THE HEART OF COHOMOLOGY

The Heart of Cohomology

by

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Marc groet 's morgens de dingen

Dag ventje met de fiets op de vaas met de bloem
ploem ploem
dag stoel naast de tafel
dag brood op de tafel
dag visserke-vis met de pijp
en
dag visserke-vis met de pet
pet en pijp
van het visserke-vis
goeiendag

Daa-ag vis
dag lieve vis
dag klein visselijn mijn

Paul van Ostaijen

Contents

Preface

The methods of (Co-) Homological Algebra provide a framework for Algebraic Geometry and Algebraic Analysis. The following two books were published during the late 1950's:

[CE] Cartan, H., Eilenberg, S., *Homological Algebra*, Princeton University Press (1956), and

[G] Godement, R., *Topologie Alg´ebraique et Th´eorie des Faisceaux*, Hermann, Paris (1958).

If you are capable of learning from either of these two books, I am afraid that *The Heart of Cohomology*, referred to hereafter as [THOC], is not for you. One of the goals of [THOC] is to provide young readers with elemental aspects of the algebraic treatment of cohomologies.

During the 1990's

[GM] Gelfand, S.I., Manin, Yu., I., *Methods of Homological Algebra*, Springer–Verlag, (1996), and

[W] Weibel, C.A., *An Introduction to Homological Algebra*, Cambridge University Press, (1994)

were published. The notion of a derived category is also treated in [GM] and [W].

In June, 2004, the author was given an opportunity to give a short course titled "Introduction to Derived Category" at the University of Antwerp, Antwerp, Belgium. This series of lectures was supported by the European Science Foundation, Scientific Programme of ESF. The handwritten lecture notes were distributed to attending members. [THOC] may be regarded as an expanded version of the Antwerp Lecture Notes. The style of [THOC] is more lecture-like and conversational. Prof. Fred van Oystaeyen is responsible for the title "The

Heart of Cohomology". In an effort to satisfy the intent of the title of this book, a more informal format has been chosen.

After each Chapter was written, the handwritten manuscript was sent to Dr. Daniel Larsson in Lund, Sweden, to be typed. As each Chapter was typed, we discussed his suggestions and questions. Dr. Larsson's contribution to [THOC] is highly appreciated.

We will give a brief introduction to each Chapter. In Chapter I we cover some of the basic notions in Category Theory. As general references we recommend

[BM] Mitchell, B., *The Theory of Categories*, Academic Press, 1965, and

[SH] Schubert, H., *Categories*, Springer-Verlag, 1972.

The original paper on the notion of a category

[EM] Eilenberg, S., MacLane, S., *General Theory of Natural Equivalences*, Trans. Amer. Math. Soc. **58**, (1945), 231–294

is still a very good reference. Our emphasis is on Yoneda's Lemma and the Yoneda Embedding. For example, for contravariant functors F and G from a category $\mathscr{C}$ to the category Set of sets, the Yoneda embedding

$$\tilde{\ } : \mathscr{C} \rightsquigarrow \hat{\mathscr{C}} := \mathsf{Set}^{\mathscr{C}^\circ}$$

gives an interpretation for the convenient notation $F(G)$ as

$$\tilde{F}(G) = \mathrm{Hom}_{\hat{\mathscr{C}}}(G, F)$$

(See Remark 5.)

We did not develop a cohomology theory based on the notion of a site. However, for a covering $\{U_i \to U\}$ of an object U in a site $\mathscr{C}$, the higher Čech cohomology with coefficient $F \in \mathrm{Ob}(\mathsf{Ab}^{\mathscr{C}^\circ})$ is the derived functor of the kernel of

$$\prod F(U_i) \xrightarrow{\mathrm{d}^0} \prod F(U_i \times U_j).$$

This higher Čech cohomology associated with the covering of U is the cohomology of the Čech complex

$$C^j(\{U_i \to U\}, F) = \prod F(U_{i_0} \times_U \cdots \times_U U_{i_j}).$$

One can continue the corresponding argument as shown in 3.4.3.

In Chapter II, the orthodox treatment of the notion of a derived functor for a left exact functor is given. In 2.11 through Note 15, a more general invariant than the cohomology is introduced. Namely for a sequence of objects and morphisms in an abelian category, when the composition $\mathrm{d}^2 = 0$ need not hold, we define two complexifying functors on the sequence. The cohomology of the complexified sequence is the notion of a precohomology generalizing cohomology. The half-exactness and the self-duality of precohomologies are proved. As a general reference for this Chapter,

[HS] Hilton, P.J., Stammbach, U., *A Course in Homological Algebra*, Graduate Texts in Mathematics, Springer-Verlag, 1971

