ADVANCES IN FRACTIONAL CALCULUS

Advances in Fractional Calculus

Theoretical Developments and Applications in Physics and Engineering

edited by

J. Sabatier

Université de Bordeaux I
Talence, France

O. P. Agrawal

Southern Illinois University
Carbondale, IL, USA

and

J. A. Tenreiro Machado

Institute of Engineering of Porto
Portugal

Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN-13 978-1-4020-6041-0 (HB)
ISBN-13 978-1-4020-6042-7 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.

www.springer.com

Printed on acid-free paper

We dedicate this book to the honorable memory of our colleague and friend Professor Peter W. Krempl

Table of Contents

7. Control

Preface

Fractional Calculus is a field of applied mathematics that deals with derivatives and integrals of arbitrary orders (including complex orders), and their applications in science, engineering, mathematics, economics, and other fields. It is also known by several other names such as Generalized Integral and Differential Calculus and Calculus of Arbitrary Order. The name "Fractional Calculus" is holdover from the period when it meant calculus of ration order. The seeds of fractional derivatives were planted over 300 years ago. Since then many great mathematicians (pure and applied) of their times, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A. K. Grünwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann M. Riesz, and H. Weyl, have contributed to this field. However, most scientists and engineers remain unaware of Fractional Calculus; it is not being taught in schools and colleges; and others remain skeptical of this field. There are several reasons for that: several of the definitions proposed for fractional derivatives were inconsistent, meaning they worked in some cases but not in others. The mathematics involved appeared very different from that of integer order calculus. There were almost no practical applications of this field, and it was considered by many as an abstract area containing only mathematical manipulations of little or no use.

Nearly 30 years ago, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade Fractional Calculus has been applied to almost every field of science, engineering, and mathematics. Some of the areas where Fractional Calculus has made a profound impact include viscoelasticity and rheology, electrical engineering, electrochemistry, biology, biophysics and bioengineering, signal and image processing, mechanics, mechatronics, physics, and control theory. Although some of the mathematical issues remain unsolved, most of the difficulties have been overcome, and most of the documented key mathematical issues in the field have been resolved to a point where many of the mathematical tools for both the integer- and fractional-order calculus are the same. The books and monographs of Oldham and Spanier (1974), Oustaloup (1991, 1994, 1995), Miller and Ross (1993), Samko, Kilbas, and Marichev (1993), Kiryakova (1994), Carpinteri and Mainardi (1997), Podlubny (1999), and Hilfer (2000) have been helpful in introducing the field to engineering, science, economics and finance, pure and applied mathematics communities. The progress in this field continues. Three recent books in this field are by West, Grigolini, and Bologna (2003), Kilbas, Srivastava, and Trujillo (2005), and Magin (2006).

One of the major advantages of fractional calculus is that it can be considered as a super set of integer-order calculus. Thus, fractional calculus has the potential to accomplish what integer-order calculus cannot. We believe that many of the great future developments will come from the applications of fractional calculus to different fields. For this reason, we are promoting this field. We recently organized five symposia (the first symposium on Fractional Derivatives and Their Applications (FDTAs), ASME-DETC 2003, Chicago, Illinois, USA, September 2003; IFAC first workshop on Fractional Differentiations and its Applications (FDAs), Bordeaux, France, July 2004; Mini symposium on FDTAs, ENOC-2005, Eindhoven, the Netherlands, August 2005; the second symposium on FDTAs, ASME-DETC 2005, Long Beach, California, USA, September 2005; and IFAC second workshop on FDAs, Porto, Portugal, July 2006) and published several special issues which include Signal Processing, Vol. 83, No. 11, 2003 and Vol. 86, No. 10, 2006; Nonlinear dynamics, Vol. 29, No. 1–4, 2002 and Vol. 38, No. 1–4, 2004; and Fractional Differentiations and its Applications, Books on Demand, Germany, 2005. This book is an attempt to further advance the field of fractional derivatives and their applications.

In spite of the progress made in this field, many researchers continue to ask: “What are the applications of this field?” The answer can be found right here in this book. This book contains 37 papers on the applications of Fractional Calculus. These papers have been divided into seven categories based on their themes and applications, namely, analytical and numerical techniques, classical mechanics and particle physics, diffusive systems, viscoelastic and disordered media, electrical systems, modeling, and control. Applications, theories, and algorithms presented in these papers are contemporary, and they advance the state of knowledge in the field. We believe that researchers, new and old, would realize that we cannot remain within the boundaries of integral order calculus, that fractional calculus is indeed a viable mathematical tool that will accomplish far more than what integer calculus promises, and that fractional calculus is the calculus for the future.

