, stressing the important role of Schouten–Van Kampen and Vrânceanu connections for understanding the geometry of distributions, in general, and the geometry of non–holonomic manifolds, in particular. When a semi–Riemannian metric is considered on the manifold, we compare the intrinsic and induced connections on a semi–Riemannian manifold, and get the local structure of the manifold when these connections coincide. By using both the Schouten–Van Kampen and Vrânceanu connections we obtain the fundamental equations and some interesting evaluations for sectional curvatures of non–holonomic manifolds. In particular, we find a large class of Riemannian non–holonomic manifolds of Vrânceanu positive constant curvature. Finally, we present a method to study the geometry of degenerate distributions of codimension one on a proper semi–Riemannian manifold.

Our approach to the geometry of distributions on a manifold via Schouten–Van Kampen and Vrânceanu connections is given, not only because of its importance for its own right, but also because of the crucial role it will play throughout the book in studying foliations on manifolds.

1.1 Distributions on a Manifold

Let $M$ be an $(n + p)$–dimensional paracompact smooth manifold and $TM$ be the tangent bundle of $M$. Denote by $\pi$ the canonical projection of $TM$ on $M$ and by $T_x M$ the fiber at $x \in M$, i.e., $T_x M = \pi^{-1}(x)$. A coordinate system (local chart) in $M$ is denoted by $\{(U, \varphi) : (x^1, ..., x^{n+p})\}$ or briefly $\{(U, \varphi) : (x^a)\}$, where $U$ is an open subset of $M$, $\varphi : U \rightarrow \mathbb{R}^{n+p}$ is a
diffeomorphism of $\mathcal{U}$ onto $\varphi(\mathcal{U})$, and $(x^1, \ldots, x^{n+p}) = \varphi(x)$ for any $x \in \mathcal{U}$. For any point $x \in \mathcal{U}$, we say that the coordinate system $\{(\mathcal{U}, \varphi) : (x^a)\}$ is about $x$. The coordinate system $\{(\mathcal{U}, \varphi) : (x^a)\}$ in $M$ defines a coordinate system $\{(\mathcal{U}^*, \Phi) : (x^1, \ldots, x^{n+p}, v^1, \ldots, v^{n+p})\}$ in $TM$, where $\mathcal{U}^* = \pi^{-1}(\mathcal{U})$ and $\Phi : \mathcal{U}^* \to \mathbb{R}^{2(n+p)}$ is a diffeomorphism of $\mathcal{U}^*$ onto $\varphi(\mathcal{U}) \times \mathbb{R}^{n+p}$ and $(x^1, \ldots, x^{n+p}, v^1, \ldots, v^{n+p}) = \Phi(v_x)$ for any $x \in \mathcal{U}$. and $v_x \in T_xM$.

Next, we consider a vector subbundle $\mathcal{D}$ of $TM$ of rank $n$. Thus for each $x \in M$ there exists a local chart $(\mathcal{U}, \varphi)$ on $M$ at $x$ such that the corresponding local chart $(\mathcal{U}^*, \Phi)$ on $TM$ satisfies the condition $\Phi(\mathcal{U}^* \cap \mathcal{D}) = \varphi(\mathcal{U}) \times \mathbb{R}^n$. Then each fiber $\mathcal{D}_x$ over $x \in M$ is an $n$–dimensional subspace of $T_xM$, and the total space of the vector bundle $\pi : \mathcal{D} \to M$ becomes a $(2n+p)$–dimensional submanifold of $TM$. We say that $\mathcal{D}$ is an $n$–distribution (n–plane field or $n$–differential system) on $M$.

A slightly different approach to distributions may be achieved by starting with the Grassmann bundle $G_n(M)$ over $M$. For any $x \in M$ the Grassmann manifold $G_n(x)$ consists of all $n$–dimensional vector subspaces of the tangent space $T_xM$. Then

$$G_n(M) = \bigcup_{x \in M} G_n(x),$$

is an $(n + p + np)$–dimensional manifold, since each fiber $G_n(x)$ is an $np$–dimensional manifold. Clearly, any smooth section of $G_n(M)$ is an $n$–distribution and conversely, any $n$–distribution defines a section of $G_n(M)$.

We do not explore here the difficult problem of the existence of distributions on a manifold. We only mention that if $M$ is a compact manifold and its Euler number $\chi(M)$ is zero, then there exists on $M$ a 1–distribution. Thus any odd–dimensional sphere $S^n$ with $m \geq 3$ admits a 1–distribution. Also, we note that the only compact surfaces with 1–distributions are the torus and the Klein bottle. Since fibers of a 1–distribution are lines, we refer to a 1–distribution as a line field.

As we have seen, a distribution on $M$ is globally given either as a vector subbundle of $TM$ or as a global section of $G_n(M)$. However, most of the problems encountering distributions have a local character. Here we present two ways to define a distribution on $M$ by some geometric objects that are locally defined on $M$.

First, suppose that on each coordinate neighbourhood $\mathcal{U}$ in $M$ there exist $n$ linearly independent smooth vector fields $\{E_1, \ldots, E_n\}$. Then the mapping

$$x \to \mathcal{D}_x = \text{span}\{E_1x, \ldots, E_nx\}, \ x \in \mathcal{U},$$

defines an $n$–distribution on $\mathcal{U}$. Now, we assume that for any two coordinate neighbourhoods $\mathcal{U}$ and $\tilde{\mathcal{U}}$ with $\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset$, the vector fields $\{\tilde{E}_1, \ldots, \tilde{E}_n\}$ and $\{E_1, \ldots, E_n\}$ are related by

$$\tilde{E}_i = a^i_j E_j, \quad (1.1)$$
where $a^j_i$ are smooth functions on $U \cap \tilde{U}$ such that $[a^j_i(x)]$ is a non-singular \(n\times n\) matrix for any $x \in U \cap \tilde{U}$. In this way the two distributions on $U$ and $\tilde{U}$ agree on $U \cap \tilde{U}$ and therefore we have a distribution on $M$. Conversely, it is easy to see that any \(n\)-distribution on $M$ is locally represented by \(n\) linearly independent smooth vector fields satisfying (1.1) on the intersection of two coordinate neighbourhoods. Though we do not write the adjective “smooth” to the noun “distribution” we always understand that all local representative vector fields of a distribution are smooth vector fields.

A distribution on a manifold can also be locally defined using a differential system. This is done as follows. We assume that on each coordinate neighbourhood $U \subset M$ there exist \(p\) linearly independent smooth \(1\)-forms $\{\omega^n+1, \ldots, \omega^n+p\}$, $\alpha \in \{n+1, \ldots, n+p\}$. Then for any $x \in U$ we consider $D_x$ as the \(n\)-dimensional subspace of $T_xM$ consisting of solutions $X$ of the system

\begin{equation}
\omega^{n+1}(X) = 0, \ldots, \omega^{n+p}(X) = 0.
\end{equation}

Next we add the condition that the \(1\)-forms $\{\omega^{n+1}, \ldots, \omega^{n+p}\}$ and $\{\tilde{\omega}^{n+1}, \ldots, \tilde{\omega}^{n+p}\}$ on $U$ and $\tilde{U}$ satisfy

\begin{equation}
\tilde{\omega}^\alpha = A^\alpha_\beta \omega^\beta, \quad \text{on } U \cap \tilde{U},
\end{equation}

where $A^\alpha_\beta$ are smooth functions on $U \cap \tilde{U}$ such that $[A^\alpha_\beta(x)]$ is a non-singular \(p\times p\) matrix for any $x \in U \cap \tilde{U}$. Then the mapping $D : x \rightarrow D_x \in G_n(x)$ defines an \(n\)-distribution on $M$. The converse is also true, that is, any \(n\)-distribution on $M$ is given locally by a differential system (1.2) whose representative \(1\)-forms are related by (1.3).

If not stated otherwise, we shall use throughout this chapter the following ranges for indices: $i, j, k, \ldots \in \{1, \ldots, n\}; \quad \alpha, \beta, \gamma, \ldots \in \{n+1, \ldots, n+p\}; \quad a, b, c, \ldots \in \{1, \ldots, n+p\}$.

The integrability problem for distributions is very important. A complete study of this problem is going to be presented in the next chapter (see Section 2.1). Here we only give some definitions and discuss their equivalence.

Let $D$ be an \(n\)-distribution on $M$. Then a \(k\)-dimensional submanifold $N$ of $M$, $0 < k \leq n$, is said to be an integral manifold of $D$, if $T_xN \subset D_x$ for any $x \in N$. Thus the maximum dimension of $N$ is $n$. Now, we say that $D$ is an integrable distribution if for any point $x \in M$ there exists a local chart $\{(U, \varphi) : (x^1, \ldots, x^n, x^{n+1}, \ldots, x^{n+p})\}$ on $M$ such that all the submanifolds of $U$ given by the equations

\begin{equation}
x^{n+1} = \text{constant}, \ldots, x^{n+p} = \text{constant},
\end{equation}

are integral manifolds of $D$. 

