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Walsh Equiconvergence of Complex Interpolating Polynomials

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DEDICATION

And one might therefore say of me that in this book, I have only made up a bunch of other people's flowers and that of my own I have only provided the string that ties them together.

(Book III, Chapter XVI
of Physiognomy)
Signeur de Montaigne

CONTENTS

PREFACE

This monograph is centered around a simple and beautiful observation of J.L. Walsh, in 1932, that if a function is analytic in a disc of radius ρ ($\rho > 1$) but not in $|z| \le \rho$, then the difference between the Lagrange interpolant to it in the n^{th} roots of unity and the partial sums of degree $n-1$ of the Taylor series about the origin, tends to zero in a larger disc of radius ρ^2, although both operators converge to $f(z)$ only for $|z| < \rho$. This result was stated by Walsh in 1932 in a short paper [304] and proved in [87]. A precise formulation of this interesting result appears in 1935 in the first edition of his book *Interpolation and Approximation by Rational Functions in the Complex Domain* [88, p. 153]).

One of the reasons why this result of Walsh was not noticed until 1980 seems to be that it is sharp in the sense that if $z = \rho^2$, then there exists a function $f(z)$ analytic in $|z| < \rho$, for which the difference, between its Lagrange interpolant on the n^{th} roots of unity and the partial sum of degree $n-1$ of its Taylor series about the origin, does not tend to zero for $z = \rho^2$. The function which provides this phenomenon is $\frac{1}{\rho - z}$. In 1980 a paper authored by A.S. Cavaretta, A. Sharma and R.S. Varga [27] gives an extension of the above result in many new directions.

The object of this monograph is to collect the various results stemming from this theorem of Walsh which have appeared in the literature, and to give as well some new results. The first work which gave publicity to this subject was a paper by R.S. Varga [82] which appeared in 1982 and later a survey paper by A. Sharma [72] in 1986. T.J. Rivlin, E.B. Saff and R.S. Varga (all students of Walsh) made significant contributions to extend this result. New directions were due to V. Totik [85], K. Ivanov and A. Sharma [43], J. Szabados [80], Lou Yuanren [202], M.P. Stojanova [76], A. Jakimovski and A. Sharma [48] and others.

T.J. Rivlin in his brief comment on the above result in the selected papers of Walsh, says that "...by the mid nineties the interest in this theorem had almost disappeared. The result was probably about 200 published papers".

This comment encouraged us to write this monograph and to present a unified presentation of the significant results and extensions of this theorem along with a complete bibliography. (How T.J. Rivlin arrived at the figure of about 200 published papers is not clear to us.)

This book is easily accessible to students who have had a course in complex variables and have gone, for example, through the book *Theory of Approximation* by P.J. Davis, or the book *Approximation of Functions* by G.G. Lorentz. Our book is divided into 12 chapters. Chapter 1 begins with elementary results on Lagrange interpolation to functions defined on $|z| \leq \rho$ and gives a proof of the Theorem of Walsh which is the object of the present study. Chapter 2 deals with an extension of Walsh's theorem to Hermite interpolation. Chapter 3 is concerned with an extension of Walsh's theorem to rational functions with given poles outside the circle $|z| < \rho$. Chapters 4 and 5 deal with sharpness and converse results respectively. Chapter 6 is concerned with Padé approximation and Walsh equiconvergence for meromorphic functions with a finite number of given poles. Chapter 7 deals with quantitative results in the overconvergence of meromorphic functions of Chapter 6. In Chapter 8, we turn to the study of equiconvergence of Lagrange and Hermite interpolation for functions analytic in an ellipse. In Chapter 9 we extend the Walsh equiconvergence by application of methods of regular summability, which was initiated by R. Brück [16] and continued by A. Jakimovski and A. Sharma [46]. Chapter 10 deals with Faber expansions of analytic functions and extensions of Walsh equiconvergence results for differences of approximation operators on Fejér and Faber nodes. Chapter 11 is concerned with corresponding results for equiconvergence on lemniscates.

We can never thank Prof. R.S. Varga enough for his kindness and constant encouragement, advice and suggestions over several years. He has been kind enough to go through the manuscript with constructive corrections and amendments.

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A. Sharma is particularly grateful to his family for their encouragement and patience with him during the preparation of this monograph. His son Raja and his wife Sarla went “the extra mile” beyond their filial duties in caring for him, and ungrudgingly endured his eccentricities. He records his gratefulness to the Good Samaritan Society (Mount Pleasant Choice Center) for his care and nursing during his illness while the work was in preparation.

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Edmonton, July 2003

A. Jakimovski, A. Sharma, and J. Szabados

Ambikeshwar Sharma died on December 22, 2003, after a long illness. It is our honor and duty to finish the work on this book he initiated so enthusiastically.

Edmonton, July 2004

A. Jakimovski and J. Szabados

CHAPTER 1

LAGRANGE INTERPOLATION AND WALSH EQUICONVERGENCE

1.1 Introduction

Let $f(z)$ be a function analytic in an open domain D and continuous on the boundary of this domain. Further let n be a positive integer, and $z_1, \ldots, z_n$ pairwise different points from D. We shall denote the (unique) Lagrange polynomial interpolant, of degree at most $n-1$, of $f(z)$ in these n zeros by $L_{n-1}(f; z)$. With the notation $\omega_n(z) := \prod_{k=1}^{n}(z - z_k)$, this polynomial can be represented in the form

$$L_{n-1}(f; z) = \frac{1}{2\pi i} \int_C \frac{\omega_n(t) - \omega_n(z)}{\omega_n(t)} \frac{f(t)}{t - z} \, dt,$$

where C may be any rectifiable Jordan curve in D containing the points $z_1, \ldots,$ z_n and z in the interior of the domain bounded by C. Indeed, this is a polynomial of degree at most $n-1$, and by Cauchy's theorem

$$L_{n-1}(f, z_k) = \frac{1}{2\pi i} \int_C \frac{f(t)}{t - z_k} \, dt = f(z_k), \qquad k = 1, \ldots, n.$$

The uniqueness of this interpolant follows from the fundamental theorem of algebra: if there existed two different interpolating polynomials, then their difference, a polynomial of degree at most $n-1$ not identically zero, would vanish at n points, which is impossible. Most often in this book, we will be concerned with the special case when the nodes of interpolation are the n^{th} roots of unity, i.e., when $\omega_n(z) = z^n - 1$. In 1884, Méray gave a very instructive example of a function whose Lagrange interpolant in the n^{th} roots of unity does not converge to it anywhere except at the point 1. Thus if $f(z) = 1/z$ then $L_{n-1}(f; z) = z^{n-1}$ is the polynomial of degree $n-1$ which interpolates $f(z)$ in the zeros of $z^n - 1$. For $|z| > 1$, $\lim_{n\to\infty} z^{n-1}$ does not exist and for $|z| < 1$, $\lim_{n\to\infty} z^{n-1} = 0$, while for $|z| = 1$, $z \neq 1$ it diverges so that $L_{n-1}(f; z)$ converges to $f(z) = z^{-1}$ only at the point 1. The same applies to the case when $f(z) = z^{-k}$, $k > 0$. Even for analytic functions in the closed unit circle $|z| \leq 1$, the condition

$$\lim_{n\to\infty} \prod_{k=1}^{n} |z - z_k|^{\frac{1}{n}} = |z| \quad \text{for} \quad |z| > 1 \tag{1.0}$$

must be satisfied for the nodes of interpolation z_k, $|z_k| = 1$, $k = 1, \dots, n$, in order to have uniform convergence of the corresponding Lagrange interpolants in $|z| \le 1$. (For the roots of unity, this is obviously satisfied.) For functions which are not analytic, we have the following theorem.

THEOREM 1. *Let $f(z)$ be defined and continuous (or R-integrable, i.e., Riemann integrable) on the circumference of the unit circle $\Gamma := \{z : |z| = 1\}$. If $L_{n-1}(f; z)$ is the Lagrange interpolant to $f(z)$ in the zeros of $z^n - 1$, then*

$$\lim_{n\to\infty} L_{n-1}(f; z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z}\, dt, \quad |z| < 1, \tag{1.1}$$

uniformly for $|z| \le \delta < 1$.