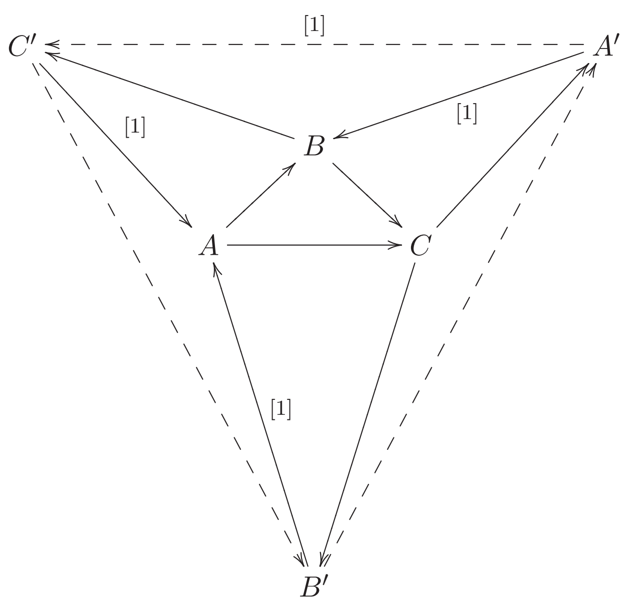
is also recommended.

In Chapter III, we focus on the spectral sequences associated with a double complex, the spectral sequences of composite functors, and the spectral sequences of hypercohomologies. For the *theory* of spectral sequences, in

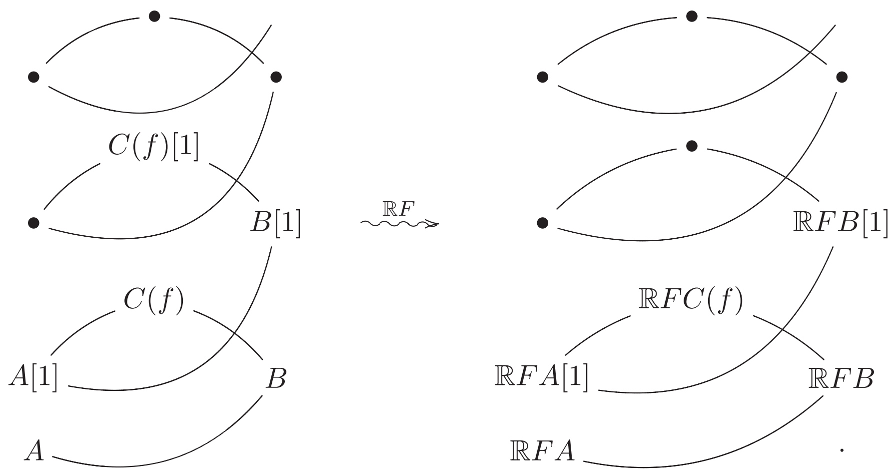
[LuCo] Lubkin, S., *Cohomology of Completions*, North-Holland, North-Holland Mathematics Studies 42, 1980

one can find the most general statements on abutments of spectral sequences. In [THOC], the interplay of the above three kinds of spectral sequences and their applications to sheaf cohomologies are given.

In Chapter IV, an elementary introduction to a derived category is given. Note that diagram (3.14) in Chapter IV comes from [GM]. The usual octahedral axiom for a triangulated category is replaced by the simpler (and maybe more natural) triangular axiom:



A schematic picture for the derived functors $\mathbb{R}F$ between derived categories carrying a distinguished triangle to distinguished triangle may be expressed as



As references for Chapter IV,

[HartRes] Hartshorne, R., *Residues and Duality*, Lecture Notes Math. 20, Springer-Verlag, 1966, and

[V] Verdier, J.L., *Catégories triangulées*, in *Cohomologie Étale*, SGA4$\frac{1}{2}$, Lecture Notes Math. 569, Springer-Verlag, 1977, 262–312.

need to be mentioned.

In Chapter V, applications of the materials in Chapters III and IV are given. The first half of Chapter V is focused on the background for the explicit computation of zeta invariances associated with the Weierstrass family. We wish to compute the homologies with compact supports of the closed fibre of the hyperplane

$$ZY^2 = 4X^3 - g_2XZ^2 - g_3Z^3$$

in $\mathbb{P}^2(\underline{A})$, $\underline{A} := \hat{\mathbb{Z}}_p[g_2, g_3]$, where X, Y, Z are homogeneous coordinates (or the open subfamily, i.e., the pre-image of $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3]_\Delta)$, i.e., localized at the discriminant $\Delta := g_2^3 - 27g_3^2$, $p \neq 2, 3$). Let U be the affine open family in the above fibre, i.e., "$Z = 1$". Then we are interested in a set of generators and relations for the $A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$-module $\mathrm{H}^1_c(U, A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$. For $\mathfrak{p}$ in the base $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3])$ (or $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3]_\Delta)$, the universal spectral sequence is induced so as to compute the zeta function of the fibre over $\mathfrak{p}$ (or elliptic curve over $\mathfrak{p}$).