If not stated otherwise, we shall use throughout this chapter the following ranges for indices: $i, j, k, \ldots \in \{1, \ldots, n\}; \quad \alpha, \beta, \gamma, \ldots \in \{n+1, \ldots, n+p\}; \quad a, b, c, \ldots \in \{1, \ldots, n+p\}$.
A connected submanifold given by (1.4) is called a **local leaf** (plaque) of \( \mathcal{D} \) (details can be seen in Section 2.1). In this case any connected integral manifold of \( \mathcal{D} \) lying in \( \mathcal{U} \) is a submanifold of one of the local leaves of \( \mathcal{D} \). Based on the above definition we can state the following.

**Theorem 1.1.** Let \( \mathcal{D} \) be an \( n \)–distribution on \( M \). Then the following assertions are equivalent:

(i) \( \mathcal{D} \) is an integrable distribution.

(ii) For any \( x \in M \) there exists a local chart \( \{ (\mathcal{U}, \varphi) : (x^a) \} \) such that \( \mathcal{D} \) is given on \( \mathcal{U} \) by the differential system

\[
dx^{n+1} = 0, \ldots, dx^{n+p} = 0.
\]

(iii) For any \( x \in M \) there exists a local chart \( \{ (\mathcal{U}, \varphi) : (x^a) \} \) such that

\[
\mathcal{D} = \text{span} \left\{ \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right\}, \quad \text{on } \mathcal{U}.
\]

Next, let \( X \) and \( Y \) be two vector fields on \( M \). Then their **Lie bracket** \([X, Y]\) is a vector field defined by

\[
[X, Y](f) = X(Y(f)) - Y(X(f)), \quad \forall f \in F(M).
\]

Locally, the Lie bracket is written as follows

\[
[X, Y] = \left( X^a \frac{\partial Y^b}{\partial x^a} - Y^a \frac{\partial X^b}{\partial x^a} \right) \frac{\partial}{\partial x^b},
\]

where \( X = X^a \frac{\partial}{\partial x^a} \) and \( Y = Y^a \frac{\partial}{\partial x^a} \). Now, we say that a vector field \( X \) on \( M \) lies in \( \mathcal{D} \) if \( X(x) \in \mathcal{D}_x \), for all \( x \in M \). If \( \Gamma(\mathcal{D}) \) denotes the \( F(M) \)–module of smooth sections of \( \mathcal{D} \), then we use the notation \( X \in \Gamma(\mathcal{D}) \) to indicate that \( X \) lies in \( \mathcal{D} \). We say that \( \mathcal{D} \) is an **involutive distribution** if \([X, Y] \in \Gamma(\mathcal{D})\) for any \( X, Y \in \Gamma(\mathcal{D}) \). At this point we only mention that \( \mathcal{D} \) is integrable if and only if it is involutive. This is the famous theorem of Frobenius which will be proved in Section 2.1.

In the present chapter we will be concerned with the geometry of distributions in general, that is, they do not need to be integrable. A pair \((M, \mathcal{D})\), where \( M \) is a manifold and \( \mathcal{D} \) is a non–integrable distribution on \( M \), is called a **non–holonomic manifold**. The concept of “non–holonomic space” in a Riemannian manifold has been introduced in 1926 by Vrânceanu [VG26a], [VG26b] and independently by Horak [Hor27] in 1927 as a need for a geometric interpretation of non–holonomic mechanical systems. In 1928 Schouten [Sch28] considered non–holonomic spaces in a manifold with a linear connection. A great deal of research has been devoted to the study of the geometry of non–holonomic spaces in Riemannian manifolds, and in manifolds with linear
connections, in general. Several references published in the first half of the 20th century can be found in Schouten [Sch54].

The purpose of this chapter is to revisit this rather forgotten area of differential geometry. In addition to the classical coordinate–base approach, we will exploit modern coordinate–free techniques. The information we present here will be used later in the book in our search for results that shed more light on the geometry of foliated manifolds. In this respect, it is worth mentioning that the linear connections introduced by Vrânceanu [VG31] and Schouten and Van Kampen [SVK30] on non–holonomic manifolds will be considered on almost product manifolds, and thus they will have an important role in studying foliations on Riemannian (semi–Riemannian) manifolds.

If a distribution $D$ on $M$ is given, then a complementary distribution $D'$ to $D$ in $TM$ can be obtained. Indeed, since $M$ is paracompact and of differentiability class $C^\infty$, there exists on $M$ a Riemannian metric of class $C^\infty$. Then we can take $D'$ as the complementary orthogonal distribution to $D$ with respect to that metric. Thus we are entitled to consider, in the first stage of our study, a pair of complementary distributions $(D, D')$ on $M$, that is, $TM$ has the decomposition

$$TM = D \oplus D'. \quad (1.9)$$

Later on (see Sections 1.5, 1.6 and 1.7) we will see the contribution of a Riemannian (semi–Riemannian) metric on $M$ to the study of the geometry of the pair $(D, D')$.

Based on the above discussion we consider on $M$ two complementary distributions $D$ and $D'$. Denote by $Q$ and $Q'$ the projection morphisms of $TM$ on $D$ and $D'$ respectively. Then we have

$$(a) \quad Q^2 = Q, \quad (b) \quad Q'^2 = Q', \quad (c) \quad QQ' = Q'Q = 0, \quad (d) \quad Q + Q' = I, \quad (1.10)$$

where $I$ is the identity morphism on $TM$. Now we define the tensor field $F$ of type $(1,1)$ by

$$F = Q - Q'. \quad (1.11)$$

It follows that $F$ is an almost product structure on $M$, that is, $F$ satisfies

$$F^2 = I. \quad (1.12)$$

For this reason we call $(M, D, D')$ an almost product manifold. Next, from (1.10d) and (1.11) we deduce that

$$(a) \quad Q = \frac{1}{2}(I + F) \quad \text{and} \quad (b) \quad Q' = \frac{1}{2}(I - F). \quad (1.13)$$

Now, we note that at any point $x \in M$, $D_x$ and $D'_x$ coincide with the eigenspaces in $T_xM$ corresponding to the eigenvalues $+1$ and $-1$ of $F_x$, respectively. Indeed, if $X_x \in T_xM$ and $F_x(X_x) = X_x$, then from (1.13a) we
deduce that $Q_x(X_x) = X_x$, that is, $X_x \in D_x$. Conversely, if $X_x \in D_x$ then there exists $Y_x \in T_xM$ such that $Q_x(Y_x) = X_x$. Then, by using (1.11), (1.10a) and (1.10c) we obtain $F_x(X_x) = X_x$. The corresponding property for $D'_x$ is obtained similarly. As a conclusion we write

$$
\begin{align*}
(a) & \quad \Gamma(D) = \{X \in \Gamma(TM) : FX = X\}, \\
(b) & \quad \Gamma(D') = \{X \in \Gamma(TM) : FX = -X\}.
\end{align*}
$$

Next, we suppose that $D$ and $D'$ are locally represented on a coordinate neighbourhood $U \subset M$ by vector fields $\{E_i\}$ and $\{E_\alpha\}$ respectively. Then we call $\{E_A\} = \{E_i, E_\alpha\}$, $A \in \{1, ..., n + p\}$, a non–holonomic frame field on $U$. Thus from now on, in this chapter, the indices $A, B, C, ...$ have the same range $\{1, ..., n + p\}$ as the indices $a, b, c, ...$, but the latter are used as indices for local components of geometric objects defined by means of the holonomic frame and coframe fields $\left\{\frac{\partial}{\partial x^a}\right\}$ and $\{dx^a\}$ on $U$. According to the definition of a distribution on a manifold, the transformation of non–holonomic frame fields on $U \cap \tilde{U} \neq \emptyset$ is given by

$$
\begin{align*}
(a) & \quad \tilde{E}_i = a^j_i E_j, & & (b) \quad \tilde{E}_\alpha = a^\beta_\alpha E_\beta, \\
\end{align*}
$$

where $[a^j_i]$ and $[a^\beta_\alpha]$ are $n \times n$ and $p \times p$ non–singular matrices respectively. Now, we consider the natural field of frames $\left\{\frac{\partial}{\partial x^a}\right\}$ on $M$ and put

$$
\begin{align*}
(a) & \quad E_A = E_A^a \frac{\partial}{\partial x^a} \quad \text{and} \quad (b) \quad \frac{\partial}{\partial x^a} = \tilde{E}_a^A E_A.
\end{align*}
$$