PROOF. Denoting $w_n = \exp\ 2\pi i/n$, the Lagrange interpolant has the following representation:

$$L_{n-1}(f; z) = \sum_{k=1}^{n} f(w_n^k) \cdot \frac{w_n^k(z^n - 1)}{(z - w_n^k)n}\ . \tag{1.2}$$

Namely, this is indeed a polynomial of degree at most $n - 1$, since each w_n^k is a root of the polynomial $z^n - 1$. Moreover,

$$\lim_{z\to w_n^j} \frac{z^n - 1}{z - w_n^k} = \begin{cases} 0 & \text{if } j \neq k, \\ \frac{n}{w_n^k} & \text{if } j = k, \end{cases}$$

i.e., $L_{n-1}(f; w_n^j) = f(w_n^j)$, $j = 1, \dots, n$ as stated. From the definition of the Riemann integral, we have

$$F(z) := \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z} dt = \lim_{n\to\infty} \frac{1}{2\pi i} \sum_{k=1}^{n} \frac{f(w_n^k)(w_n^{k+1} - w_n^k)}{w_n^k - z}\ , \quad |z| < 1 \tag{1.3}$$

and

$$\lim_{n\to\infty} [F(z) - L_n(f; z)] = \lim_{n\to\infty} \left[\frac{1}{2\pi i} + \frac{z^n - 1}{n(w_n - 1)}\right] \sum_{k=1}^{n} \frac{w_n^k(w_n - 1) f(w_n^k)}{w_n^k - z}\ .$$

Since $\lim\limits_{n\to\infty} n(w_n - 1) = 2\pi i$, we see that for $|z| < 1$, we have (1.1). The uniform convergence for $|z| \le \delta < 1$ is also clear from the last formula. □

If $f(z)$ is analytic for $|z| < 1$ and continuous for $|z| = 1$, then $f(z) = F(z)$. In order to extend this result to other operators, we shall need the following

LEMMA 1. *Let $f(z)$ be R-integrable on Γ and let $L_{n-1}(f;z)$ be the Lagrange interpolant to f on the zeros of z^n-1. Then, for any fixed nonnegative integer p*

$$\lim_{n\to\infty} L_{n-1}^{(p)}(f;z) = \frac{p!}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)^{p+1}}\,dt, \quad |z|<1 \tag{1.4}$$

the convergence being uniform for $|z|\le\delta<1$.

PROOF. Since

$$L_{n-1}(f;z) = \frac{1}{n}\sum_{k=0}^{n-1} f(w_n^k)\sum_{j=0}^{n-1} w_n^{-kj}z^j.$$

Differentiating the above p times with respect to z, we get

$$\begin{aligned} L_{n-1}^{(p)}(f;z) &= \frac{p!}{n}\sum_{k=0}^{n-1}\frac{f(w_n^k)w_n^k}{(w_n^k-z)^{p+1}} \\ &\quad - \frac{z^n}{n}\sum_{k=0}^{p}\binom{p}{k}(n)_k z^{-k}(p-k)!\sum_{\ell=0}^{n-2}\frac{f(w_n^\ell)w^\ell}{(w_n^\ell-z)^{p-k+1}} \end{aligned} \tag{1.5}$$

where $(n)_k = n(n-1)\dots(n-k+1)$. We notice that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\frac{f(w_n^k)w_n^k}{(w_n^k-z)^{p-k+1}} = \frac{1}{2\pi i}\int_\Gamma\frac{f(t)}{(t-z)^{p-k+1}}\,dt,$$

and that for any $k>0$, $|z|^n n^k\to 0$ uniformly for $|z|\le\delta<1$ as $n\to\infty$. (1.4) now follows from (1.5). $\square$

If $f^{(j)}(z)$ exists along Γ for $j=0,1,\dots,r-1$, we denote by $h_{rn-1}(f;z)$ the polynomial of degree $rn-1$ which satisfies the conditions:

$$h_{rn-1}^{(j)}(f;w_n^k) = f^{(j)}(w_n^k), \quad k=1,\dots,n; \quad j=0,1,\dots,r-1. \tag{1.6}$$

Then we have

THEOREM 2. *Let $f^{(r-1)}(z)$ exist and be R-integrable along Γ. If $h_{rn-1}(f;z)$ is the Hermite interpolant to f satisfying* (1.6)*, then*

$$\lim_{n\to\infty} h_{rn-1}(f;z) = \frac{1}{2\pi i}\int_\Gamma\frac{f(t)}{t-z}\,dt, \quad |z|<1 \tag{1.7}$$

and uniformly for $|z|\le\delta<1$.

PROOF. For $r=1$, the theorem is the same as Theorem 1; so it is enough to consider the case when $r>1$. Set

$$h_{rn-1}(f;z) = L_{n-1}(f;z) + \sum_{j=1}^{r-1}(1-z^n)^j P_{n,j}(f;z) \tag{1.8}$$

where each $P_{n,j}(f;z)$ is a polynomial of degree $\le n-1$. Thus it is enough to prove that

$$\lim_{n\to\infty} P_{n,j}(f;z) = 0, \quad j = 1,\dots,r-1, \quad |z|<1.$$

However we shall prove the stronger result that

$$\lim_{n\to\infty} P^{(\ell)}_{n,j}(f;z) = 0, \quad j = 1,2,\dots,r-1, \quad \ell = 0,1,\dots, \quad |z|<1. \tag{1.9}$$

We use induction on j. First let $j=1$. Differentiating (1.8) at $z = w_n^k$, we obtain

$$w_n h'_{rn-1}(f;w_n^k) = w_n f'(w_n^k) = w_n L'_{n-1}(f;w_n^k) - nP_{n,1}(f;w_n^k)$$
$$(k = 0,1,\dots,n-1)$$

whence we have

$$P_{n,1}(f;z) = \frac{1}{n}\,[zL'_{n-1}(f;z) - L_{n-1}(zf';z)]. \tag{1.10}$$

Differentiating this ℓ times gives

$$P^{(\ell)}_{n,1}(f;z) = \frac{1}{n}[zL^{(\ell+1)}_{n-1}(f;z) + \ell L^{(\ell)}_{n-1}(f;z) - L^{(\ell)}_{n-1}(zf';z)] \tag{1.11}$$

so that by Lemma 1, we see that (1.9) holds for $j=1$. Now suppose that (1.9) has been proved for $j,\ 1\le j\le r-2$. From (1.8), we deduce

$$w_n^{(j+1)k} h^{(j+1)}_{rn-1}(f;w_n^k) = w_n^{(j+1)k} L^{(j+1)}_{n-1}(f;w_n^k) + (-1)^{j+1}(j+1)!n^{j+1}\times$$
$$\times P_{n,j+1}(f;w_n^k)$$
$$+ w_n^{(j+1)k}\sum_{\ell=1}^{j}\sum_{s=\ell}^{j+1}\binom{j+1}{s}\Big(\frac{d^s}{dz^s}(1-z^n)^\ell\Big)_{z=w_n^k} P^{(j+1-s)}_{n,\ell}(f;w_n^k)$$
$$(k = 0,1,\dots,n-1).$$

Because of (1.6), we obtain

$$(-1)^{j+1}(j+1)!P_{n,j+1}(f;z) = \frac{1}{n^{j+1}}L_{n-1}(z^{j+1}f^{(j+1)};z) - (\frac{z}{n})^{j+1}L^{(j+1)}_{n-1}(f;z)$$
$$-\sum_{\ell=1}^{j}\sum_{s=j}^{r-1}\binom{j+1}{s}\frac{z^{j+1-s}}{n^{j+1}}\,P^{(j+1-s)}_{n,\ell}(f;z)\sum_{t=1}^{\ell}\binom{\ell}{t}(-1)^t(nt)_s.$$

Differentiating ℓ times, using the induction hypothesis and Lemma 1, we see that (1.9) holds for $j+1$ and the proof is complete. □

If we set

$$h_{rn-1}(f;z) = \sum_{k=0}^{rn-1}\gamma_k z^k,$$

then we can define the average of the partial sums of $h_{rn-1}(f;z)$ and set

$$A_{rn-1}(f;z) = \frac{1}{rn}\sum_{j=0}^{rn-1}\sum_{k=0}^{j}\gamma_k z^k.$$

In a similar fashion, one can establish

THEOREM 3. *Let $f^{(r-1)}$ exist along Γ and be R-integrable on Γ. Let $A_{rn-1}(f;z)$ be the average of the partial sums of the Hermite interpolant of $f(z)$ satisfying* (1.6). *Then*

$$\lim_{n\to\infty} A_{rn-1}(f;z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,dt,$$

for $|z|<1$ and uniformly for $|z|\le\delta<1$.