Most of the papers in this book are expanded and improved versions of the papers presented at the Mini symposium on FDTAs, ENOC-2005, Eindhoven, The Netherlands, August 2005, and the second symposium on FDTAs, ASME-DETC 2005, Long Beach, California, USA, September 2005. We sincerely thank the ASME for allowing the authors to submit modified versions of their papers for this book. We also thank the authors for submitting their papers for this book and to Springer-Verlag for its

publication. We hope that readers will find this book useful and valuable in the advancement of their knowledge and their field.

Dr. J. Sabatier
LAPS – CRONE group
Bordeaux 1 University
France

Prof. O. P. Agrawal
College of Engineering
Southern Illinois
University
USA

Prof. J. A. Tenreiro Machado
Institute of Engineering of
Porto
Dept. of Electrotech.
Portugal

Part 1

Analytical and Numerical Techniques

THREE CLASSES OF FDEs AMENABLE TO APPROXIMATION USING A GALERKIN TECHNIQUE

Satwinder Jit Singh and Anindya Chatterjee

Mechanical Engineering Department, Indian Institute of Science, Bangalore 560012, India

Abstract

We have recently presented elsewhere a Galerkin approximation scheme for fractional order derivatives, and used it to obtain accurate numerical solutions of second-order (mechanical) systems with fractional-order damping terms. Here, we demonstrate how that approximation can be used to find accurate numerical solutions of three different classes of fractional differential equations (FDEs), where for simplicity we assume that there is a single fractional-order derivative, with order between 0 and 1. In the first class of FDEs, the highest derivative has integer order greater than one. An example of a traveling point load on an infinite beam resting on an elastic, fractionally damped, foundation is studied. The second class contains FDEs where the highest derivative has order 1. Examples of the so-called generalized Basset's equation are studied. The third class contains FDEs where the highest derivative is the fractional-order derivative itself. Two specific examples are considered. In each example studied in the paper, the Galerkin-based numerical approximation is compared with analytical or semi-analytical solutions obtained by other means. In each case, the Galerkin approximation is found to be very good. We conclude that the Galerkin approximation can be used with confidence for a variety of FDEs, including possibly nonlinear ones for which analytical solutions may be difficult or impossible to obtain.

Keywords

Fractional derivative, Galerkin, finite element, Basset's problem, relaxation, creep.

1 Introduction

A fractional derivative of order α is given using the Riemann–Louville definition [1, 2], as

$$\mathrm{D}^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \left[\int_0^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau \right] ,$$

J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, 3–14.

where $0 < \alpha < 1$. Two equivalent forms of the above with zero initial conditions (as in, e.g., [3]) are given as

$$\mathrm{D}^{\alpha}[x(t)] = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(\tau)}{(t-\tau)^{\alpha}} d\tau = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{x}(t-\tau)}{\tau^{\alpha}} d\tau \,. \qquad (1)$$

Differential equations with a single-independent variable (usually "time"), which involve fractional-order derivatives of the dependent variable(s) are called fractional differential equations or FDEs. In this work, we consider FDEs where the fractional derivative has order between 0 and 1 only. Such FDEs, for our purposes, are divided into three categories, depending on whether the highest-order derivative in the FDE is an integer greater than 1, is exactly equal to 1, or is a fraction between 0 and 1.

In this article, we will demonstrate three strategies for these three classes of FDEs, whereby a new Galerkin technique [4] for fractional derivatives can be used to obtain simple, quick, and accurate numerical solutions. The Galerkin approximation scheme of [4] involves two calculations:

$$\mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\,\mathbf{a} = \mathbf{c}\,\dot{x}(t) \qquad (2)$$

and

$$\mathrm{D}^{\alpha}[x(t)] \approx \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\,\mathbf{c}^T\mathbf{a}, \qquad (3)$$

where $\mathbf{A}$ and $\mathbf{B}$ are $n \times n$ matrices (specified by the scheme; see [4]), $\mathbf{c}$ is an $n \times 1$ vector also specified by the scheme[1], and $\mathbf{a}$ is an $n \times 1$ vector n internal variables that approximate the infinite-dimensional dynamics of the actual fractional order derivative. The T superscript in Eq. (3) denotes matrix transpose.