We also decided to include a letter from Prof. Dwork in 5.2.4 in Chapter V since we could not find the contents of this letter elsewhere.

In the second half of Chapter V, only some of the cohomological aspects of $\mathscr{D}$-modules are mentioned. None of the microlocal aspects of $\mathscr{D}$-modules are treated in this book. One may consider the latter half materials of Chapter V as examples and exercises of the spectral sequences and derived categories in Chapters III and IV.

Lastly, I would like to express my gratitude to my mathematician friends in the U.S.A., Japan and Europe. I will not try to list the names of these people here fearing that the names of significant people might be omitted. However, I would like to mention the name of my teacher and Ph.D. advisor, Prof. Saul Lubkin. I would like to apologize to him, however, because I was not able to learn as much as he exposed me to during my student years in the late 1970's. (I wonder where my Mephistopheles is.) In a sense, this book is my humble delayed report to Prof. Lubkin.

Tomo enpouyori kitari
mata tanoshi karazuya. . .

Goro Kato

Thanksgiving Holiday with my Family and Friends, 2005

Chapter 1

CATEGORY

1.1 Categories and Functors

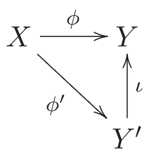
The notion of a category is a concise concept shared among "groups and group homomorphisms", "set and set-theoretic mappings", "topological spaces and continuous mappings", e t c.

Definition 1. A *category* $\mathscr{C}$ consists of *objects*, denoted as $X, Y, Z, \ldots$, and *morphisms*, denoted as $f, g, \phi, \psi, \alpha, \beta, \ldots$. For objects X and Y in the category $\mathscr{C}$, there is induced the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms from X to Y. If $\phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ we write $\phi : X \to Y$ or $X \xrightarrow{\phi} Y$. Then, for $\phi : X \to Y$ and $\psi : Y \to Z$, the composition $\psi \circ \phi : X \to Z$ is defined. Furthermore, for $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\gamma} W$, the associative law $\gamma \circ (\psi \circ \phi) = (\gamma \circ \psi) \circ \phi$ holds. For each object X there exists a morphism $1_X : X \to X$ such that for $f : X \to Y$ and for $g : Z \to X$ we have $f \circ 1_X = f$ and $1_X \circ g = g$. Lastly, the sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are pairwise disjoint. Namely, if $\mathrm{Hom}_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X', Y')$, then $X = X'$ and $Y = Y'$.

Note 1. When X is an object of a category $\mathscr{C}$ we also write $X \in \mathrm{Ob}(\mathscr{C})$, the class of objects in $\mathscr{C}$. Note that a category is said to be *small* if $\mathrm{Ob}(\mathscr{C})$ is a set.

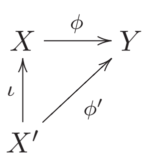
Example 1. The category Ab of abelian groups consists of abelian groups and group homomorphisms as morphisms. The category Set of sets consists of sets and set-theoretic maps as morphisms. Next let T be a topological space. Then there is an induced category $\mathscr{T}$ consisting of the open sets of T as objects. For open sets $U, V \subset T$, the induced set $\mathrm{Hom}_{\mathscr{T}}(U, V)$ of morphisms from U and V consists of the inclusion map $\iota : U \hookrightarrow V$ if $U \subset V$, and $\mathrm{Hom}_{\mathscr{T}}(U, V)$ an empty set if $U \not\subseteq V$.

Remark 1. For the category Ab we have the familiar element-wise definitions of the kernel and the image of a group homomorphism f from a group G to a group H. We also have the notions of a monomorphism, called an injective homomorphism, and of an epimorphism, called a surjective homomorphism in the category Ab. For a general category $\mathscr{C}$ we need to give appropriate definitions without using elements for the above mentioned concepts. For example, $\phi : X \to Y$ in $\mathscr{C}$ is said to be an *epimorphism* if $f \circ \phi = g \circ \phi$ implies $f = g$ where $f, g : Y \to Z$. (This definition of an epimorphism is reasonable since the agreement $f \circ \phi = g \circ \phi$ only on the set-theoretic image of ϕ guarantees that $f = g$.) Similarly, $\phi : X \to Y$ is said to be a *monomorphism* if $\phi \circ f = \phi \circ g$ implies $f = g$ where $f, g : W \to X$. (This is reasonable since there can not be two different paths from W to Y.) In order to give a categorical definition of an image of a morphism, we need to define the notion of a subobject. Let $W \xrightarrow{\phi} X$ and $W' \xrightarrow{\phi'} X$ be monomorphisms. Then define a pre-order $(W', \phi') \leq (W, \phi)$ if and only if there exists a morphism $\psi : W' \to W$ satisfying $\phi \circ \psi = \phi'$. Notice that ψ is a uniquely determined monomorphism. If $(W, \phi) \leq (W', \phi')$ also holds, we have a monomorphism $\psi' : W \to W'$ satisfying $\phi' \circ \psi' = \phi$ and so $\phi \circ \psi \circ \psi' = \phi' \circ \psi' = \phi = \phi \circ 1_W$. Since ϕ is a monomorphism we have $\psi \circ \psi' = 1_W$. Similarly, we also have $\psi' \circ \psi = 1_{W'}$. This means that ψ is an isomorphism, and (W, ϕ), (W', ϕ') are said to be equivalent. A *subobject* of X is defined as an equivalence class of such pairs (W, ϕ). A categorical, i.e., element-free, definition of the image of a morphism $\phi : X \to Y$ may be given as follows. Consider a factorization of ϕ