PROOF. By a change in the order of summation we see that

$$A_{rn-1}(f;z) = zL'_{n-1}(f;z) + z\sum_{k=0}^{r-1}(rn-k)\gamma_k z^k = h_{rn-1}(f;z) - \frac{1}{nr} z h'_{rn-1}(f;z).$$

Now we see from (1.8) that

$$\begin{aligned} zh'_{rn-1}(f;z) =& zL'_{n-1}(f;z) + z\sum_{k=0}^{r-1}(1-z^n)^j P'_{n,j}(f;z) - nz^n\sum_{j=1}^{r-1} j(1-z^n)^{j-1}\times \\ &\times P_{n,j}(f;z). \end{aligned}$$

Thus, using Lemma 1 and

$$\lim_{n\to\infty}[h_{rn-1}(f;z) - L_{n-1}(f;z)] = 0, \qquad |z|<1$$

(which follows from (1.8)-(1.9)), we see that

$$\lim_{n\to\infty} \frac{z}{n} h'_{rn-1}(f;z) = 0.$$

This, combined with Theorem 1 proves Theorem 3.

1.2. Least-Square Minimization

For $m\ge n$, let $Q_{n-1}(f;z)$ denote the unique polynomial of degree $\le n-1$ which minimizes

$$\sum_{k=0}^{m-1} |f(w_m^k) - p(w_m^k)|^2, \quad w_m^m = 1, \tag{2.1}$$

over all $p(z)\in\pi_{n-1}$. If $m=n$, $Q_{n-1}(f;z)$ is the Lagrange interpolant to f at the n^{th} roots of unity. If $m>n$, then it is easy to see that $Q_{n-1}(f;z)$ is obtained by truncating $L_{m-1}(f;z)$. More precisely if

$$L_{m-1}(f;z) = \sum_{k=0}^{m-1} c_k z^k, \quad m>n \quad \text{then} \quad Q_{n-1}(f;z) = \sum_{\nu=0}^{n-1} c_\nu z^\nu,$$

where

$$c_\nu = \frac{1}{m} \sum_{k=0}^{m-1} f(w_m^k) w_m^{-\nu k}, \quad \nu = 0, 1, \ldots, n-1. \tag{2.2}$$

To see this, we first observe that from

$$L_{m-1}(f; z) = \frac{1}{m} \sum_{n=0}^{m-1} \frac{f(w_m^k)(z^m - 1)w^k}{(z - w_m^k)}$$

it follows that the coefficient of z^ν in $L_{m-1}(f; z)$ is given by (2.2). If we want to minimize (2.1) and set $p(z) = \sum_{\nu=0}^{n-1} p_\nu z^\nu$, then in order to minimize

$$\sum_{k=0}^{m-1} \left| f(w_m^k) - \sum_{\nu=0}^{n-1} p_\nu w_m^{k\nu} \right|^2$$

we need the orthogonality conditions

$$\sum_{k=0}^{m-1} \left(f(w_m^k) - \sum_{\nu=0}^{n-1} p_\nu w_m^{\nu k} \right) w_m^{-\mu k} = 0, \quad \mu = 0, 1, \ldots, n-1.$$

Simplifying, we see that

$$\sum_{k=0}^{m-1} f(w_m^k) w_m^{-\mu k} = \sum_{\nu=0}^{m-1} p_\nu \sum_{k=0}^{m-1} w_m^{\nu k - \mu k} = m p_\mu$$

which proves that $p_\mu = c_\mu$ in (2.2) and proves the assertion. From (2.2) we can see that

$$Q_{n-1}(f; z) = \frac{1}{m(w_m - 1)} \sum_{k=0}^{m-1} \frac{f(w_m^k)(w_m^{k+1} - w_m^k)}{w_m^k - z} + \frac{z^n}{m} \sum_{k=0}^{m-1} \frac{f(w_m^k) w_m^{-(n-1)k}}{w_m^k - z} = S_1 + S_2.$$

We notice that as $n \to \infty$

$$|S_2| = O(|z|^n) = o(1) \quad \text{uniformly for} \quad |z| \le \delta < 1.$$

Since $m > n$, and $\lim_{n\to\infty} m(w_m - 1) = 2\pi i$ we have proved

THEOREM 4. *If $f(z)$ is R-integrable on Γ and if $Q_{n-1}(f; z)$ is the unique polynomial which minimizes* (2.1), *then*

$$\lim_{n\to\infty} Q_{n-1}(f; z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - z}\, dt, \quad |z| < 1 \tag{2.3}$$

uniformly for $|z| \le \delta < 1$.

The above theorems have a corresponding analogue for Laurent development.

THEOREM 5. *Let $f(z)$ be R-integrable on the unit circle Γ and let $Q_{n,n}(z)$ be the polynomial in z and $\frac{1}{z}$ of degree n in each, which interpolates $f(z)$ in the zeros of $z^{2n+1}-1$. If $Q_{n,n}(z)=q_n(z)+r_n(z^{-1})$, where*

$$q_n(z)=a_0+a_1z+\cdots+a_nz^n, \quad r_n(z^{-1})=a_{-1}z^{-1}+a_{-2}z^{-2}+\cdots+a_{-n}z^{-n},$$

then

$$\begin{cases} \lim\limits_{n\to\infty} q_n(z)=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,dt, & |z|<1 \\ \lim\limits_{n\to\infty} r_n(z^{-1})=\frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t-z}\,dt, & |z|>1. \end{cases} \tag{2.4}$$

The convergence is uniform in $\frac{1}{\delta}\le|z|\le\delta<1$.

If $f(z)$ is analytic in an annulus $\rho^{-1}<|z|<\rho,\ \rho>1$ then the equations (2.4) are valid respectively for $|z|<\rho$ and for $|z|>\frac{1}{\rho}$ and uniformly for $|z|\le R<\rho$ and $|z|\ge\frac{1}{R}>\frac{1}{\rho}$. Moreover

$$q_n(z)+r_n(z^{-1})\to f(z) \quad \text{for} \quad \frac{1}{\rho}<|z|<\rho,$$

and uniformly in

$$\frac{1}{R}\le|z|\le R<\rho.$$

1.3. Functions Analytic in $\Gamma_\rho=\{z:|z|=\rho\}$

We shall now consider functions which are analytic in the disc of radius ρ ($\rho>1$) but not in Γ_ρ. We shall denote this class of functions by A_ρ. It is known that if $f(z)\in A_\rho$ and if

$$f(z)=\sum_{k=0}^{\infty}a_kz^k$$

is the power-series expansion of $f(z)$, then the right side converges in $|z|<\rho$ and

$$\overline{\lim_{n\to\infty}}\ |a_n|^{1/n}=\frac{1}{\rho}\ .$$

If we set $p_{n-1}(f;z)=\sum\limits_{k=0}^{n-1}a_kz^k$, the Taylor expansion of f then $p_{n-1}(f;z)$ converges to $f(z)$ for $|z|<\rho$, if $f(z)\in A_\rho$. Similarly $L_{n-1}(f;z)$ (the Lagrange interpolant to f on the zeros of z^n-1) also converges to $f(z)$ only for $|z|<\rho$. However the difference of $L_{n-1}(f;z)$ and $p_{n-1}(f;z)$ converges to 0 for $|z|<\rho^2$. This beautiful observation is formulated as

THEOREM 6. *Let $f(z) \in A_\rho$ $(\rho > 1)$ and let $L_{n-1}(f;z)$ be the Lagrange interpolant to f on the zeros of $z^n - 1$. Then the sequence $L_{n-1}(f;z)$ converges geometrically to $f(z)$ in any closed subdomain of $|z| < \rho$. Moreover if $p_{n-1}(f;z)$ is the Taylor section of $f(z)$ of degree $n-1$, then*

$$\lim_{n\to\infty} [L_{n-1}(f;z) - p_{n-1}(f;z)] = 0, \tag{3.1}$$

geometrically for any closed subdoamin of $|z| < \rho^2$.