As will be seen below, the first category of FDEs (section 2) poses no real problem over and above the examples already considered in [4]. That is, in [4], the highest derivatives in the examples considered had order 2; while in the example considered in section 2 below, the highest derivative will be or order 4. However, the example of section 2 is a boundary-value problem on an infinite domain. Our approximation scheme provides significant advantages for this problem. The second category of FDEs (section 3) also leads to numerical solution of ODEs (not FDEs). The specific example considered here is relevant to the physical problem of a sphere falling slowly under gravity through a viscous liquid, but not yet at steady state. Again, the approximation scheme leads to an algorithmically simple, quick and accurate solution. However, the equations are stiff and suitable for a routine that can handle stiff systems, such as Matlab's "ode23t". Finally, the third category of FDEs (section 4) leads to a system of differential algebraic equations (DAEs), which can be solved simply and accurately using an index one DAE solver such as Matlab's "ode23t".

[1] A Maple-8 worksheet to compute the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{c}$ is available on [5].

We emphasize that we have deliberately chosen linear examples below so that analytical or semi-analytical alternative solutions are available for comparing with our results using the Galerkin approximation. However, it will be clear that the Galerkin approximation will continue to be useful for a variety of nonlinear problems where alternative solution techniques might run into serious difficulties.

2 Traveling Load on an Infinite Beam

The governing equation for an infinite beam on a fractionally damped elastic foundation, and with a moving point load (see Fig. 1), is

$$u_{xxxx} + \frac{\bar{m}}{EI} u_{tt} + \frac{c}{EI} \mathrm{D}_t^{1/2} u + \frac{k}{EI} u = -\frac{1}{EI} \delta(x - vt) \,, \tag{4}$$

where $\mathrm{D}^{1/2}$ has a t-subscript to indicate that x is held constant. The boundary conditions of interest are

$$u(\pm\infty, t) \equiv 0.$$

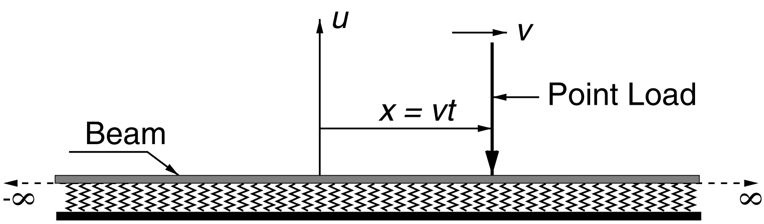

Fig. 1. Traveling point load on an infinite beam with a fractionally damped elastic foundation.

We seek steady-state solutions to this problem.

2.1 With Galerkin

With the Galerkin approximation of the fractional derivative, we get the new PDEs

$$u_{xxxx} + \frac{\bar{m}}{EI} u_{tt} + \frac{c}{EI\,\Gamma(1/2)\Gamma(3/2)} \mathbf{c}^T \mathbf{a} + \frac{k}{EI} u = -\frac{1}{EI} \delta(x - vt)$$

and

$$\mathbf{A}\dot{\mathbf{a}} + \mathbf{B}\mathbf{a} = \mathbf{c} u_t \,,$$

where $\mathbf{a}$ is now a function of both x and t, and the overdot denotes a partial derivative with respect to t. Changing variables to $\xi = x - vt$ and $\tau = t$ to shift to a steadily moving coordinate system, we get

$$u_{\xi\xi\xi\xi} + \frac{\bar{m}}{EI}\left(v^2\, u_{\xi\xi} - 2\, v\, u_{\xi\tau} + u_{\tau\tau} + \frac{c}{\Gamma(1/2)\,\Gamma(3/2)}\mathbf{c}^T\mathbf{a} + k\, u\right) = -\frac{1}{EI}\delta(\xi) \tag{5}$$

and

$$\mathbf{A}(\mathbf{a}_\tau - v\,\mathbf{a}_\xi) + \mathbf{B}\mathbf{a} = \mathbf{c}\,(u_\tau - v\, u_\xi)\,. \tag{6}$$

Now, seeking a steady-state solution, Eqs. (5) and (6) become

$$u_{\xi\xi\xi\xi} + \frac{\bar{m}}{EI}\left(v^2\, u_{\xi\xi} + \frac{c}{\Gamma(1/2)\,\Gamma(3/2)}\mathbf{c}^T\mathbf{a} + k\, u\right) = -\frac{1}{EI}\delta(\xi) \tag{7}$$

and

$$-v\mathbf{A}\mathbf{a}_\xi + \mathbf{B}\mathbf{a} = -v\,\mathbf{c}\, u_\xi\,. \tag{8}$$

The solution will be discussed later.