Then taking into account that the $(n + p)\times(n + p)$ matrices $[E_A^a]$ and $[\tilde{E}_a^A]$ are inverses for each other we deduce that

$$
\begin{align*}
(a) & \quad \tilde{E}_i^b E_i^b + \tilde{E}_\alpha^\beta E_\alpha^\beta = \delta^b_a, \\
(b) & \quad E_\alpha^a \tilde{E}_\beta^\alpha = \delta^\beta_\beta, \\
(c) & \quad E_\alpha^a \tilde{E}_j^\alpha = \delta^j_i, \\
(d) & \quad E_\alpha^a \tilde{E}_\alpha^a = 0, \\
(e) & \quad E_\alpha^a \tilde{E}_i^a = 0.
\end{align*}
$$

The dual frame field $\{\omega^A\} = \{\omega^i, \omega^\alpha\}$ to the non–holonomic frame field $E_A = \{E_i, E_\alpha\}$ is called the dual non–holonomic coframe field to $\{E_A\}$. Then the distributions $D$ and $D'$ are locally defined by the differential systems

$$
\begin{align*}
\omega^\alpha = 0, \quad & \alpha \in \{n + 1, ..., n + p\}, \\
\omega^i = 0, \quad & i \in \{1, ..., n\},
\end{align*}
$$

respectively.
1.2 Adapted Linear Connections on Almost Product Manifolds

Let $D$ be an $n$–distribution on an $(n+p)$–dimensional manifold $M$. A linear connection $\nabla^*$ on $M$ is said to be adapted to $D$ if

$$\nabla^*_X U \in \Gamma(D), \quad \forall X \in \Gamma(TM), \ U \in \Gamma(D).$$

Now, if $D'$ is a $p$–distribution on $M$ complementary to $D$, then $(M, D, D')$ is an almost product manifold as we have seen in Section 1.1. We call $D$ the structural distribution and $D'$ a transversal distribution. These names were introduced by Vaisman [Vai71] when $D$ is a distribution on a Riemannian manifold and $D'$ is its orthogonal complement.

A linear connection $\nabla^*$ on an almost product manifold $(M, D, D')$ is said to be an adapted linear connection if it is adapted to both distributions $D$ and $D'$. Thus $\nabla^*$ is adapted if and only if the following conditions are satisfied:

$$\nabla^*_X QY \in \Gamma(D), \quad \forall X,Y \in \Gamma(TM),$$

and

$$\nabla^*_X Q'Y \in \Gamma(D'), \quad \forall X,Y \in \Gamma(TM),$$

where $Q$ and $Q'$ stand, as in the first section, for projection morphisms of $TM$ on $D$ and $D'$ respectively. It is easy to see that an adapted linear connection $\nabla^*$ defines two linear connections $\nabla$ and $\nabla'$ on $D$ and $D'$ respectively, by

(a) $\nabla_X QY = \nabla^*_X QY,$ \quad and\n
(b) $\nabla'_X Q'Y = \nabla^*_X Q'Y, \forall X,Y \in \Gamma(TM).$  \quad (2.3)

Conversely, if $\nabla$ and $\nabla'$ are two linear connections on $D$ and $D'$ respectively, then we construct an adapted linear connection $\nabla^*$ on $(M, D, D')$, by the formula

$$\nabla^*_X Y = \nabla_X QY + \nabla'_X Q'Y, \quad \forall X,Y \in \Gamma(TM).$$

Moreover, the restrictions of $\nabla^*_X$ to $\Gamma(D)$ and $\Gamma(D')$ are exactly $\nabla_X$ and $\nabla'_X$ respectively. Thus, by the above discussion we state the following.

**Theorem 2.1.** There exists on $(M, D, D')$ an adapted linear connection $\nabla^*$ if and only if there exists a pair $(\nabla, \nabla')$, where $\nabla$ and $\nabla'$ are linear connections on $D$ and $D'$ respectively.

An adapted linear connection on $(M, D, D')$ can also be characterized by means of the almost product structure $F$ given by (1.11) and as well by the projection morphisms $Q$ and $Q'$. To state this we give the following definition. We say that $F$ is parallel with respect to a linear connection $\tilde{\nabla}$ on $M$ if its covariant derivative with respect to $\tilde{\nabla}$ vanishes, i.e., we have
\[(\bar{\nabla}_X F)Y = \bar{\nabla}_X FY - F(\bar{\nabla}_X Y) = 0, \quad \forall X, Y \in \Gamma(TM). \tag{2.5}\]

The same definition applies for \(Q\) and \(Q'\). Then the following theorem can be easily proved.

**Theorem 2.2.** Let \(\nabla^*\) be a linear connection on the almost product manifold \((M, D, D')\). Then the following assertions are equivalent:

(i) \(\nabla^*\) is an adapted linear connection.

(ii) The almost product structure \(F\) is parallel with respect to \(\nabla^*\).

(iii) The projection morphisms \(Q\) and \(Q'\) are parallel with respect to \(\nabla^*\).

Next, we would like to present some local characterizations of the linear connections on \(D\) and \(D'\), and therefore of the adapted linear connections on \((M, D, D')\). To this end, we consider the non–holonomic frame field \(\{E_A\} = \{E_i, E_\alpha\}\) on \(U \subset M\). Then for any smooth function \(f\) on \(M\) we define

(a) \(f|_\alpha = E_\alpha(f) = E^a_\alpha \frac{\partial f}{\partial x^a}\), and

(b) \(f||_i = E_i(f) = E^a_i \frac{\partial f}{\partial x^a}\). \tag{2.6}

We call \(f|_\alpha\) and \(f||_i\) the **transversal non–holonomic derivative** and **structural non–holonomic derivative** of \(f\) with respect to the non–holonomic frame field \(\{E_A\}\). Now, let \(\nabla\) and \(\nabla'\) be linear connections on \(D\) and \(D'\) respectively. Then, locally on \(U \subset M\) we put

(a) \(\nabla E_j E_i = \Gamma^k_{ij} E_k\), \quad (b) \(\nabla E_\alpha E_i = \Gamma^k_{i\alpha} E_k\), \tag{2.7}

and

(a) \(\nabla' E_j E_\alpha = \Gamma^\beta_{\alpha j} E_\beta\), \quad (b) \(\nabla' E_\alpha E_\beta = \Gamma^\gamma_{\alpha \beta} E_\gamma\). \tag{2.8}

We perform a transformation of non–holonomic frame fields, and by using (1.15), (2.7) and (2.8) we obtain

(a) \(\bar{\nabla}^h_{\gamma a_h} = (\Gamma^k_{ij} a^i_s + (a^k_s)_{||j}) a^j_i\),

(b) \(\bar{\nabla}^h_{\beta a_h} = (\Gamma^k_{i\alpha} a^i_s + (a^k_s)_{|\alpha}) a^\alpha_i\), \tag{2.9}

and

(a) \(\bar{\nabla}'^\varepsilon_{\gamma a_\varepsilon} = (\Gamma^\beta_{\alpha i} a^\alpha_\varepsilon + (a^\beta_\varepsilon)_{||i}) a^i_j\),

(b) \(\bar{\nabla}'^\varepsilon_{\mu a_\varepsilon} = (\Gamma^\beta_{\alpha \gamma} a^\alpha_\varepsilon + (a^\beta_\varepsilon)_{|\gamma}) a^\gamma_\mu\). \tag{2.10}

Conversely, if on each \(U \subset M\) there exist functions \((\Gamma^k_{ij}, \Gamma^k_{i\alpha})\) and \((\Gamma^\beta_{\alpha i}, \Gamma^\beta_{\alpha \gamma})\) satisfying (2.9) and (2.10) with respect to the transformation (1.15) of non–holonomic frame fields, then the differential operators \(\nabla\) and \(\nabla'\) given by (2.7) and (2.8) define linear connections on \(D\) and \(D'\) respectively. Thus we may state the following.
Theorem 2.3.

(i) There exists a linear connection on $\mathcal{D}$ if and only if on each coordinate neighbourhood $U \subset M$ there exist $n^2(n + p)$ functions $(\Gamma^k_i, \Gamma^k_i \alpha)$ satisfying (2.9) with respect to (1.15).

(ii) There exists a linear connection on $\mathcal{D}'$ if and only if on each coordinate neighbourhood $U \subset M$ there exist $p^2(n + p)$ functions $(\Gamma'_{\alpha \beta}, \Gamma'_{\alpha \beta \gamma})$ satisfying (2.10) with respect to (1.15).

The next corollary follows from Theorems 2.1 and 2.3.

Corollary 2.4. The exists an adapted linear connection on $M$ if and only if on each $U \subset M$ there exist $(n^2 + p^2)(n + p)$ functions $(\Gamma^k_i, \Gamma^k_i \alpha, \Gamma'_{\alpha \beta}, \Gamma'_{\alpha \beta \gamma})$ satisfying (2.9) and (2.10) with respect to (1.15).