(1.1)

where (Y', ι) is a subobject of Y. For another such factorization (Y'', ι'), if there exists a morphism $j : Y' \to Y''$ satisfying $\iota = \iota' \circ j$, then (Y', ι) is said to be the *image* of ϕ. Intuitively speaking, shrink Y as much as possible to Y' so that factorization is still possible. Namely, the image of ϕ is the smallest subobject (Y', ι) to satisfy the commutative diagram (1.1). On the other hand, the *kernel* of $\phi : X \to Y$ can be characterized as the largest subobject (X', ι) of satisfying $\phi \circ \iota = 0$ in



(1.2)

1.1.1 Cohomology in Ab

For a sequence

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

in Ab, the *cohomology group* at Y is defined as the quotient group of Y

$$\ker\psi \big/ \operatorname{im}\phi \tag{1.3}$$

provided $\operatorname{im}\phi \subset \ker\psi$, i.e., for $y = \phi(x) \in \operatorname{im}\phi$ we have $\psi(y) = 0$, or in still other words, $\psi(y) = \psi(\phi(x)) = (\psi \circ \phi)(x) = 0$.

1.1.2 The functor $\mathrm{Hom}_{\mathscr{C}}(\cdot,\cdot)$

Let us take a close look at the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ in Definition 1. First consider $\mathrm{Hom}_{\mathscr{C}}(X, X)$. Recall that there is a special morphism from X to X, call it 1_X, satisfying the following. For any $\phi : X \to Y$ and $\psi : Z \to X$ we have $1_X \circ \psi = \psi$ and $\phi \circ 1_X = \phi$ in

$$Z \xrightarrow{\psi} X \xrightarrow{1_X} X \xrightarrow{\phi} Y. \tag{1.4}$$

Then 1_X is said to be an *identity morphism* as in Definition 1, (i).

Next delete Y in the expression $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to get $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$. Then, regard $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$ as an assignment

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} &\longrightarrow \mathsf{Set} \\ Y &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{aligned} \tag{1.5}$$

Similarly we can consider

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(\cdot, Y) : \mathscr{C} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{aligned} \tag{1.6}$$

That is, when you substitute Y in the deleted spot of $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$, you get the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms. For two objects Y and Y' we have two sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $\mathrm{Hom}_{\mathscr{C}}(X, Y')$. Then for a morphism $\beta : Y \to Y'$ consider the diagram

$$\begin{array}{ccc} & X & \\ {\scriptstyle\phi}\swarrow & & \searrow{\scriptstyle\beta\circ\phi} \\ Y & \xrightarrow{\quad\beta\quad} & Y' \end{array} \tag{1.7}$$

This diagram indicates that for $\phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, we get $\beta \circ \phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y')$. Schematically, we express this situation as:

$$\begin{array}{ccc} \beta : Y & \longrightarrow & Y' \quad \text{in } \mathscr{C} \\ & \mathrm{Hom}_{\mathscr{C}}(X,\cdot) \big\wr\!\!\downarrow & \\ \mathrm{Hom}_{\mathscr{C}}(X, \beta) : \mathrm{Hom}_{\mathscr{C}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, Y') \quad \text{in Set} \end{array} \tag{1.8}$$

where $\mathrm{Hom}_{\mathscr{C}}(X, \beta)(\phi) := \beta \circ \phi$.

On the other hand, when X is deleted from $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, we get (1.6). But for $X \xrightarrow{\alpha} X'$, i.e., considering

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & {\scriptstyle \psi\circ\alpha} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & Y & \end{array} \tag{1.9}$$

$\psi \in \mathrm{Hom}_{\mathscr{C}}(X', Y)$ induces $\psi \circ \alpha \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Schematically,

$$\begin{array}{ccc} \alpha : X & \longrightarrow & X' \quad \text{in } \mathscr{C} \\ & \mathrm{Hom}_{\mathscr{C}}(\cdot,Y) \big\wr\!\!\downarrow & \\ \mathrm{Hom}_{\mathscr{C}}(\alpha, Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}}(X', Y) \quad \text{in Set} \end{array} \tag{1.10}$$

Notice that the direction of the morphism in (1.10) is changed as compared with $\mathrm{Hom}_{\mathscr{C}}(X, \beta)$ in (1.8).