PROOF. Since $f(z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)}{t-z}\,dt$ where $R < \rho$, and since

$$L_{n-1}(f;z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)(t^n - z^n)}{(t^n-1)(t-z)}\,dt,$$

we obtain

$$f(z) - L_{n-1}(f;z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{(z^n-1)f(t)}{(t^n-1)(t-z)}\,dt, \quad |z| < R.$$

We see from the above that

$$\overline{\lim}_{n\to\infty}|f(z) - L_{n-1}(f;z)|^{1/n} \le \frac{|z|}{R},$$

which proves the geometric convergence for closed subdomains of $|z| < \rho$ (since $R < \rho$ was arbitrary). Similarly, we have

$$L_{n-1}(f;z) - p_{n-1}(f;z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{(t^n - z^n)f(t)}{t^n(t^n-1)(t-z)}\,dt. \tag{3.2}$$

Hence

$$\overline{\lim}_{n\to\infty}|L_{n-1}(f;z) - p_{n-1}(f;z)|^{1/n} \le \frac{\max\{R, |z|\}}{R^2}, \quad R < \rho.$$

The result follows from this immediately. □

The quantity ρ^2 is the best possible, in the sense that for any point z on $|z| = \rho^2$, there is a function $f(z) \in A_\rho$ for which (3.1) does not hold. The function $f(z) = \frac{1}{z-\rho}$ is a natural example since in this case

$$L_{n-1}(f;z) - p_{n-1}(f;z) = \frac{\rho^n - z^n}{\rho^n(\rho^n-1)(z-\rho)}$$

when $z = \rho^2$, and we see that this difference becomes $1/(\rho - \rho^2)$. Many extensions of Theorem 6 have recently been given. We begin with a straightforward extension. Let us set

$$p_{n-1,j}(f;z) := \sum_{k=0}^{n-1} a_{k+jn} z^k, \quad j = 0, 1, 2, \ldots \tag{3.3}$$

where the function $f(z) \in A_\rho$ has the Taylor-series expansion $\sum_0^\infty a_k z^k$. We shall prove below the following

THEOREM 7. *If $f(z) \in A_\rho$ and if $\ell \geq 1$ is any given integer, then*

$$\lim_{n\to\infty} \max_{|z|\leq\mu} \left| L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z) \right|^{1/n} \leq \frac{\mu}{\rho^{\ell+1}}, \quad \mu < \rho^{\ell+1} \tag{3.4}$$

i.e. the convergence is uniform and geometric for all $|z| \leq \mu < \rho^{\ell+1}$. Moreover the region $|z| < \rho^{\ell+1}$ is best possible in the sense that for any point z_0 with $|z_0| = \rho^{\ell+1}$, there exists a function $f_0(z) \in A_0$ for which (3.4) *does not hold for $z = z_0$.*

Thus if we take $z_0 = \rho$, and $f_0(z) = (\rho - z)^{-1}$, then

$$p_{n-1,j}(f_0, z) = \frac{\rho^n - z^n}{(\rho - z)\rho^{(j+1)n}}$$

and

$$\sum_{j=0}^{\ell-1} p_{n-1,j}(f_0, z) = \frac{(\rho^n - z^n)(\rho^{\ell n} - 1)}{(\rho - z)\rho^{\ell n}(\rho^n - 1)} .$$

It is easy to see that

$$\lim_{n\to\infty} \min_{|z|=\rho^{\ell+1}} \left| L_{n-1}(f_0;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f_0;z) \right|^{1/n} \geq \frac{1}{\rho^{\ell+1} + \rho} > 0.$$

PROOF. As in the proof of Theorem 6, we can express the difference on the left in (3.4) as a contour integral

$$\frac{1}{2\pi i} \int_{\Gamma_R} \frac{f(t)(t^n - z^n)}{(t - z)(t^n - 1)t^{\ell n}}\, dt. \tag{3.5}$$

For $|t| = R$ and for all $|z| \leq \mu < R < \rho^{\ell+1}$ $(\mu \geq \rho)$, we have

$$\left|\frac{t^n - z^n}{t - z}\right| \leq \frac{\mu^n + R^n}{R - \mu} ,$$

so that the above integral is bounded above in modulus by

$$\frac{MR(\mu^n + R^n)}{(R - \mu)(R^n - 1)R^{\ell n}}$$

where $M := \max_{z\in\Gamma_R} |f(z)|$. Taking n^{th} roots we see that

$$\overline{\lim}_{n\to\infty} \left\{ \max_{|z|\leq\mu} \left| L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z) \right| \right\}^{1/n} \leq \frac{\mu}{R^{\ell+1}} ,$$

which proves the desired uniform and geometric convergence of (2.3). □

On letting $\ell \to \infty$ in (3.4), we see that

$$L_{n-1}(f;z) = \sum_{j=0}^{\infty} p_{n-1,j}(f;z)$$

which shows that if $L_{n-1}(f;z) = \sum_{\nu=0}^{n-1} c_\nu z^\nu$, then, it can be verified that

$$c_\nu = \sum_{\lambda=0}^{\infty} a_{\nu+\lambda n}.$$

In Theorems 6 and 7, we compared two processes of interpolation each of which separately converges to $f(z)$ only for $|z| < \rho$, while their difference converges to zero in a larger region. In view of this, the above phenomenon is often termed as "overconvergence" or "equiconvergence." It is natural to ask whether the Taylor polynomial $p_{n-1}(f;z)$ can be replaced by the polynomial $\widehat{p}_{n-1}(f;z)$ which is the polynomial of best uniform approximation to $f(z)$ in $|z| \le 1$. If $f_0(z) = (\rho - z)^{-1}$, then

$$\widehat{p}_{n-1}(f_0;z) = \frac{\rho^{n-1} - z^{n-1}}{(\rho - z)\rho^{n-1}} + \frac{z^{n-1}}{(\rho^2 - 1)\rho^{n-2}}$$

for all $n \ge 2$. Then

$$L_{n-1}(f_0;z) - \widehat{p}_{n-1}(f_0;z) = \frac{\rho^{n-1} - z^{n-1}}{(\rho - z)(\rho^n - 1)\rho^{n-1}} - \frac{z^{n-1}(\rho^{n-2} - 1)}{(\rho^n - 1)(\rho^2 - 1)\rho^{n-2}}$$

and

$$\widehat{p}_{n-1}(f_0;z) - p_{n-1}(f_0;z) = \frac{1}{\rho(\rho^2 - 1)} \left(\frac{z}{\rho}\right)^{n-1}$$

which converges to zero only for $|z| < \rho$. If $f(z) \in A_\rho$ and is also continuous in $D_\rho := \{z : |z| \le \rho\}$, it is natural to ask if this stronger hypothesis on the function would make the equiconvergence region larger. The answer to this question is given by

Theorem 8. *Let $f(z) \in A_\rho \cap C(D_\rho)$. Then for each positive integer ℓ, we have*

$$\lim_{n\to\infty} \left\{ L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z) \right\} = 0, \quad |z| \le \rho^{\ell+1},$$

the convergence being uniform for all $|z| \le \rho^{\ell+1}$ and geometric for all $|z| \le r < \rho^{\ell+1}$.