Thus an adapted linear connection $\nabla^*$ on $M$ is locally given by

\[
\begin{align*}
(a) \quad \nabla^*_{E_j} E_i &= \Gamma^k_i E_k, \\
(b) \quad \nabla^*_{E_\alpha} E_i &= \Gamma^k_i \alpha E_k, \\
(c) \quad \nabla^*_{E_\alpha} E_\alpha &= \Gamma'_{\alpha \beta} E_\beta, \\
(d) \quad \nabla^*_{E_\gamma} E_\alpha &= \Gamma'_{\alpha \beta \gamma} E_\beta,
\end{align*}
\]

where the non–holonomic coefficients satisfy the conditions from Corollary 2.4.

Next, we consider an adapted linear connection $\nabla^* = (\Gamma^k_i A, \Gamma'_{\alpha \beta} A)$ on $(M, \mathcal{D}, \mathcal{D}')$ and look for the non–holonomic local components of its torsion and curvature tensor fields with respect to a non–holonomic frame field. To achieve this we put:

\[
\begin{align*}
(a) \quad Q[E_j, E_i] &= V^k_i E_k, \\
(b) \quad Q[E_\beta, E_\alpha] &= V^k_\alpha E_k, \\
(c) \quad Q[E_\alpha, E_i] &= -Q[E_i, E_\alpha] = V^k_\alpha E_k = -V^k_i E_k,
\end{align*}
\]

and

\[
\begin{align*}
(a) \quad Q'[E_j, E_i] &= V'_{\alpha \beta} E_\beta, \\
(b) \quad Q'[E_\gamma, E_\alpha] &= V'_{\alpha \beta \gamma} E_\beta, \\
(c) \quad Q'[E_i, E_\alpha] &= -Q'[E_\alpha, E_i] = V'_{\alpha \beta} E_\beta = -V'_{\alpha \beta \gamma} E_\beta.
\end{align*}
\]

Then we recall that the torsion tensor field $T^*$ of the linear connection $\nabla^*$ is given by (cf. Kobayashi–Nomizu [KN63], p. 133)

\[
T^*(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y], \quad \forall X,Y \in \Gamma(TM).
\]

By using the decomposition (1.9) and the non–holonomic frame field $\{E_A\}$ we set:

\[
\begin{align*}
(a) \quad T^*(E_j, E_i) &= T^k_i j E_k + T^i_\alpha j E_\alpha, \\
(b) \quad T^*(E_\alpha, E_i) &= -T^*(E_i, E_\alpha) = T^k_\alpha E_k + T^i_\alpha \beta E_\beta \\
&= -T^k_\alpha E_k - T'_\alpha \beta \gamma E_\beta, \\
(c) \quad T^*(E_\gamma, E_\alpha) &= T^k_\alpha \gamma E_k + T'_\alpha \beta \gamma E_\beta.
\end{align*}
\]
Then by direct calculations using (2.11)–(2.15) we obtain all non–holonomic components of $T^*$ as in the next theorem.

**Theorem 2.5.** Let $\nabla^* = (\Gamma^k_i, \Gamma'^\beta_i^\alpha \alpha)$ be an adapted linear connection on the almost product manifold $(M, \mathcal{D}, \mathcal{D}')$. Then the local components of its torsion tensor field with respect to a non–holonomic frame field $\{E_A\}$ are given by

(a) $T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} - V^k_{ij}$,

(b) $T'_{i}^{\alpha j} = -V'_{i}^{\alpha j}$,

(c) $T^k_{i}^{\alpha} = -T^k_{i}^{\alpha} + \Gamma^k_{i}^{\alpha} - V^k_{i}^{\alpha}$,

(d) $T'^{\beta i}_{\alpha} = -T'^{\beta i}_{\alpha} + \Gamma'^{\beta i}_{\alpha} - V'^{\beta i}_{\alpha}$,

(e) $T'_{\alpha}^{k \beta} = -V'_{\alpha}^{k \beta}$,

(f) $T'^{\alpha \beta}_{\gamma} = \Gamma'^{\alpha \beta}_{\gamma} - \Gamma'^{\alpha \beta}_{\gamma} - V'^{\alpha \beta}_{\gamma}$.

We now look for the non–holonomic local components of the curvature tensor field $R^*$ of $\nabla^*$, given by (cf. Kobayashi–Nomizu [KN63], p. 133)

$$R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_Z [X,Y],$$

(2.17)

for any $X,Y,Z \in \Gamma(TM)$. To this end we first note that the $F(M)$–linear operator $R^*(X,Y)$ on $\Gamma(TM)$ induces $F(M)$–linear operators on both $\Gamma(D)$ and $\Gamma(D')$. This enables us to set:

(a) $R^*(E_k, E_j)E_i = R^h_{i j k} E_h$,

(b) $R^*(E_k, E_\alpha)E_i = -R^*(E_\alpha, E_k)E_i = R^h_{i k \alpha} E_h = -R^h_{i \alpha k} E_h$,

(c) $R^*(E_\beta, E_\alpha)E_i = R^h_{i \alpha \beta} E_h$,

and

(a) $R^*(E_k, E_j)E_\alpha = R'^{\alpha \beta}_{j k} E_\beta$,

(b) $R^*(E_k, E_\gamma)E_\alpha = -R^*(E_\gamma, E_k)E_\alpha = R'^{\alpha \beta \gamma}_{k} E_\beta = -R'^{\alpha \beta}_{k \gamma} E_\beta$,

(c) $R^*(E_\mu, E_\gamma)E_\alpha = R'^{\alpha \beta \mu}_{\gamma} E_\beta$.

(2.19)

The proof of the next theorem follows by direct calculations using (2.11)–(2.13) and (2.17)–(2.19).
Theorem 2.6. Let $\nabla^* = (\Gamma^k_{\alpha \beta})$ be an adapted linear connection on the almost product manifold $(M, D, D')$. Then the local components of its curvature tensor field with respect to a non–holonomic frame field $\{E_A\}$ are given by

(a) $R^h_{ijk} = \Gamma^h_{ij}k - \Gamma^h_{ik}j + \Gamma^s_{ij}k \Gamma^s_{ik}j - \Gamma^s_{ik}k \Gamma^s_{ij}j - \Gamma^s_{js}k \Gamma^s_{shk} - \Gamma^s_{isk} \Gamma^s_{shj} - \Gamma^s_{ihs} V^s_{jsk} - \Gamma^s_{ih\alpha} V^s_{j\alpha k}$,

(b) $R^h_{i\alpha k} = \Gamma^h_{i\alpha k} - \Gamma^h_{i\alpha k} + \Gamma^s_{i\alpha k} \Gamma^s_{h\alpha} - \Gamma^s_{ish\alpha} - \Gamma^s_{ihs} V^s_{\alpha sk} - \Gamma^s_{ih\epsilon} V^s_{j\epsilon k}$,

(c) $R^h_{i\alpha\beta} = \Gamma^h_{i\alpha\beta} - \Gamma^h_{i\alpha\beta} + \Gamma^s_{i\alpha\beta} \Gamma^s_{h\alpha} - \Gamma^s_{ish\alpha} - \Gamma^s_{ihs} V^s_{\alpha\beta k}$,

and

(a) $R'^{\beta}_{\alpha jk} = \Gamma'^{\beta}_{\alpha j}k - \Gamma'^{\beta}_{\alpha k}j + \Gamma'^{\epsilon}_{\alpha j}k \Gamma'^{\epsilon}_{\alpha k}j - \Gamma'^{\epsilon}_{\alpha k}j \Gamma'^{\epsilon}_{\alpha j}k - \Gamma'^{\epsilon}_{\alpha \beta} s V^s_{\epsilon j k} - \Gamma'^{\epsilon}_{\alpha \beta} V^s_{\epsilon k}$,

(b) $R'^{\beta}_{\alpha \gamma k} = \Gamma'^{\beta}_{\alpha \gamma}k - \Gamma'^{\beta}_{\alpha k} \Gamma'^{\gamma}_{\alpha \gamma} - \Gamma'^{\epsilon}_{\alpha \gamma} \Gamma'^{\beta}_{\alpha \gamma} - \Gamma'^{\epsilon}_{\alpha k} \Gamma'^{\beta}_{\gamma \gamma} - \Gamma'^{\epsilon}_{\alpha \beta} s V^s_{\epsilon \gamma k}$,

(c) $R'^{\beta}_{\alpha \gamma \mu} = \Gamma'^{\beta}_{\alpha \gamma \mu} - \Gamma'^{\beta}_{\alpha \mu} \Gamma'^{\gamma}_{\alpha \gamma} - \Gamma'^{\epsilon}_{\alpha \gamma} \Gamma'^{\beta}_{\alpha \gamma} - \Gamma'^{\epsilon}_{\alpha \mu} \Gamma'^{\beta}_{\gamma \gamma} - \Gamma'^{\epsilon}_{\alpha \beta} s V^s_{\epsilon \gamma \mu} - \Gamma'^{\epsilon}_{\alpha \beta} V^s_{\epsilon \gamma \mu}$.