2.2 Without Galerkin

Without the Galerkin approximation, the fractional term in Eq. (4) can be written as

$$\mathrm{D}_t^{1/2} u(t,x) = \frac{1}{\Gamma(1/2)}\int_0^t \frac{\dot{u}(z,x)}{\sqrt{t-z}}\, dz\,.$$

On letting $w = t - z$ in the above we get

$$\mathrm{D}_t^{1/2} u(t,x) = \frac{1}{\Gamma(1/2)}\int_0^t \frac{\dot{u}(t-w,x)}{\sqrt{w}}\, dw\,. \tag{9}$$

After the change of variables $\xi = x - vt$ and $\tau = t$, we get $\dot{u} = -v\, u_\xi + u_\tau$, which gives $\dot{u} = -v\, u_\xi$ for the steady state (τ independent) solution. Hence, $\dot{u}(t-w,x) = -v\, u_\xi(\xi + v\, w)$, because $\xi = x - vt \Longrightarrow x - v(t-w) = \xi + v\, w$. On substituting in Eq. (9) we get (with incomplete incorporation of steady state conditions)

$$\begin{aligned}\mathrm{D}_t^{1/2} u(t,x) &= \frac{-v}{\Gamma(1/2)}\int_0^\tau \frac{u_\xi(\xi + v\, w)}{\sqrt{w}}\, dw \\ &= \frac{-v}{\Gamma(1/2)}\left(\int_0^\infty \frac{u_\xi(\xi + v\, w)}{\sqrt{w}}\, dw - \int_\tau^\infty \frac{u_\xi(\xi + v\, w)}{\sqrt{w}}\, dw\right).\end{aligned}$$

In the above, steady state is achieved as $\tau \to \infty$, and we get

$$\mathrm{D}_t^{1/2} u(t,x) = \frac{-v}{\Gamma(1/2)}\int_0^\infty \frac{u_\xi(\xi + v\, w)}{\sqrt{w}}\, dw\,.$$

Substituting $y = \xi + v\, w$ above for later convenience, we get

$$\mathrm{D}_t^{1/2} u(t,x) = \frac{-\sqrt{v}}{\Gamma(1/2)} \int_{\xi}^{\infty} \frac{u'(y)}{\sqrt{y-\xi}} dy = \frac{-\sqrt{v}}{\Gamma(1/2)} \int_{-\infty}^{\infty} \frac{\mathrm{H}(y-\xi)\, u'(y)}{\sqrt{y-\xi}} dy \,,$$

where $\mathrm{H}(y-\xi)$ is the Heaviside step function, with $\mathrm{H}(s) = 1$ if $s > 0$, and 0 otherwise.

Thus, the steady state version of Eq. (4) *without* approximation is

$$u_{\xi\xi\xi\xi} + \frac{\bar{m}v^2}{EI} u_{\xi\xi} - \frac{c\sqrt{v}}{EI\,\Gamma(1/2)} \int_{-\infty}^{\infty} \frac{\mathrm{H}(y-\xi)\, u'(y)}{\sqrt{y-\xi}} dy + \frac{k}{EI} u = -\frac{1}{EI}\delta(\xi)\,. \quad (10)$$

2.3 Solutions, with Galerkin and without

Solution of Eq. (7) and (8) is straightforward and quick. An algebraic eigenvalue problem is solved and a jump condition imposed. The details are as follows. For $\xi = \emptyset$, the system reduces to a homogeneous first-order system with constant coefficients. The eigenvalues of this system have nonzero real parts, and are found numerically. Those with negative real parts contribute to the solution for $\xi > 0$, while those with positive real parts contribute to the solution for $\xi < 0$. There is a jump in the solution at $\xi = 0$. The jump occurs only in $u_{\xi\xi\xi}$, and equals $-1/EI$. All other state variables are continuous at $\xi = 0$. These jump/continuity conditions provide as many equations as there are state variables; and these equations can be used to solve for the same number of unknown coefficients of eigenvectors in the solution. The overall procedure is straightforward, and can be implemented in, say, a few lines of Matlab code. Numerical results obtained will be presented below.