Definition 2. Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. A *covariant functor* from $\mathscr{C}$ to $\mathscr{C}'$ denoted as $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$, is an assignment of an object FX in $\mathscr{C}'$ to each object X in $\mathscr{C}$ and a morphism $F\alpha$ from FX to FX' to each morphism $\alpha : X \to X'$ in $\mathscr{C}$ satisfying:

(Func1) For $X \xrightarrow{\alpha} X' \xrightarrow{\alpha'} X''$ in $\mathscr{C}$ we have

$$F(\alpha' \circ \alpha) = F\alpha' \circ F\alpha.$$

(Func2) For $1_X : X \to X$ we have $F1_X = 1_{FX} : FX \to FX$.

Condition (Func1) may schematically be expressed as the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & {\scriptstyle \alpha'\circ\alpha} \searrow & \downarrow {\scriptstyle \alpha'} \\ & & X'' \end{array} \qquad\qquad \begin{array}{ccc} FX & \xrightarrow{F\alpha} & FX' \\ & {\scriptstyle F(\alpha'\circ\alpha)} \searrow & \downarrow {\scriptstyle F\alpha'} \\ & & FX'' \end{array} \tag{1.11}$$

in $\mathscr{C}$ in $\mathscr{C}'$

Example 2. In Definition 2, let $\mathscr{C}' = \mathsf{Set}$ and let $F = \mathrm{Hom}_{\mathscr{C}}(X, \cdot)$. Then one notices from (1.8) that $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$ is a covariant functor.

Note 2. Similarly, a *contravariant functor* $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ can be defined as in Definition 2 with the following exception: For $\alpha : X \to X'$ in $\mathscr{C}$, $F\alpha$ is a morphism from FX' to FX in $\mathscr{C}'$, i.e., as in (1.10) the direction of the morphism is changed. Notice that $\mathrm{Hom}_{\mathscr{C}}(\cdot, Y)$ is a contravariant functor from $\mathscr{C}$ to Set.

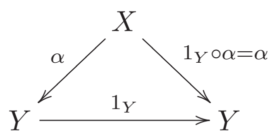
Before we begin the next topic, let us confirm that the covariant functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$ satisfies Condition (Func2) of Definition 2. To demonstrate this: for $1_Y : Y \to Y$, indeed

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

is to be the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) = 1_{\mathrm{Hom}_{\mathscr{C}}(X,Y)}.$$

Let $\alpha \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ be an arbitrary morphism. Then consider



(1.12)

which is a special case of (1.7). As shown in (1.8), the definition of

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

is $\alpha \mapsto 1_Y \circ \alpha = \alpha$. Namely, $\mathrm{Hom}_{\mathscr{C}}(X, 1_Y)$ is an identity on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.

1.2 Opposite Category

Next, we will define the notion of an *opposite category* (or *dual category*). Let $\mathscr{C}$ be a category. Then the opposite category $\mathscr{C}^\circ$ has the same objects as $\mathscr{C}$. This means that the dual object X° in $\mathscr{C}^\circ$ of an object X in $\mathscr{C}$ satisfies $X^\circ = X$. We will use the same X even when X is an object of $\mathscr{C}^\circ$. Let X and Y be objects in $\mathscr{C}^\circ$, then the set of morphisms from X to Y in $\mathscr{C}^\circ$ is defined as the set of morphisms from Y to X in $\mathscr{C}$, i.e.,

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X). \tag{2.1}$$

Note that $\mathscr{C}^\circ$ is also called the *dual category* of $\mathscr{C}$. Recall that

$$\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$$

is a covariant functor. Let us replace $\mathscr{C}$ by $\mathscr{C}^\circ$. Then we have

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot):\mathscr{C}^\circ \rightsquigarrow \mathsf{Set}.$$

Let $Y \xrightarrow{\phi} Y'$ be a morphism in $\mathscr{C}$. Then in $\mathscr{C}^\circ$ we have $Y \xleftarrow{\phi^\circ} Y'$. The covariant functor $\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot)$ takes $Y \xleftarrow{\phi^\circ} Y'$ in $\mathscr{C}^\circ$ without changing the direction of ϕ° to

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,Y) \longleftarrow \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y')$$

in Set. From (2.1) we get

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,Y)=\mathrm{Hom}_{\mathscr{C}}(Y,X) \longleftarrow \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y')=\mathrm{Hom}_{\mathscr{C}}(Y',X)\ .$$