Taking into account that $\nabla^*$ induces a linear connection $\nabla = (\Gamma^k_{\alpha \beta})$ on $D$ and a linear connection $\nabla' = (\Gamma'^{\alpha}_{\alpha \beta})$ on $D'$, by Theorem 2.6 we may state the following.

Corollary 2.7. The local components of the curvature tensor fields of $\nabla$ and $\nabla'$ with respect to a non–holonomic frame field $\{E_A\}$ are given by (2.20) and (2.21) respectively.

As it is well known, a torsion tensor field is not defined, in general, for a linear connection on a vector bundle. However, by using the notion of general connection introduced by Otsuki [Ots61] we will define here a torsion tensor field for a linear connection on a distribution. To achieve this we consider a vector bundle $E$ over $M$ and a vector bundle morphism $P : E \longrightarrow E$. Then according to Abe [Abe85] an Otsuki connection (general connection) on $E$ with respect to the vector bundle morphism $P$ is a mapping $\tilde{\nabla}$ that assigns to any $X \in \Gamma(TM)$ the differential operator

$$\tilde{\nabla}_X : \Gamma(E) \longrightarrow \Gamma(E); \quad S \longrightarrow \tilde{\nabla}_X S, \quad \forall S \in \Gamma(E),$$
satisfying the following conditions:

\[ \tilde{\nabla}_{fX + Y}(S) = f\tilde{\nabla}_X S + \tilde{\nabla}_Y S, \]

and

\[ \tilde{\nabla}_X (fS + S') = X(f)P(S) + f\tilde{\nabla}_X S + \tilde{\nabla}_X S', \]

for any \( f \in F(M) \), \( X, Y \in \Gamma(TM) \) and \( S, S' \in \Gamma(E) \). It is easy to see that \( \tilde{\nabla} \)
becomes a linear connection on \( E \) when \( P \) is the identity morphism on \( E \).

The above operator \( \tilde{\nabla}_X \) can be extended to \( F(M) \)-linear mappings \( N : (\Gamma(E))^r \rightarrow \Gamma(E) \) for any positive integer \( r \). In particular, for the identity morphism \( I_E \) on \( E \) we have

\[ (\tilde{\nabla}_X I_E)(S) = \tilde{\nabla}_X P(S) - P(\tilde{\nabla}_X S), \quad \forall X \in \Gamma(TM), \ S \in \Gamma(E). \]

The curvature form \( \tilde{\Omega} \) of \( \tilde{\nabla} \) is defined as follows (cf. Abe [Abe85])

\[ \tilde{\Omega}(X, Y)S = \tilde{\nabla}_X \tilde{\nabla}_Y P(S) - \tilde{\nabla}_Y \tilde{\nabla}_X P(S) - P(\tilde{\nabla}_{[X,Y]}P(S)) - (\tilde{\nabla}_X I_E)(\tilde{\nabla}_Y S) + (\tilde{\nabla}_Y I_E)(\tilde{\nabla}_X S), \]

for any \( X, Y \in \Gamma(TM) \) and \( S \in \Gamma(E) \). For the particular case \( E = TM \), an Otsuki connection \( \tilde{\nabla} \) has a torsion tensor field \( \tilde{T} \) given by (cf. Nemoto [Nem85])

\[ \tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - P([X, Y]), \quad \forall X, Y \in \Gamma(TM). \quad (2.22) \]

Now, we show that starting with a linear connection \( \nabla \) on a vector bundle \( E \) we can obtain an Otsuki connection \( \tilde{\nabla} \) on a vector bundle \( G \) that is larger than \( E \) and \( \tilde{\nabla} = \nabla \) on \( E \). Indeed, suppose \( G = E \oplus F \), where \( F \) is another vector bundle over \( M \), and denote by \( P \) the projection morphism of \( G \) on \( E \). Then for any \( X \in \Gamma(TM) \) we define the differential operator

\[ \tilde{\nabla}_X : \Gamma(G) \rightarrow \Gamma(G); \tilde{\nabla}_X S = \nabla_X P(S), \quad \forall S \in \Gamma(G). \quad (2.23) \]

It is easy to check that \( \tilde{\nabla} \) is an Otsuki connection on \( G \) with respect to the vector bundle morphism \( P \) and \( \tilde{\nabla} = \nabla \) on \( E \). Moreover, the following has been proved.

**Theorem 2.8.** (Bejancu–Otsuki [BO87]). The restriction of the curvature form \( \tilde{\Omega} \) of \( \tilde{\nabla} \) to the sections of \( E \) coincides with the curvature form \( \Omega \) of \( \nabla \).

Next, we apply the theory of Otsuki connections to the study of an almost product manifold \( (M, D, D') \). First, suppose that \( \nabla \) is a linear connection on \( D \) and consider the Otsuki connection \( \tilde{\nabla} \) on \( TM \) with respect to the decomposition (1.9) such that \( \tilde{\nabla} = \nabla \) on \( D \). Then according to (2.23) we have

\[ \tilde{\nabla}_X Y = \nabla_X QY, \quad \forall X, Y \in \Gamma(TM). \quad (2.24) \]
Taking into account the relationship between the curvature forms of $\tilde{\nabla}$ and $\nabla$ stated in Theorem 2.8, we define a torsion tensor field $T$ of $\nabla$ as the restriction of the torsion tensor field $\tilde{T}$ of $\tilde{\nabla}$ to $\Gamma(TM) \times \Gamma(D)$. It is noteworthy that $T$ is $\Gamma(D)$–valued. More precisely, by using (2.22) and (2.24) we obtain

$$T(X, QY) = \tilde{T}(X, QY) = \nabla_X QY - \nabla_{QY} QX - Q[X, QY],$$  \hspace{1cm} (2.25)

for any $X, Y \in \Gamma(TM)$. As $T$ depends on $D'$ we call it the $D'$–torsion tensor field of $\nabla$. Similarly, a linear connection $\nabla'$ on $D'$ has a $D$–torsion tensor field $T'$ given by

$$T'(X, Q'Y) = \nabla'_X Q'Y - \nabla'_{Q'Y} Q'X - Q'[X, Q'Y], \ \forall X, Y \in \Gamma(TM).$$  \hspace{1cm} (2.26)

Finally, with respect to a non–holonomic frame field $\{E_A\}$ on $U \subset M$ we put:

(a) $T(E_j, E_i) = T_{ij}^k E_k$, \hspace{1cm} (b) $T(E_\alpha, E_i) = T_{i\alpha}^k E_k$, \hspace{1cm} (2.27)

and

(a) $T'(E_\gamma, E_\alpha) = T_{\alpha\beta\gamma}^\nu E_\nu$, \hspace{1cm} (b) $T'(E_i, E_\alpha) = T_{i\alpha}^\nu E_\nu$. \hspace{1cm} (2.28)

Then by using (2.7), (2.8), (2.12), (2.13) and (2.25)–(2.28) we deduce the local components of $T$ and $T'$ with respect to $\{E_A\}$ as they are expressed in the next theorem.

**Theorem 2.9.** Let $\nabla$ and $\nabla'$ be linear connections on the complementary distributions $D$ and $D'$ on $M$. Then the local components of $T$ and $T'$ with respect to the non–holonomic frame field $\{E_A\}$ are given by

(a) $T_{ij}^k = \Gamma_{ij}^k - \Gamma_{ji}^k - V_{ij}^k$, \hspace{1cm} (b) $T_{i\alpha}^k = \Gamma_{i\alpha}^k - V_{i\alpha}^k$, \hspace{1cm} (2.29)

and

(a) $T'_{\alpha\beta\gamma} = \Gamma'_{\alpha\beta\gamma} - \Gamma'_{\gamma\beta\alpha} - V'_{\alpha\beta\gamma}$, \hspace{1cm} (b) $T'_{i\alpha}^\nu = \Gamma'_{i\alpha}^\nu - V'_{i\alpha}^\nu$, \hspace{1cm} (2.30)

respectively.

As the pair $(\nabla, \nabla')$ defines an adapted linear connection $\nabla^*$ on $M$ we should see what relationship exists (if any) between their torsion tensor fields. First, by (2.14), (2.25) and (2.26) we deduce that $T$ and $T'$ are not equal to the restrictions of $T^*$ on $\Gamma(TM) \times \Gamma(D)$ and $\Gamma(TM) \times \Gamma(D')$ respectively. However, comparing Theorems 2.5 and 2.9 we see that the local components of $T$ and $T'$ form a part of the local components of $T^*$ with respect to a non–holonomic frame field $\{E_A\}$ on $M$. 
1.3 The Schouten–Van Kampen and Vrăanceanu Connections

In the first part of this section we study the existence of adapted linear connections on an almost product manifold \((M, \mathcal{D}, \mathcal{D}')\). More precisely, we construct two adapted linear connections which were first introduced by Schouten and Van Kampen [SVK30] and Vrăanceanu [VG31] for studying non–holonomic manifolds. Then we determine the general form of all adapted linear connections on \((M, \mathcal{D}, \mathcal{D}')\) and present these two special connections in an invariant form.