PROOF. For any $f(z) \in A_\rho \cap C(D_\rho)$, let $s_{n-1}(f;z)$ be the polynomial of best approximation to f from π_{n-1} on the circle $D_\rho = \{z : |z| \le \rho\}$. Then

$$E_{n-1}(f) := \inf_{q \in \pi_{n-1}} \|f - q\|_{D_n} = \|f - s_{n-1}\|_{D_\rho}$$

and it is known that $\lim\limits_{n\to\infty} E_{n-1}(f) = 0$. From the linearity of the Lagrange and Taylor polynomials, we have

$$L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z) = L_{n-1}(f - s_{n-1};z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f - s_{n-1};z),$$

so that from (3.4), we obtain for $R < \rho$

$$L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\big(f(t) - s_{n-1}(f;t)\big)\cdot(t^n - z^n)}{(t-z)(t^n-1)t^{\ell n}}\, dt.$$

This shows that

$$\max_{|z|\le\rho^{\ell+1}} \left| L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z)\right| \le \frac{E_{n-1}(f)(\rho^{n(\ell+1)} + R^n)R}{(\rho^{\ell+1} - R)(R^n - 1)R^{\ell n}} \, .$$

Since the left side is independent of R, we get on letting R tend to ρ,

$$\max_{|z|\le\rho^{\ell+1}} \left| L_{n-1}(f;z) - \sum_{j=0}^{\ell-1} p_{n-1,j}(f;z)\right| \le \frac{E_{n-1}(f)(1 + \rho^{-n\ell})}{\rho^{\ell}(1-\rho^{-\ell})(1-\rho^{-n})} \, .$$

But the right side tends to zero as $n \to \infty$, which proves the result. Uniform and geometric convergence for $|z| \le r < \rho^{\ell+1}$ follows as in Theorem 7.

1.4. An extension of Walsh's Theorem

We claim that the sum

$$\sum_{j=0}^{\ell-1} p_{n-1,j}(f;z)$$

in (3.3) (Theorem 7) is the Lagrange interpolant in the n^{th} roots of unity of the polynomial $p_{\ell n-1}(f;z) = \sum\limits_{k=0}^{\ell n-1} a_k z^k$. This is easily seen since

$$\begin{aligned} p_{\ell n-1}(f;z) = & \sum_{\lambda=0}^{\ell-1}\sum_{k=0}^{n-1} a_{k+\lambda n} z^{k+\lambda n} = \sum_{\lambda=0}^{\ell-1}\sum_{k=0}^{n-1} a_{k+\lambda n}(z^{\lambda n} - 1)z^k \\ & + \sum_{\lambda=0}^{\ell-1}\sum_{k=0}^{n-1} a_{k+\lambda n} z^k \end{aligned}$$

so that

$$L_{n-1}(p_{\ell n-1};z)=\sum_{\lambda=0}^{\ell-1}\sum_{k=0}^{n-1}a_{k+\lambda n}z^k=\sum_{\lambda=0}^{\ell-1}p_{n-1,\lambda}(f;z).$$

With this simple observation, one can write the formula (3.3) in the equivalent form

$$\lim_{n\to\infty}\big[L_{n-1}(f;z)-L_{n-1}\big(p_{\ell_{n-1}}(f;z);z\big)\big]=0\quad\text{for}\quad |z|<\rho^{\ell+1}. \tag{4.1}$$

If we denote by $L_{n-1}(f;\alpha,z)$ the Lagrange interpolant in the zeros of $z^n-\alpha^n$, when $\alpha\neq 0$, and the Hermite interpolant of order n at 0 when $\alpha=0$, then (4.1) is also equivalent to

$$\lim_{n\to\infty}\big[L_{n-1}(f;1,z)-L_{n-1}\big(L_{\ell n-1}(f;0,z);1,z\big)\big]=0,\quad |z|<\rho^{\ell+1}.$$

This train of ideas amply justifies the following

THEOREM 9. *If $m=rn+q$, $s\le\frac{q}{n}<1$ and $\frac{q}{n}=s+O(\frac{1}{n})$ then for each $f(z)\subset A_\rho$ and for each $\alpha,\beta\in D_\rho$ $(\alpha\neq\beta)$, we have*

$$\lim_{n\to\infty}\Delta^{\alpha,\beta}_{n,m}(f;z):=\lim_{n\to\infty}\big[L_{n-1}(f,\alpha,z)-L_{n-1}\big(L_{m-1}(f,\beta,z),\alpha,z\big)\big]=0 \tag{4.2}$$

for $|z|<\sigma$, where

$$\sigma:=\rho/\max\Big(\Big(\frac{|\alpha|}{\rho}\Big)^r,\Big(\frac{|\beta|}{\rho}\Big)^{r+s}\Big). \tag{4.3}$$

More precisely, for any μ with $\rho<\mu<\infty$, we have

$$\lim_{n\to\infty}\{\max_{z\in D_\mu}|\Delta^{\alpha,\beta}_{n,m}(f;z)|\}^{1/n}\le\frac{\mu}{\sigma}\,.$$

Moreover if α,β,m satisfy neither $\alpha=\beta=0$ nor $\alpha^r=\beta^r$ when $m=rn$, then (4.3) is best possible in the sense that for any z_0 with $|z_0|=\sigma$, there is a function $f_0\in A_\rho$ such that (4.2) fails to hold for f_0 at z_0.

When $\alpha=1$, $\beta=0$ and $m=\ell n$, (4.2) yields Theorem 7.

PROOF. Since $\alpha,\beta\in D_\rho$, we may write

$$L_{m-1}(f,\beta,z)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{f(t)(t^m-z^m)}{(t-z)(t^m-\beta^m)}\,dt.$$

In order to find a similar representation for $L_{n-1}\big(L_{m-1}(f,\beta,z),\alpha,z\big)$, it is enough to evaluate

$$L_{n-1}\left(\frac{t^m-z^m}{t-z},\alpha,z\right).$$

Since $m = rn + q$, we have

$$\begin{aligned}\frac{t^m - z^m}{t-z} &= \frac{t^{rn+q} - z^{rn+q}}{t-z} \\ &= \frac{t^{nr} - z^{rn}}{t-z}\, t^q + z^{rn}\, \frac{t^q - z^q}{t-z} \\ &= t^q \cdot \frac{t^{rn} - z^{rn}}{t^n - z^n} \cdot \frac{t^n - z^n}{t-z} + z^{rn} \cdot \frac{t^q - z^q}{t-z}\,.\end{aligned}$$

From this it is clear that the Lagrange interpolant of $(t^m - z^m)/(t-z)$ in the zeros of $z^n - \alpha^n$ will be given as below:

$$L_{n-1}\left(\frac{t^m - z^m}{t-z}, \alpha, z\right) = t^q \left(\frac{t^{rn} - \alpha^{rn}}{t^n - \alpha^n}\right) \frac{t^n - z^n}{t-z} + \alpha^{rn} \cdot \left(\frac{t^q - z^q}{t-z}\right), \text{ as } q < n.$$

Hence

$$\begin{aligned}L_{n-1}(f,\alpha,z) &= \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)(t^n - z^n)}{(t-z)(t^n - \alpha^n)}\, dt, \\ L_{n-1}\big(L_{m-1}(f,\beta,z),\alpha,z\big) &= \frac{1}{2\pi i}\int \frac{f(t)}{t^m - \beta^m} L_{n-1}\left(\frac{t^m - z^m}{t-z}, \alpha, z\right) dt.\end{aligned}$$

From this we obtain the representation

$$\Delta_{n,m}^{\alpha,\beta}(f;z) = \frac{1}{2\pi i}\int_{\Gamma_R} f(t)K(t,z)dt \tag{4.4}$$

where

$$\begin{aligned}K(t,z) :=& \frac{1}{t^{rn+q} - \beta^{rn+q}}\left[t^q \cdot \frac{t^{rn} - \alpha^{rn}}{t^n - \alpha^n} \cdot \frac{t^n - z^n}{t-z} + \alpha^{rn}\, \frac{t^q - z^q}{t-z}\right] \\ &- \frac{t^n - z^n}{t^n - \alpha^n} \cdot \frac{1}{t-z} \\ \\ =& \frac{\beta^{rn+q} - \alpha^{rn} \cdot t^q}{(t^{rn+q} - \beta^{rn+q})(t^n - \alpha^n)} \cdot \frac{t^n - z^n}{t-z} \\ &+ \frac{\alpha^{rn}}{t^{nr+q} - \beta^{rn+q}} \cdot \frac{t^q - z^q}{t-z}\,.\end{aligned}$$