Schematically, we have

$$\begin{array}{lccc} \text{In } \mathscr{C}^\circ: & Y & \xleftarrow{\ \phi^\circ\ } & Y' \ . \\ & {\scriptstyle\circ}\big\downarrow\wr & & \wr\big\uparrow{\scriptstyle\circ} \\ \text{In } \mathscr{C}: & Y & \xrightarrow{\ \phi\ } & Y' \end{array} \tag{2.2a}$$

Applying $\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot)$ to the top row and $\mathrm{Hom}_{\mathscr{C}}(\cdot,X)$ to the bottom row, we get:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y') \\ \Big\| & & \Big\| \\ \mathrm{Hom}_{\mathscr{C}}(Y,X) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}}(Y',X) \end{array} \tag{2.2b}$$

in Set. Generally, for a covariant functor $F:\mathscr{C} \rightsquigarrow \mathscr{C}'$, there is induced a contravariant functor $F:\mathscr{C}^\circ \rightsquigarrow \mathscr{C}'$. On the other hand, $F:\mathscr{C} \rightsquigarrow \mathscr{C}'^\circ$ becomes contravariant.

1.2.1 Presheaf on $\mathscr{T}$

In Example 1, we defined the category $\mathscr{T}$ associated with a topological space T. Let us consider a contravariant functor F from $\mathscr{T}$ to a category $\mathscr{A}$. Namely, for $U \hookrightarrow V$ in $\mathscr{T}$, we have $FU \leftarrow FV$ in $\mathscr{A}$. (As noted, $F:\mathscr{T}^\circ \rightsquigarrow \mathscr{A}$ is a covariant functor.) Then F is said to be a *presheaf* defined on $\mathscr{T}$ with values in $\mathscr{A}$. In the category of presheaves on $\mathscr{T}$

$$\hat{\mathscr{T}} := \mathscr{A}^{\mathscr{T}^\circ}, \tag{2.3}$$

an object is a covariant functor (presheaf) from $\mathscr{T}^\circ$ to $\mathscr{A}$, and a morphism f of presheaves F and G is defined as follows. To every object U of $\mathscr{T}$, f assigns a morphism

$$f_U : FU \to GU \tag{2.4}$$

in $\mathscr{A}$. Generally, for categories $\mathscr{C}$ and $\mathscr{C}'$, let

$$\hat{\mathscr{C}} = \mathscr{C}'^{\mathscr{C}} \tag{2.5}$$

be the category of (covariant) functors as its objects. For functors F and G, a morphism $f : F \to G$ is called a *natural transformation* from F to G and is defined as an assignment $f_U : FU \to GU$ for an object U in $\mathscr{C}$. Additionally f must satisfy the following condition: for every $U \xrightarrow{\alpha} V$ in $\mathscr{C}$, the diagram

$$\begin{array}{ccc} FU & \xrightarrow{f_U} & GU \\ \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle G\alpha} \\ FV & \xrightarrow{f_V} & GV \end{array} \tag{2.6a}$$

commutes, i.e., $f_V \circ F\alpha = G\alpha \circ f_U$ in $\mathscr{C}'$. Therefore, a morphism $f : F \to G$ in $\hat{\mathscr{T}} = \mathscr{A}^{\mathscr{T}^\circ}$ must satisfy the following in addition to (2.4). For $\iota : U \hookrightarrow V$ in $\mathscr{T}$ (i.e., $U \hookleftarrow V$ in $\mathscr{T}^\circ$),

$$\begin{array}{ccc} FU & \xrightarrow{f_U} & GU \\ {\scriptstyle F\iota}\uparrow & & \uparrow{\scriptstyle G\iota} \\ FV & \xrightarrow{f_V} & GV \end{array} \tag{2.6b}$$

must commute. Important examples of $\hat{\mathscr{T}}$ are the cases when $\mathscr{A} = \mathsf{Set}$ and $\mathscr{A} = \mathsf{Ab}$. We will return to this topic when the notion of a site is introduced.

1.3 Forgetful Functors

Let A be an abelian group. By forgetting the abelian group structure, A can be regarded as just a set. Namely, we have an assignment $S : \mathsf{Ab} \rightsquigarrow \mathsf{Set}$. For a group homomorphism $\phi : A \to B$ in Ab, assign the set-theoretic map $S\phi : SA \to SB$. One may wish to check axioms (Func1) and (Func2) of Definition 2 for the assignment S. Consequently S is a covariant functor from Ab to Set. This functor S is said to be a forgetful functor from Ab to Set.

Definition 3. Let $\mathscr{C}'$ and $\mathscr{C}$ be categories. Then $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$ when the following conditions are satisfied.