As \(M\) is supposed to be paracompact, by a result stated in Brickell–Clark [BC70], p. 154, there exists a linear connection \(\nabla\) on \(M\). Then, locally we set

\[
\nabla_{E_{ib}} E_A = F_A^C B E_C, \tag{3.1}
\]

where \(\{E_A\} = \{E_i, E_\alpha\}\) is a non–holonomic frame field on \(\mathcal{U} \subset M\). By direct calculations using (3.1) with respect to two non–holonomic frame fields \(\{E_A\}\) and \(\{\tilde{E}_A\}\) on \(\mathcal{U}\) and \(\tilde{\mathcal{U}}\) we obtain the following transformations of non–holonomic coefficients of \(\nabla\) on \(\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset\):

\[
(a) \quad \tilde{F}_s^k i a^k_h = (F_i^k j a^i_s + (a^k_s)_{\|j}) a^j_t, \quad (b) \quad \tilde{F}_s^k i a^\alpha_\beta = F_i^k j a^\alpha_s a^j_t, \\
(c) \quad \tilde{F}_s^k \gamma a^k_\alpha = (F_i^k \alpha a^i_s + (a^k_s)_{\|\alpha}) a^\alpha_\beta, \quad (d) \quad \tilde{F}_s^k \gamma a^\alpha_\beta = F_i^k \gamma a^\alpha_s a^\beta_\gamma, \\
(e) \quad \tilde{F}_\nu^s j a^\beta_\gamma = (F_{\nu}^k \beta a^\nu_\gamma + (a^\beta_\gamma)_{\|i}) a^i_j, \quad (f) \quad \tilde{F}_\nu^s j a^\alpha_\beta = F_{\nu}^k i a^\alpha_s a^\beta_j, \\
(g) \quad \tilde{F}_\nu^s \mu a^\alpha_\beta = (F_{\nu}^{\alpha} \beta a^\nu_\mu + (a^\alpha_\beta)_{\|\gamma}) a^\gamma_\mu, \quad (h) \quad \tilde{F}_\nu^s \mu a^k_i = F_{\nu}^{\alpha} \beta a^\alpha_s a^\beta_\mu, \\
\]

with respect to (1.15). From (3.2a), (3.2c), (3.2e) and (3.2g) we deduce that \((\Gamma_i^k A, \Gamma_\alpha^\beta A)\) given by

\[
(a) \quad \Gamma_i^k A = F_i^k A, \quad (b) \quad \Gamma_\alpha^\beta A = F_\alpha^\beta A, \tag{3.3}
\]

satisfy the conditions of Corollary 2.4. Hence they define an adapted linear connection \(\nabla^o\) on \(M\). With respect to this connection we have to note that the formulas (60) from the book of Vrăanceanu [VG57], p.235, are the same as our (3.3). As these formulas were first obtained by Schouten and Van Kampen [SVK30], we call the adapted linear connection \(\nabla^o = (\Gamma_i^k A, \Gamma_\alpha^\beta A)\) given by (3.3) the Schouten–Van Kampen connection.

In order to define another adapted linear connection on \(M\) we consider (2.12c) on \(\tilde{\mathcal{U}} \subset M\) and \(\mathcal{U} \cap \tilde{\mathcal{U}} \neq \emptyset\). Then by using elementary properties of the Lie bracket and taking into account (1.15), (2.6a) and (2.12c) we obtain

\[
Q[\tilde{E}_\gamma, \tilde{E}_s] = Q[a^\alpha_\gamma E_\alpha, a^s_i E_i] = a^\alpha_\gamma a^s_i Q[E_\alpha, E_i] + a^\alpha_\gamma (a^s_i)_{\|\alpha} E_i \\
= (V_i^k \alpha a^i_s + (a^s_i)_{\|\alpha}) a^\alpha_\gamma E_k.
\]

On the other hand, by (2.12c) on \(\tilde{\mathcal{U}}\) and (1.15a) we have
Comparing these equalities we deduce that \( V^i_k \alpha \) satisfy (2.9b) with respect to (1.15). In a similar way it follows that \( V^{\prime \alpha \beta} i \) satisfy (2.10a). Hence, according to (3.2a) and (3.2g) the functions \( (\Gamma^* i k A, \Gamma^* \alpha \beta A) \) given on each \( U \subset M \) by

\[
\begin{align*}
(a) \quad &\Gamma^* i k j = F^i k j, \quad (b) \quad \Gamma^* i k \alpha = V^i k \alpha, \\
(c) \quad &\Gamma^* \alpha \beta i = V^{\prime \alpha \beta} i, \quad (d) \quad \Gamma^* \alpha \beta \gamma = F^\alpha \gamma, \\
\end{align*}
\]

also satisfy the conditions of Corollary 2.4. The above adapted linear connection was first introduced by Vr˘anceanu [VG31]. Indeed, it is easy to see that formulas (21) of Vr˘anceanu [VG31], p. 199, are the same as our (3.4). The same formulas can be found in the book of Vr˘anceanu [VG57] (see formulas (61) at p. 235). Thus we are entitled to call the adapted linear connection \( \nabla^* = (\Gamma^* i k A, \Gamma^* \alpha \beta A) \) the Vr˘anceanu connection.

Next, consider the torsion tensor field \( \tilde{T} \) of \( \tilde{\nabla} \) and by using (3.1) and (2.12)–(2.14) we obtain its local components with respect to the non–holonomic frame field \( \{E_A\} \):

\[
\begin{align*}
(a) \quad &\tilde{T} A k B = F A k B - F B k A - V A k B, \\
(b) \quad &\tilde{T} A \alpha B = F A \alpha B - F B \alpha A - V^\prime A \alpha B. \\
\end{align*}
\]

Also, by using (3.3), (3.4) and Theorem 2.5 we obtain the following.

**Theorem 3.1.** The local components of the torsion tensor fields \( T^\circ \) and \( T^* \) of Schouten–Van Kampen and Vr˘anceanu connections with respect to the non–holonomic frame field \( \{E_A\} \) are given by

\[
\begin{align*}
(a) \quad &T i k j = F i k j - F j k \ i - V i k j, \quad (b) \quad T i \alpha j = -V^\prime i k \alpha, \\
(c) \quad &T i k \alpha = -T \alpha k i = F i k \alpha - V i k \alpha, \quad (d) \quad T \alpha \beta i = -T \alpha \beta i = F \alpha \beta i - V^\prime \alpha \beta i, \\
(e) \quad &T \alpha k \beta = -V \alpha k \beta, \quad (f) \quad T \alpha \gamma \beta = F \alpha \gamma \beta - F \beta \gamma \alpha - V^\prime \alpha \gamma \beta, \\
\end{align*}
\]

and

\[
\begin{align*}
(a) \quad &T^* i k j = F^* i k j - F^* j k \ i - V^* i k j, \quad (b) \quad T^* i \alpha j = -V^\prime i \alpha, \\
(c) \quad &T^* i k \alpha = -T^* \alpha k i = 0, \quad (d) \quad T^* \alpha \beta i = -T^* \alpha \beta i = 0, \\
(e) \quad &T^* \alpha k \beta = -V^* \alpha k \beta, \quad (f) \quad T^* \alpha \gamma \beta = F^* \alpha \gamma \beta - F^* \beta \gamma \alpha - V^\prime \alpha \gamma \beta, \\
\end{align*}
\]

respectively.

**Corollary 3.2.** The Schouten–Van Kampen and Vr˘anceanu connections coincide if and only if they have the same torsion tensor fields.

From (3.5)–(3.7) we see that even when \( \tilde{\nabla} \) is torsion–free, the Schouten–Van Kampen and Vr˘anceanu connections are not necessarily torsion–free. Related to this, by using (3.5) and (3.7) we obtain the following.
Theorem 3.3. (Vrăanceanu [VG57], p. 235). Let $\tilde{\nabla}$ be a torsion–free linear connection on $(M, D, D')$. Then the Vrăanceanu connection determined by $\tilde{\nabla}$ is torsion–free if and only if both distributions $D$ and $D'$ are involutive.

This made Vrăanceanu ([VG57], p. 236) remark that the connection $\nabla^*$ is more intimately related to the properties of the manifold than $\nabla^\circ$. This remark will become more evident as we go further into the study of non–holonomic semi–Riemannian manifolds and semi–Riemannian foliations.