Since

$$|K(t,z)| \le c\, \frac{R^n - |z|^n}{R - |z|} \cdot \frac{\max(|\beta|^{rn+q}, |\alpha|^{rn} R^q)}{R^{rn+q} R^n} + c\, \frac{|\alpha|^{rn}}{R^{rn+q}} \cdot \frac{R^q - |z|^q}{R - |z|} \quad (|z| < R),$$

it follows that (4.2) will be proved if

$$|z|^n\, \frac{\max(|\beta|^{rn+q}, |\alpha|^{rn} R^q)}{R^{rn+q+n}} < 1 \quad \text{and} \quad \frac{|z|^q |\alpha|^{rn}}{R^{rn+q}} < 1,$$

where $q = sn + O(1)$. In other words taking the n^{th} roots of both sides above, and letting $n \to \infty$, we see that (4.2) is proved if

$$|z| < \rho / \max\left(\left(\frac{|\beta|}{\rho}\right)^{r+s}, \left(\frac{|\alpha|}{\rho}\right)^{r}\right) =: \sigma_1 \quad \text{and} \quad |z| < \rho / \left(\frac{|\alpha|}{\rho}\right)^{r/s} =: \sigma.$$

Since $\frac{|\alpha|}{\rho} < 1$ and $s < 1$, we have

$$\left(\frac{|\alpha|}{\rho}\right)^{r/s} > \left(\frac{|\alpha|}{\rho}\right)^{r}$$

so that $\sigma_1 < \sigma$ and this completes the proof. □

COROLLARY. *Let $m = rn + q$, $s \le \frac{q}{n} < 1$ and $\frac{q}{n} = s + O(\frac{1}{n})$. If $p_{n,m(n)}(f; z)$ denotes the polynomial of degree $n - 1$ which minimizes*

$$\sum_{k=0}^{m-1} |f(\beta w_m^k) - p(\beta w_m^k)|^2, \quad \beta \in D_\rho,$$

over all polynomials $p(z) \in \pi_{n-1}$, then

$$p_{n,m(n)}(f; z) - S_{n-1}(f; z) \to 0 \quad \textit{for} \quad |z| < \rho / \left(\frac{|\beta|}{\rho}\right)^{r+s}$$

and the bound for $|z|$ above is best possible in the same sense as in Theorem 9.

This corollary follows from Theorem 8 on taking $\alpha = 0$, $\beta = 1$.

1.5. Multivariate Extensions of Walsh's Theorem

In the multivariate case the domain of analyticity of a function $f(\boldsymbol{z})$, where $\boldsymbol{z} = (z_1, \dots, z_m) \in \mathbb{C}^m$, can be defined in two different ways. One possibility is to consider the *ball*, i.e., the set defined by $\sum_{j=1}^{m} |z_j|^2 < \rho^2$. The other possibility is to take the *polydisc* $|z_j| < \rho_j$, $j = 1, \dots, m$. These two definitions lead to entirely different theories, since there is no equivalence (i.e. holomorphic mapping) between the ball and the polydisc. For our purposes, the setup based on a polydisc is more suitable and convenient. We begin with some fundamental definitions. Let

$$1 < \rho_1 \le \rho_2 \le \cdots \le \rho_m, \quad \boldsymbol{\rho} = (\rho_1, \rho_2, \dots, \rho_m); \tag{5.1}$$

we remark that the ordering in (5.1) can be achieved, without loss of generality, by simply renumbering the components of $\boldsymbol{\rho}$. Then, denote by $A(\boldsymbol{\rho})$ the set of functions analytic in the polydisc

$$D(\boldsymbol{\rho}) := \{\boldsymbol{z} = (z_1, \dots, z_m) : \ |z_j| < \rho_j, \ j = 1, \dots, m\}.$$

where each such function has a singularity on each of the circles $|z_j| = \rho_j$, $j = 1, \dots, m$. (Here singularity may involve either poles or branchpoints on the circle $|\boldsymbol{z}| = \boldsymbol{\rho}$.) The multivariate Cauchy formula

$$f(\boldsymbol{z}) = \frac{1}{(2\pi i)^m} \int_D \frac{f(\boldsymbol{t})}{\prod_{j=1}^{m} (t_j - z_j)} \, d\boldsymbol{t}, \quad \boldsymbol{z} \in D(\boldsymbol{\rho}) \tag{5.2}$$

where the integration is taken over a polydisc D in $D(\boldsymbol{\rho})$ which contains the point $\boldsymbol{z}$ and, with $d\boldsymbol{t} := dt_1 \dots dt_m$, is valid for all $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$. Let Γ_n^m denote set of all complex polynomials $p(\boldsymbol{z})$ of m variables which are of degree at most n in each of the variables z_j, $j = 1, \dots, m$. (This set differs from the usual definition of a polynomial of several variables, having degree at most n, which means that the total degree of each term is at most n, but our definition here serves a more useful purpose later.) The $(n-1)^{\text{th}}$ Taylor section of an $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$ is then defined as

$$S_{n-1}(f; \boldsymbol{z}) := \frac{1}{(2\pi i)^m} \int_D f(\boldsymbol{t}) \prod_{j=1}^{m} \frac{z_j^n - t_j^n}{t_j^n (z_j - t_j)} \, d\boldsymbol{t} \tag{5.3}$$

which, in the sense of the above definition, is an element of Γ_{n-1}^m.

THEOREM 10. *For any $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$, the Taylor sections $S_{n-1}(f; \boldsymbol{z})$ of* (5.3) *converge to $f(\boldsymbol{z})$, uniformly and geometrically in each closed subset of $D(\boldsymbol{\rho})$.*

PROOF. We have from (5.2) and (5.3) that

$$f(\boldsymbol{z}) - S_{n-1}(f; \boldsymbol{z}) = \frac{1}{(2\pi i)^m} \int_D \frac{f(\boldsymbol{t})}{\prod_{j=1}^{m} (t_j - z_j)} \left[1 - \prod_{j=1}^{m} \left(1 - \frac{z_j^n}{t_j^n} \right) \right] d\boldsymbol{t}.$$

Here,

$$\left| 1 - \prod_{j=1}^{m} \left(1 - \frac{z_j^n}{t_j^n} \right) \right| \le C \max_{1 \le j \le m} \left| \frac{z_j}{t_j} \right|^n \to 0 \quad \text{as} \quad n \to \infty$$

in any closed subset of D, and this proves the theorem. □

We now turn to the definition of the interpolation operator. The problem of interpolation in the multivariate case is more difficult (in general, existence and uniqueness are not guaranteed), but, with our definition of the set Γ_n^m, the situation simplifies. Consider the polynomial

$$L_{n-1}(f; \boldsymbol{z}) = \frac{1}{(2\pi i)^m} \int_D f(\boldsymbol{t}) \prod_{j=1}^{m} \frac{z_j^n - t_j^n}{(t_j^n - 1)(z_j - t_j)} \, d\boldsymbol{t} \in \Gamma_{n-1}^m \tag{5.4}$$

where $L_{n_1}(f;\boldsymbol{z}) \in \Gamma^m_{n-1}$, for any $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$. As usual, let ω be a primitive n^{th} root of unity. From the representation (5.4) we can see that, at the points $\boldsymbol{z} = (\omega^{k_1}, \dots, \omega^{k_m})$ where $0 \le k_j \le n-1$, $j = 1, \dots, m$, are arbitrary integers, the polynomial (5.4) has the same values as $f(\boldsymbol{z})$. It will follow from the next lemma that this interpolation polynomial $L_{n-1}(f;\boldsymbol{z})$ is uniquely determined