(Subcat1) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$ and for all objects X and Y in $\mathscr{C}'$,

$$\mathrm{Hom}_{\mathscr{C}'}(X, Y) \subset \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

(Subcat2) The composition of morphisms in $\mathscr{C}'$ is coming from the composition of morphisms in $\mathscr{C}$, and for all objects X in $\mathscr{C}'$ the identity morphisms 1_X in $\mathscr{C}'$ are also identity morphisms in $\mathscr{C}$.

Example 3. Let $\mathcal{V}'$ be the category of finite-dimensional vector spaces over a field ℓ and let $\mathcal{V}$ be the category of vector spaces over ℓ and where the morphisms are the ℓ-linear transformations. Then $\mathcal{V}'$ is a subcategory of $\mathcal{V}$. Let Top be the category of topological spaces where the morphisms are continuous mappings. Then Top is a subcategory of Set.

Remark 2. Note that we have $\mathrm{Hom}_{\mathcal{V}'}(X, Y) = \mathrm{Hom}_{\mathcal{V}}(X, Y)$, since the ℓ-linearity has nothing to do with dimensions. In general, when a subcategory $\mathscr{C}'$ of a category $\mathscr{C}$ satisfies $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X, Y)$ for all X and Y in $\mathscr{C}'$, $\mathscr{C}'$ is said to be a *full subcategory* of $\mathscr{C}$.

1.4 Embedddings

Let $\mathscr{B}$ and $\mathscr{C}$ be categories. Even though $\mathscr{B}$ is not a subcategory of $\mathscr{C}$, one can ask whether $\mathscr{B}$ can be embedded in $\mathscr{C}$ (whose definition will be given in the following). Let F be a covariant functor from $\mathscr{B}$ to $\mathscr{C}$. Then for $f : X \to Y$ in $\mathscr{B}$ we have $FX \to FY$ in $\mathscr{C}$. Namely, for an element f of $\mathrm{Hom}_{\mathscr{B}}(X, Y)$ we obtain Ff in $\mathrm{Hom}_{\mathscr{C}}(FX, FY)$. That is we have the following map $\bar{F}$:

$$\begin{aligned} \bar{F} : \mathrm{Hom}_{\mathscr{B}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(FX, FY) \\ f &\longmapsto \bar{F}(f) = Ff \end{aligned} \tag{4.1}$$

If $\bar{F}$ is injective, $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to be *faithful*, and if $\bar{F}$ is surjective, F is said to be *full*. Furthermore, F is said to be an *embedding* (or *imbedding*) if $\bar{F}$ is not only injective on morphisms, but also F is injective on objects. That is, $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to be an embedding if F is a faithful functor and if $FX = FY$ implies $X = Y$. Then $\mathscr{B}$ may be regarded as a subcategory of $\mathscr{C}$. We also say that $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is *fully faithful* when F is full and faithful. A functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to *represent* $\mathscr{C}$ when the following condition is satisfied: For every object X' of $\mathscr{C}$ there exists an object X in $\mathscr{B}$ so that there exists an isomorphism from FX to X'. If a fully faithful functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ represents $\mathscr{C}$ then F is said to be an *equivalence*. Furthermore, an equivalence F is said to be an *isomorphism* if F induces an injective correspondence between the objects of $\mathscr{B}$ and $\mathscr{C}$. The notion of an equivalence F can be characterized by the following.

Proposition 3. *A functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is an equivalence if and only if there exists a functor $F' : \mathscr{C} \rightsquigarrow \mathscr{B}$ satisfying*

(Eqv) *$F' \circ F$ and $F \circ F'$ are isomorphic to the identity functors $1_{\mathscr{B}}$ and $1_{\mathscr{C}}$, respectively.*

Proof. Let $f : Z \to Z'$ be a morphism in $\mathscr{C}$. Since F represents $\mathscr{C}$, there are objects X and X' in $\mathscr{B}$ so that $FX \xrightarrow{i} Z$ and $FX' \xrightarrow{j} Z'$ are isomorphisms in $\mathscr{C}$.

Then we have the morphism $j^{-1} \circ f \circ i : FX \to FX'$. Define $\tilde{f} := j^{-1} \circ f \circ i$. Since F is fully faithful there exists a unique morphism $\tilde{f}' : X \to X'$ in $\mathscr{B}$ satisfying $F\tilde{f}' = \tilde{f}$. Then define $F'f := \tilde{f}'$. Namely, we have $F'Z = X$ and $F'Z' = X'$. Note that F' becomes a functor from $\mathscr{C}$ to $\mathscr{B}$. From the commutative diagram

$$\begin{CD} FX @>{\approx}>{i}> Z \\ @V{\tilde{f}:=j^{-1}\circ f\circ i}VV @VV{f}V \\ FX' @>{\approx}>{j}> Z' \end{CD} \tag{4.2}$$

in $\mathscr{C}$, we get the commutative diagram in $\mathscr{B}$

$$\begin{CD} F'FX @>{\approx}>{F'i}> F'Z = X \\ @VVV @VV{F'f:=\tilde{f}'}V \\ F'FX' @>{\approx}>{F'j}> F'Z' = X'. \end{CD} \tag{4.3}$$

From the definition of F', i.e., $F'Z = X$ and (4.2), we also get

$$\begin{CD} FF'Z @>{\approx}>{i}> Z \\ @VVV @VV{f}V \\ FF'Z' @>{\approx}>{j}> Z'. \end{CD} \tag{4.4}$$

We obtain $F' \circ F \approx 1_{\mathscr{B}}$ and $F \circ F' \approx 1_{\mathscr{C}}$.