According to Theorem 2.6 we may write down all the local components of curvature tensor fields of $\nabla^\circ$ and $\nabla^*$ with respect to a non–holonomic frame field. However, since for $\nabla^\circ$ we just replace $\Gamma$ and $\Gamma'$ from (2.20) and (2.21) by $F$, we omit them here. We only apply Theorem 2.6 for $\nabla^*$ and obtain the following.

Theorem 3.4. The local components of the curvature tensor field of the Vrăanceanu connection $\nabla^*$ with respect to a non–holonomic frame field $\{E_A\}$ are given by

\[
(a) \, R^*_{\ i \ j k} = F^*_{ i \ j k} - F^*_{ i \ k j} + F^*_{ i \ j} F^*_{ k} - F^*_{ i \ k} F^*_{ j},
\]
\[
- F^*_{ i \ j} V^*_{ k} - V^*_{ i \ k} V^*_{ j},
\]

\[
(b) \, R^*_{ i \ a k} = V^*_{ i \ a k} - F^*_{ i \ k a} + V^*_{ i \ a} F^*_{ k} - F^*_{ i \ k} V^*_{ a},
\]
\[
- F^*_{ i \ k} V^*_{ a} - V^*_{ i \ a} V^*_{ k},
\]

\[
(c) \, R^*_{ i \ a \beta} = V^*_{ i \ a \beta} - V^*_{ i \ \beta a} + V^*_{ i \ a} V^*_{ \beta} - V^*_{ i \ \beta} V^*_{ a},
\]
\[
- F^*_{ i \ a} V^*_{ \beta} - V^*_{ i \ \beta} V^*_{ a},
\]

and

\[
(a) \, R^*_{ \alpha \beta \ j k} = V^*_{ \alpha \beta \ j k} - F^*_{ \alpha \beta \ j k} + V^*_{ \alpha \beta} V^*_{ j k} - V^*_{ \alpha \beta} V^*_{ j} V^*_{ k},
\]
\[
- V^*_{ \alpha \beta} V^*_{ j} V^*_{ k} - F^*_{ \alpha \beta} V^*_{ j k},
\]

\[
(b) \, R^*_{ \alpha \gamma \ j k} = F^*_{ \alpha \gamma \ j k} - V^*_{ \alpha \gamma \ j k} + F^*_{ \alpha \gamma} V^*_{ j k} - V^*_{ \alpha \gamma} V^*_{ j} V^*_{ k},
\]
\[
- V^*_{ \alpha \gamma} V^*_{ j} V^*_{ k} - F^*_{ \alpha \gamma} V^*_{ j k},
\]

\[
(c) \, R^*_{ \alpha \beta \gamma \mu} = F^*_{ \alpha \beta \gamma \mu} - F^*_{ \alpha \beta \mu \gamma} + F^*_{ \alpha \beta} F^*_{ \gamma \mu} - F^*_{ \alpha \beta} F^*_{ \gamma \mu},
\]
\[
- V^*_{ \alpha \beta} F^*_{ \gamma \mu} - F^*_{ \alpha \beta} V^*_{ \gamma \mu}.
\]

Now, we want to express the general form of all adapted linear connections on $(M, D, D')$ and then to describe the Schouten–Van Kampen and Vrăanceanu connections in an invariant form. First we prove the following general result.

Theorem 3.5. Let $(M, D, D')$ be an almost product manifold and $\tilde{\nabla}$ be a linear connection on $M$. Then all the adapted linear connections on $M$ are given by

\[
\nabla_X Y = Q\tilde{\nabla}_X QY + Q'\tilde{\nabla}_X Q'Y + QS(X, QY) + Q'S(X, Q'Y),
\]

(3.10)
for any \( X, Y \in \Gamma(TM) \), where \( S \) is an arbitrary tensor field of type \((1, 2)\) on \( M \).

**Proof.** It is easy to check that \( \nabla \) given by (3.10) is an adapted linear connection on \( M \). Conversely, suppose that \( \nabla \) is an adapted linear connection on \( M \). Then we put

\[
\nabla_X Y - \tilde{\nabla}_X Y = S(X, Y), \quad \forall X, Y \in \Gamma(TM),
\]

(3.11)

where \( S \) is a tensor field of type \((1, 2)\) on \( M \). Next, by using (2.1) and (2.2), we have

\[
Q'(\nabla_X QY) = 0 \quad \text{and} \quad Q(\nabla_X Q'Y) = 0, \quad \forall X, Y \in \Gamma(TM).
\]

Thus by (3.11) we deduce that

\[
Q'(\tilde{\nabla}_X QY + S(X, QY)) = 0 \quad \text{and} \quad Q(\tilde{\nabla}_X Q'Y + S(X, Q'Y)) = 0,
\]

(3.12)

for any \( X, Y \in \Gamma(TM) \). Finally, by using (3.12) in (3.11) we obtain (3.10).

Next, we define:

\[
S^\circ(X, Y) = Q'\tilde{\nabla}_X QY + Q\tilde{\nabla}_X Q'Y,
\]

and

\[
S^*(X, Y) = Q([Q'X, QY] - \tilde{\nabla}_{Q'X} QY) + Q'([QX, Q'Y] - \tilde{\nabla}_{QX} Q'Y),
\]

for any \( X, Y \in \Gamma(TM) \). It is easy to check that both \( S^\circ \) and \( S^* \) are tensor fields of type \((1, 2)\) on \( M \). Then, by direct calculations we deduce that

\[
(a) \quad QS^\circ(X, QY) = 0, \quad (b) \quad Q'S^\circ(X, Q'Y) = 0,
\]

(3.13)

and

\[
(a) \quad QS^*(X, QY) = Q([Q'X, QY] - \tilde{\nabla}_{Q'X} QY), \quad \text{and} \quad (b) \quad Q'S^*(X, Q'Y) = Q'([QX, Q'Y] - \tilde{\nabla}_{QX} Q'Y),
\]

(3.14)

for any \( X, Y \in \Gamma(TM) \). Finally, by using in turn (3.13) and (3.14) in the general form (3.10) we obtain two adapted linear connections \( \nabla^\circ \) and \( \nabla^* \) given by

\[
\nabla^\circ_X Y = Q\tilde{\nabla}_X QY + Q'\tilde{\nabla}_X Q'Y,
\]

(3.15)

and

\[
\nabla^*_X Y = Q\tilde{\nabla}_{QX} QY + Q'\tilde{\nabla}_{Q'X} Q'Y + Q[Q'X, QY] + Q'[QX, Q'Y],
\]

(3.16)

for any \( X, Y \in \Gamma(TM) \). Moreover, we prove the following theorem.
Theorem 3.6. The adapted linear connections given by (3.15) and (3.16) are the Schouten–Van Kampen and Vrânceanu connections respectively.

Proof. Replace the pair \((X,Y)\) from (3.15) and (3.16) in turn by \((E_j, E_i)\), \((E_\alpha, E_i)\), \((E_i, E_\alpha)\) and \((E_\gamma, E_\alpha)\) and using (3.1), (2.12) and (2.13) we obtain the local coefficients of Schouten–Van Kampen and Vrânceanu connections given by (3.3) and (3.4) respectively.

The coordinate-free forms (3.15) and (3.16) of Schouten–Van Kampen and Vrânceanu connections were first obtained by Ianuș [Ian71] and then used by Bădițoiu, Buchner and Ianuș [BBI98] for studying semi–Riemannian submersions.

1.4 From Semi–Euclidean Algebra to Semi–Riemannian Geometry

For the sake of completeness of the book, and to present our terminology, we start with some basic notions and results about semi–Euclidean spaces.

Let \(V\) be a real \(m\)–dimensional vector space and \(g : V \times V \to \mathbb{R}\) be a symmetric bilinear mapping. We say that \(g\) is a scalar product on \(V\) if it is non–degenerate, that is, whenever \(g(u, v) = 0\) for all \(v \in V\), then \(u = 0\). The vector space \(V\) endowed with a scalar product \(g\) is denoted by \((V, g)\) and it is called a semi–Euclidean (pseudo–Euclidean) space. Let \(q\) be the dimension of the largest subspace \(W\) of \((V, g)\) now which \(g\) is negative definite, i.e., \(g(w, w) < 0\) for any non-zero vector \(w \in W\). Then we say that \(g\) is of index \(q\). When \(q = 0\) (resp. \(q = 1\)), \((V, g)\) is called a Euclidean space (resp. Lorentz (Minkowski) space). If \(0 < q < m\), then we say that \((V, g)\) is a proper semi–Euclidean space. In such a vector space we have three categories of vectors as follows. A vector \(v \in V\) is called:

- space–like, if \(g(v, v) > 0\) or \(v = 0\),
- light–like (null), if \(g(v, v) = 0\) and \(v \neq 0\),
- time–like, if \(g(v, v) < 0\).