LEMMA 2. *If $p(\boldsymbol{z}) \in \Gamma^m_{n-1}$ has n^m different roots $\boldsymbol{z} = (z_1, \dots, z_m)$ such that each z_j takes n different values, then $p(\boldsymbol{z}) \equiv 0$.*

PROOF. We use induction on m. For $m = 1$, the statement follows from the fundamental theorem of algebra. Assume it is true for $m-1$, and represent $p(\boldsymbol{z})$ in the form

$$p(\boldsymbol{z}) = \sum_{k=0}^{n-1} z_1^k p_k(\boldsymbol{z}^*) \tag{5.5}$$

where $\boldsymbol{z}^* = (z_2, \dots, z_p) \in \mathbb{C}^{m-1}$ and $p_k(\boldsymbol{z}^*) \in \Gamma^{m-1}_{n-1}$. Fixing an arbitrary $\boldsymbol{z}^* = (z'_2, \dots, z'_m)$ where z'_j, $j = 2, \dots, m-1$, are coordinates of the roots of $p(\boldsymbol{z})$, then according to our assumption (5.5) vanishes for n different values of z_1. But then

$$p_k(\boldsymbol{z}^*) = 0, \quad k = 0, \dots, n-1.$$

Here, by our assumption, $\boldsymbol{z}^*$ takes n^{m-1} different values, and thus, by the induction hypothesis, the p_k are identically zero for $k = 0, \dots, n-1$. This proves the statement for m. □

Since the interpolation points for the polynomial (5.4) satisfy the condition of Lemma 2, $L_{n-1}(f, \boldsymbol{z})$ is uniquely determined. The uniform convergence of $L_{n-1}(f, \boldsymbol{z})$ to $f(\boldsymbol{z})$ in every closed subset of $D(\boldsymbol{\rho})$ will follow from Theorem 10 and the following overconvergence theorem:

THEOREM 11. *We have*

$$\overline{\lim_{n\to\infty}} |L_{n-1}(f, \boldsymbol{z}) - S_{n-1}(f, \boldsymbol{z})|^{1/n} \le \frac{1}{\rho_1} \prod_{|z_j|>\rho_j} \frac{|z_j|}{\rho_j} \tag{5.6}$$

for all $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$ and $z \in \mathbb{C}^m$. (Here, the empty product is defined as unity.)

REMARKS. 1. In particular if $\boldsymbol{z} \in D(\boldsymbol{\rho})$, then, as the product is unity in (5.6), the right hand side of (5.6) is $1/\rho_1 < 1$ which, coupled with Theorem 10, yields the uniform convergence of $L_{n-1}(f;\boldsymbol{z})$ to $f(\boldsymbol{z})$.

2. If

$$\prod_{|z_j|>\rho_j} \frac{|z_j|}{\rho_j} < \rho_1, \tag{5.7}$$

then we have the overconvergence of the difference $L_{n-1} - S_{n-1}$. Condition (5.7) gives an intrinsic relation between the coordinates $z_1, \dots, z_m$. The larger we choose some $|z_j|$'s, the smaller we have to make the remaining $|z_j|$'s. In order to see more clearly how this works, consider the special case $\rho_1 = \cdots = \rho_m := \rho$. Then note that (5.7) allows us to select either

$$|z_j| < \rho^{1+\frac{1}{m}}, \quad j = 1, \dots, m,$$

or

$$|z_1| = \cdots = |z_{m-1}| = \rho, \quad |z_m| < \rho^2.$$

In the first case, one has overconvergence in *each* coordinates (but with a smaller radius), while the second case gives *no* overconvergence in $m-1$ variables, but *optimal* overconvergence in the final coordinate. Of course, other choices are also possible.

PROOF OF THEOREM 11. Equations (5.3) and (5.4) imply

$$\begin{aligned}\Delta_{n-1}(f;\boldsymbol{z}) &:= L_{n-1}(f;\boldsymbol{z}) - S_{n-1}(f;\boldsymbol{z}) \\ &= \frac{1}{(2\pi i)^m} \int_D f(\boldsymbol{t}) \prod_{j=1}^m \frac{z_j^n - t_j^n}{z_j - t_j} \left(\prod_{j=1}^m \frac{1}{t_j^n - 1} - \prod_{j=1}^m \frac{1}{t_j^n} \right) d\boldsymbol{t}. \end{aligned} \tag{5.8}$$

Here,

$$\begin{aligned}\prod_{j=1}^m \frac{1}{t_j^n - 1} - \prod_{j=1}^m \frac{1}{t_j^n} &= \prod_{j=1}^m \frac{1}{t_j^n} \left[\prod_{j=1}^m \left(1 + \frac{1}{t_j^n - 1}\right) - 1 \right] \\ &= O\left(\frac{1}{(\rho_1 - \varepsilon)^n \prod_{j=1}^n (\rho_j - \varepsilon)^n} \right)\end{aligned}$$

where $\varepsilon > 0$ is an arbitrary small fixed number. Thus, we obtain from (5.8) that

$$\Delta_{n-1}(t;\boldsymbol{z}) = O\left(\left(\frac{\sum_{j=1}^m \max(|z_j|, \rho_j - \varepsilon)}{(\rho_1 - \varepsilon) \prod_{j=1}^m (\rho_j - \varepsilon)} \right)^n \right),$$

i.e.,

$$\overline{\lim_{n\to\infty}} |\Delta_{n-1}(f;\boldsymbol{z})|^{1/n} \le \frac{1}{\rho_1 - \varepsilon} \prod_{|z_j| > \rho_j - \varepsilon} \frac{|z_j|}{\rho_j - \varepsilon}$$

whence, the statement of the theorem follows, since $\varepsilon > 0$ was arbitrary. □

The estimate in (5.6) of Theorem 11 is *sharp* in the following sense. Consider the function

$$f_0(\boldsymbol{z}) = \prod_{j=1}^{m} \frac{1}{z_j - \rho_j} \in A(\boldsymbol{\rho}).$$

Evidently

$$L_{n-1}(f_0; \boldsymbol{z}) = \prod_{j=1}^{m} \frac{z_j^m - \rho_j^n}{(1-\rho_j^n)(z_j - \rho_j)}\ ,$$

and from (5.3)

$$S_{n-1}(f_0; \boldsymbol{z}) = \prod_{j=1}^{m} \frac{z_j^n - \rho_j^n}{\rho_j^n(\rho_j - z_j)}\ .$$

Thus

$$\begin{aligned}\Delta_{n-1}(f_0; \boldsymbol{z}) &= \prod_{j=1}^{m} \frac{z_j^n - \rho_j^n}{\rho_j - z_j} \left(\prod_{j=1}^{m} \frac{1}{\rho_j^n - 1} - \prod_{j=1}^{m} \frac{1}{\rho_j^n} \right) \\ &= \prod_{j=1}^{m} \frac{z_j^n - \rho_j^n}{(\rho_j - z_j)\rho_j^n}\, O\left(\frac{1}{\rho_1^{2n} \rho_2^n \cdots \rho_m^n} \right),\end{aligned}$$

whence by (5.1)

$$\overline{\lim_{n\to\infty}}\, |\Delta_{n-1}(f; \boldsymbol{z})|^{1/n} = \frac{1}{\rho_1} \prod_{|z_j|>\rho_j} \frac{|z_j|}{\rho_j}\ .$$

Thus for *some* functions in $A(\boldsymbol{\rho})$, the result of (5.6) is sharp. However, we can ask for the following stronger version of sharpness: is it true that

$$\overline{\lim_{n\to\infty}} \max_{\substack{|z_j|=r_h \\ j=1,\ldots,m}} |\Delta_{n-1}(f; \boldsymbol{z})|^{1/n} = \frac{1}{\rho_1} \prod_{r_j>\rho_j} \frac{r_j}{\rho_j}$$

for any $r_j > 0$, $j = 1, \ldots, m$ and $f(z) \in A(\boldsymbol{\rho})$? The answer to this question is *no*, and this is in sharp contrast to the univariate case (cf. Chapter 4). This can be seen from the following example.