Conversely, assume (Eqv). For an object Z of $\mathscr{C}$ we have an isomorphism $(F \circ F')Z \xrightarrow{\approx} 1_{\mathscr{C}}Z = Z$. Let $X = F'Z$. Then $FX \xrightarrow{\approx} Z$. Therefore, F represents $\mathscr{C}$. Consider $\bar{F}$ of (4.1), i.e.,

$$\bar{F} : \mathrm{Hom}_{\mathscr{B}}(X, X') \to \mathrm{Hom}_{\mathscr{C}}(FX, FX').$$

Suppose that $\bar{F}f = \bar{F}g$ for $f, g \in \mathrm{Hom}_{\mathscr{B}}(X, X')$. We have $Ff = Fg$ which implies $F'Ff = F'Fg$. Since $F' \circ F \xrightarrow{\approx} 1_{\mathscr{B}}$, $f = g$. Therefore F is faithful. Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(FX, FX')$. Since F represents $\mathscr{C}$, we have isomorphisms $F(F'FX) \xrightarrow[i]{\approx} FX$ and $F(F'FX') \xrightarrow[j]{\approx} FX'$. That is, we have the commutative diagram

$$\begin{CD} FF'FX @>{\approx}>{i}> FX \\ @V{F(F'\phi)}VV @VV{\phi}V \\ FF'FX' @>{\approx}>{j}> FX'. \end{CD} \tag{4.5}$$

Then $F'\phi : F'FX \to F'FX'$, i.e., $F'\phi \in \mathrm{Hom}_{\mathscr{B}}(X, X')$ satisfying

$$\bar{F}(F'\phi) = (F \circ F')\phi = 1_{\mathscr{C}}\phi = \phi.$$

Therefore, F is full.

Remark 3. When there is an equivalence $F : \mathscr{B} \rightsquigarrow \mathscr{C}$, $\mathscr{B}$ may be identified with $\mathscr{C}$ in the following sense. If there are objects X and X' in $\mathscr{B}$ having isomorphisms $FX \xrightarrow{i} Z$ and $FX' \xrightarrow{j} Z$ then we get the isomorphisms $F'FX \xrightarrow{F'i} F'Z$ and $F'FX' \xrightarrow{F'j} F'Z$. Namely,

$$X \xrightarrow[F'i]{\approx} F'Z \xleftarrow[F'j]{\approx} X' \; .$$

Considering Z' as isomorphic to Z we can conclude that there is a bijective correspondence between isomorphic classes of $\mathscr{B}$ and $\mathscr{C}$.

1.5 Representable Functors

First recall from (1.9) that $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a contravariant functor from $\mathscr{C}$ to Set. Let G also be a contravariant functor from $\mathscr{C}$ to Set. Namely, $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ and G are objects of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$ as in (2.5) and (2.6a). For $G \in \mathrm{Ob}(\hat{\mathscr{C}})$, if there exists an object X in $\mathscr{C}$ so that $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is isomorphic to G in the category $\hat{\mathscr{C}}$, then G is said to be a *representable functor*. We also say that G and $\widetilde{X} := \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ are *naturally equivalent*. That is, there is a natural transformation $\alpha : \widetilde{X} \to G$ (i.e., α is a morphism in $\hat{\mathscr{C}}$) which gives an isomorphism for every object Y in $\mathscr{C}$

$$\alpha_Y : \widetilde{X}(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X) \to GY. \tag{5.1}$$

Such an α is said to be a *natural equivalence*.

1.5.1 Yoneda's Lemma

Let F be an arbitrary contravariant functor from a category $\mathscr{C}$ to Set. For two objects F and $\widetilde{X} = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$, consider the set $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ of all morphisms in $\hat{\mathscr{C}}$ from $\widetilde{X}$ to F, i.e., $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ is the set of all the natural transformations from $\widetilde{X}$ to F. The Yoneda Lemma asserts that there is an isomorphism (i.e., a bijection) between the sets $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ and FX. If an element of $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ is written vertically as

$$\begin{array}{c} F \\ \uparrow \\ \widetilde{X} \end{array} \tag{5.2}$$