The length (norm) of \(v \in V\) is the non–negative number \(|v| = |g(v, v)|^{1/2}\). When \(|v| = 1\) we say that \(v\) is a unit vector. Two vectors \(v\) and \(w\) are orthogonal if \(g(v, w) = 0\). Contrary to the case of Euclidean geometry, a light–like vector of a proper semi–Euclidean space is a non-zero vector that is orthogonal to itself. A basis of \((V, g)\) formed by \(m\) mutually orthogonal unit vectors is called an orthonormal basis. The existence of such bases is ensured by the following.

Lemma 4.1. (O’Neill [O83], p. 50).
Lemma 4.1. (O’Neill [O83], p. 50).

(i) Any semi–Euclidean space \((V, g)\) with \(V \neq \{0\}\) has an orthonormal basis \(B = \{e_1, ..., e_m\}\).

(ii) Any vector \(v \in V\) has a unique expression

\[
v = \sum_{i=1}^{m} \varepsilon_i g(v, e_i) e_i,
\]

where \(\varepsilon_i = g(e_i, e_i)\).

Next, we consider a subspace \(W\) of a semi–Euclidean space \((V, g)\). Then the restriction of \(g\) to \(W\) is a symmetric bilinear form on \(W\) which we also denote by \(g\). If \(g\) is non–degenerate on \(W\), then \((W, g)\) is also a semi–Euclidean space. Any subspace \(W \neq \{0\}\) of a Euclidean space \((V, g)\) is a Euclidean space too. However, when \((V, g)\) is a proper semi–Euclidean space \(g\) might be degenerate on \(W\), that is, there exists a non-zero vector \(u \in W\) such that

\[
g(u, w) = 0, \text{ for all } w \in W.
\]

When \(g\) is degenerate (resp. non–degenerate) on a subspace \(W\) of \((V, g)\) we say that \(W\) is a degenerate (resp. non–degenerate) subspace of \((V, g)\).

Lemma 4.2. Any \(m\)–dimensional proper semi–Euclidean space with \(m \geq 2\) has both degenerate and non–degenerate subspaces.

Proof. According to (i) of Lemma 4.1 we consider an orthonormal basis \(B\) of \((V, g)\). If \(u \in B\), then \(W = \text{span}\{u\}\) is a non–degenerate subspace of \((V, g)\). Since \(g\) is of index \(0 < q < m\), there exist in \(B\) at least one time–like vector \(u\) and one space–like vector \(v\). Then \(W' = \text{span}\{u+v\}\) is a degenerate subspace of \((V, g)\).

To discuss the degree of degeneracy of a subspace \(W\) we define the orthogonal subspace \(W^\perp\) to \(W\) in \((V, g)\) by

\[
W^\perp = \{u \in V : g(u, w) = 0, \forall w \in W\}.
\]

In general, \(W^\perp\) is not complementary to \(W\) in \(V\), but the following equalities are true:

\[
\dim W + \dim W^\perp = m,
\]

and

\[
(W^\perp)^\perp = W.
\]

Moreover, we have the following.

Lemma 4.3. \(W\) is a non–degenerate subspace of a semi–Euclidean space if and only if \(W^\perp\) is non–degenerate too.
Proof. Suppose $W$ is non-degenerate and $W^\perp$ is degenerate. Then there exists $u \in W^\perp$, $u \neq 0$, such that
\[ g(u, w^\perp) = 0, \quad \text{for all } w^\perp \in W^\perp. \quad (4.5) \]
On the other hand, by the definition of $W^\perp$ we have
\[ g(u, w) = 0, \quad \text{for all } w \in W. \quad (4.6) \]
From (4.5) and (4.4) it follows that $u \in W$. Then by (4.6) we deduce that $W$ is degenerate, which is a contradiction. Thus $W^\perp$ must be non-degenerate. Conversely, if $W^\perp$ is non-degenerate, then by the above reason we infer that $(W^\perp)^\perp$ is non-degenerate. Hence by (4.4), $W$ is non-degenerate. \[ \square \]

Corollary 4.4. $W$ is a degenerate subspace of a semi-Euclidean space $(V, g)$ if and only if $W^\perp$ is degenerate too.

Now, we consider the null subspace of $W \subset (V, g)$ with respect to $g$, denoted by $\mathcal{N}(W, g)$ and defined by
\[ \mathcal{N}(W, g) = \{ u \in W : g(u, w) = 0, \quad \forall w \in W \}. \quad (4.7) \]
By using (4.4) and (4.7) we deduce that
\[ \mathcal{N}(W, g) = \mathcal{N}(W^\perp, g) = W \cap W^\perp. \quad (4.8) \]
The dimension of the null subspace of $W$ is called the nullity degree of $W$ with respect to $g$, and it is denoted by null $(W, g)$. Then the following can be easily proved.

Lemma 4.5. Let $(V, g)$ be a semi-Euclidean space and $W$ be a subspace of $V$. Then we have the assertions:

(i) $W$ is a degenerate subspace of $(V, g)$ if and only if null $(W, g) > 0$.

(ii) $W$ is a non-degenerate subspace of $(V, g)$ if and only if null $(W, g) = 0$.

Let null$(W, g) = r$. If $r > 0$ we say that $(W, g)$ is an $r$–degenerate subspace of $(V, g)$. According to Walker [Wal50a] the $n$–dimensional subspace $(W, g)$ is called:

- partially–null, if $0 < r < n$,
- totally–null, if $r = n$,
- non–null, if $r = 0$.

Lemma 4.6. Let $(W, g)$ be a partially–null subspace of a semi–Euclidean space $(V, g)$. Then any complementary subspace to $\mathcal{N}(W, g)$ in $W$ is non–degenerate.

Proof. Let $S(W, g)$ be a complementary subspace to $\mathcal{N}(W, g)$ in $W$. Suppose that $S(W, g)$ is degenerate. Then there exists a non–zero vector $v \in S(W, g)$
1.4 From Semi–Euclidean Algebra to Semi–Riemannian Geometry

such that \( g(v, w) = 0 \) for any \( w \in S(W, g) \). As \( v \in W \) and \( \mathcal{N}(W, g) \) is the null subspace of \( W \), we have \( g(v, u) = 0 \), for any \( u \in \mathcal{N}(W, g) \). Thus \( v \) is orthogonal to all vectors of \( W \) and hence it is a vector in \( \mathcal{N}(W, g) \). This is a contradiction because \( v \neq 0 \) and \( \mathcal{N}(W, g) \) and \( S(W, g) \) are complementary subspaces of \( W \). Therefore, \( S(W, g) \) must be non–degenerate.

A complementary subspace to \( \mathcal{N}(W, g) \) in a partially–null subspace \( W \) of \( (V, g) \) is called a screen subspace. Later on, in Sections 1.8 and 3.5 we shall see that screen subspaces are fibers of some distributions which play an important role in studying degenerate distributions (resp. foliations).

Finally, we define the light–like (null) cone of a proper semi–Euclidean space \((V, g)\) as the set \( A \) of all light–like vectors in \( V \), that is, we have

\[
A = \{ v \in V \setminus \{0\} : g(v, v) = 0 \}.
\]

Clearly \( A \) is not a subspace of \( V \), but it contains \( \mathcal{N}(W, g) \setminus \{0\} \) for any degenerate subspace \( W \) of \((V, g)\).

**Example 4.1.** Let \( \mathbb{R}^m \) be the space of \( m \)–tuples \((x^1, ..., x^m) = \mathbf{x}\) of real numbers. For any \( 0 < q < m \) define \( g : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R} \) by

\[
g(x, y) = -\sum_{t=1}^{q} x^t y^t + \sum_{s=q+1}^{m} x^s y^s. \tag{4.9}
\]

Then \( \mathbb{R}^m_q = (\mathbb{R}^m, g) \) is a proper semi–Euclidean space of index \( q \). In particular, \( \mathbb{R}^m_1 \) is a Lorentz (Minkowski) vector space with \( g \) given by

\[
g(x, y) = -x^1 y^1 + \sum_{s=2}^{m} x^s y^s. \tag{4.10}
\]

Finally, \( \mathbb{R}^m \) becomes a Euclidean space with respect to the usual inner product

\[
g(x, y) = \sum_{s=1}^{m} x^s y^s. \tag{4.11}
\]

**Example 4.2.** Consider in \( \mathbb{R}^4_1 \) the subspaces:

\[
W = \{ x \in \mathbb{R}^4 : x^1 + x^2 + x^3 + x^4 = 0 \},
\]

\[
W' = \{ x \in \mathbb{R}^4 : x^1 = x^2 \},
\]

\[
W'' = \{ x \in \mathbb{R}^4 : x^1 = x^2, x^3 = x^4 = 0 \}.
\]

Then it is easy to see that \( W, W' \) and \( W'' \) are non–null, partially–null and totally–null subspaces of \( \mathbb{R}^4_1 \), respectively. Moreover, we have