Example. Let $m = 2$, $1 < \rho_1 < \rho_2$, and consider the function

$$f_1(\boldsymbol{z}) = \sum_{k=0}^{\infty} \left(\frac{z_1}{\rho_1}\right)^{3^k} \sum_{k=0}^{\infty} \left(\frac{z_2}{\rho_2}\right)^{3^k} \in A(\boldsymbol{\rho}).$$

We shall examine the overconvergence case of $|z_1| = r_1 > \rho_1$, $|z_2| = r_2 > \rho_2$.

Formula (5.8) in this case gives

$$\Delta_{n-1}(f_1;\boldsymbol{z}) = \frac{1}{(2\pi i)^2}\int_D f_1(\boldsymbol{t})\prod_{j=1}^{2}\frac{z_j^n - t_j^n}{(z_j-t_j)t_j^n}\Big[\frac{1}{t_1^n} + O\Big(\frac{1}{(\rho_1-\varepsilon)^{2n}} + \frac{1}{(\rho_2-\varepsilon)^n}\Big)\Big]d\boldsymbol{t}$$
$$= \frac{1}{(2\pi i)^2}\int_D \prod_{j=1}^{2}\Big[\sum_{k=0}^{\infty}\Big(\frac{t_j}{\rho_j}\Big)^{3^k}\sum_{k=0}^{n-1}\frac{z_j^k}{t_j^{k+1}}\Big]\frac{1}{t_1^n}d\boldsymbol{t} + O\left(\left(\frac{r_1r_2}{(\rho_1-\varepsilon)^3(\rho_2-\varepsilon)} + \frac{r_1r_2}{(\rho_1-\varepsilon)(\rho_2-\varepsilon)^2}\right)^n\right)$$
$$= \sum_{n\le 3^k\le 2n-1}\frac{z_1^{3^k-n}}{\rho_1^{3^k}}\cdot\sum_{3^k\le n-1}\Big(\frac{z_2}{\rho_2}\Big)^{3^k} + o\left(\left(\frac{r_1r_2}{\rho^2\rho_2}\right)^n\right),$$

provided $\varepsilon > 0$ is small enough. Now, assume that the integers λ_n, μ_n satisfy

$$3^{\lambda_n} < 2n < 3^{\lambda_n+1} \quad \text{and} \quad 3^{\mu_n} < n \le 3^{\mu_n+1}. \tag{5.9}$$

Evidently, in the sum $\sum_{n\le 3^k\le 2n-1}$ above there is at most one term (for $k=\lambda_n$; otherwise it may be empty). Then we can write

$$|\Delta_{n-1}(f_1;\boldsymbol{z})| = O\left(\frac{r_1^{3^{\lambda_n}-n}r_2^{3^{\mu_n}}}{\rho_1^{3^{\lambda_n}}\rho_2^{3^{\mu_n}}}\right) + o\left(\left(\frac{r_1r_2}{\rho^2\rho_2}\right)^n\right). \tag{5.10}$$

By (5.9), $\mu_n \le \lambda_n - 1$, and therefore $3^{\mu_n} \le 3^{\lambda_n-1} < \frac{2n}{3}$, whence

$$\frac{r_1^{3^{\lambda_n}-n}r_2^{3^{\mu_n}}}{\rho_1^{3^{\lambda_n}}\rho_2^{3^{\mu_n}}} \le \left(\frac{r_1r_2^{2/3}}{\rho_1^2\rho_2^{2/3}}\right)^n = o\left(\left(\frac{r_1r_2}{\rho^2\rho_2}\right)^n\right).$$

Thus, (5.10) yields

$$|\Delta_{n-1}(f_1;\boldsymbol{z})| = o\left(\left(\frac{r_1r_2}{\rho^2\rho_2}\right)^n\right),$$

i.e., for this function the error estimate in case $r_1 > \rho_1$, $r_2 > \rho_2$ is indeed better than the one provided by Theorem 11.

If we iterate interpolation operators and Taylor series, we can obtain different types of overconvergence results. (For a detailed account on this subject in the univariate case, see Ch. 2.) Here we restrict ourselves to one particular case. Instead of the interpolating polynomial (5.4), let us introduce the operator

$$L_{n-1}(f;\boldsymbol{\alpha};\boldsymbol{z}) := \frac{1}{(2\pi i)^m}\int_D f(\boldsymbol{t})\prod_{j=1}^{m}\frac{z_j^n - t_j^n}{(t_j^n-\alpha_j^n)(z_j-t_j)}\,d\boldsymbol{t} \in \Gamma_{n-1}^m, \tag{5.11}$$

where $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m) \in D(\boldsymbol{\rho})$, $\alpha_j > 0$, $j = 1, \ldots, m$, for any $f(\boldsymbol{z}) \in A(\boldsymbol{\rho})$. (5.11) interpolates f at the points $\boldsymbol{z} = (\alpha_1\omega^{k_1}, \ldots, \alpha_m\omega^{k_m})$, where $0 \leq k_j \leq n-1$, $j = 1, \ldots, m$, are arbitrary integers. This polynomial, just like (5.4), is uniquely determined. Using also the notation (5.3), we now state

THEOREM 12. *If $\lambda > 1$ is a fixed integer, then we have*

$$\limsup_{n\to\infty} |L_{n-1}(f - S_{\lambda n-1}(f); \boldsymbol{\alpha}; \boldsymbol{z})|^{1/n} \leq \max_{1\leq j\leq m} \left(\frac{\alpha_j}{\rho_j}\right)^{\lambda} \cdot \prod_{j=1}^{m} \frac{|z_j|}{\alpha_j}$$

for any $f \in A(\boldsymbol{\rho})$ and any $\boldsymbol{z} = (z_1, \ldots, z_m)$.

The result shows that we have convergence if

$$\prod_{j=1}^{m} |z_j| < \frac{\prod_{j=1}^{m} \alpha_j}{\max_{1\leq j\leq m} \left|\frac{\alpha_j}{\rho_j}\right|^{\lambda}}.$$

In particular, if $0 < \alpha_1 = \cdots = \alpha_m = \alpha < \rho_1 = \cdots = \rho_m = \rho$, then this condition takes the form

$$|z_j| < \frac{\rho^{\lambda/m}}{\alpha^{\lambda/m-1}}, \qquad j = 1, \ldots, m,$$

i.e. we have overconvergence provided $\lambda > m$.

PROOF OF THEOREM 12. (5.2), (5.3) and (5.11) yield

$$\begin{aligned} L_{n-1}(f - S_{\lambda n-1}(f); \boldsymbol{\alpha}; \boldsymbol{z}) &= \frac{1}{(2\pi i)^{2m}} \int_{D_2}\int_{D_1} \frac{f(\boldsymbol{u})}{\prod_{j=1}^{m}(u_j - t_j)} \times \\ &\times \left[1 - \prod_{j=1}^{m}\left(1 - \frac{t_j^{\lambda n}}{u_j^{\lambda n}}\right)\right] d\boldsymbol{u} \prod_{j=1}^{m} \frac{z_j^m - t_j^m}{(t_j^n - \alpha_j^n)(z_j - t_j)}\, d\boldsymbol{t} \\ &= \frac{1}{(2\pi i)^{2m}} \int_{D_1} f(\boldsymbol{u}) \int_{D_2} \prod_{j=1}^{m} \frac{z_j^n - t_j^n}{(t_j^n - \alpha_j^n)(u_j - t_j)(z_j - t_j)} \times \\ &\times \left[1 - \prod_{j=1}^{m}\left(1 - \frac{t_j^{\lambda n}}{u_j^{\lambda n}}\right)\right] d\boldsymbol{t}\, d\boldsymbol{u}, \end{aligned}$$

where

$$D_1 = \{(t_1, \ldots, t_m) : \ |t_j| = \alpha_j + \varepsilon,\ j = 1, \ldots, m\}$$

and

$$D_2 = \{(u_1, \ldots, u_m) : \ |u_j| = \rho_j - \varepsilon,\ j = 1, \ldots, m\}$$