Equation (10) cannot, as far as we know, be solved in closed form. It can be solved numerically using Fourier transforms. The Fourier transform of $u(\xi)$ is given by

$$U(\omega) = \frac{\sqrt{-i\omega}}{-EI\omega^4\sqrt{-i\omega} + \bar{m}v^2\omega^2\sqrt{-i\omega} - ic\sqrt{v}\,\omega + k\sqrt{-i\omega}} \quad (11)$$

The inverse Fourier transform of the above was calculated numerically, pointwise in ξ. The integral involved in inversion is well behaved and convergent. However, due to the presence of the oscillatory quantity $\exp(i\omega\xi)$ in the integrand, some care is needed. In these calculations, we used numerical observation of antisymmetry in the imaginary part, and symmetry in the real part, to simplify the integrals; and then used MAPLE to evaluate the integrals numerically.

2.4 Results

Results for $\bar{m} = 1$, $EI = 1$, $k = 1$ and various values of v and c are shown in Fig. 2. The Galerkin approximation is very good.

The agreement between the two solutions (Galerkin and Fourier) provides support for the correctness of both. In a problem with several unequally spaced traveling loads, the Galerkin technique will remain straightforward while the Fourier approach will become more complicated. Our point here is not that the Fourier solution is intellectually inferior (we find it elegant). Rather, straightforward application of the Galerkin technique requires less problem-specific ingenuity and effort.

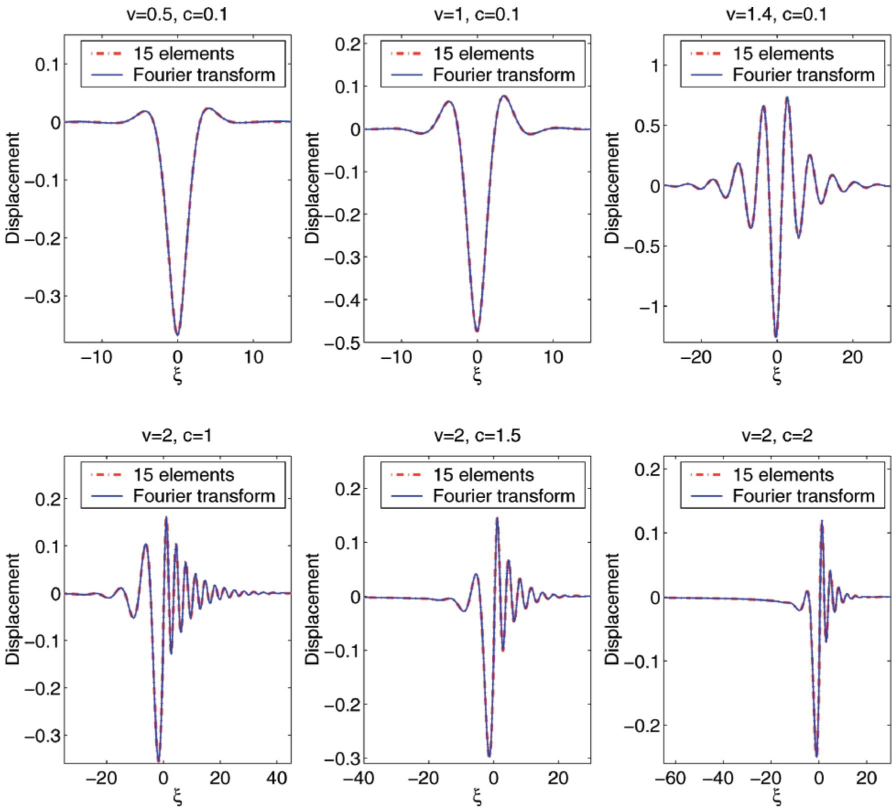

Fig. 2. Numerical results for a traveling point load on an infinite beam at steady state.

3 Off Spheres Falling Through Viscous Liquids

A sphere falling slowly under its own weight through a viscous liquid will approach a steady speed [6]. The approach is described by a FDE where the highest derivative has order 1. Here, we study no fluid mechanics issues. Rather, we consider two such FDEs with, for simplicity, zero initial conditions. Such problems have been referred to as examples of the generalized Basset's problem [7]. Our aim is to demonstrate the use of our Galerkin approximation for such problems.

Consider

$$\dot{v}(t) + \mathrm{D}^{\alpha} v(t) + v(t) = 1\,, \quad v(0) = 0, \tag{12}$$

$0 < \alpha < 1$. Here, for demonstration, we will consider $\alpha = 1/2$ and $1/3$. The solution methods discussed below will work for any reasonable α between 0 and 1.

3.1 With Galerkin

The fractional derivative is approximated as before to give

$$\dot{v}(t) + \frac{1}{\Gamma(1-\alpha)\,\Gamma(1+\alpha)} \mathbf{c}^T \mathbf{a} + v(t) = 1 \tag{13a}$$

and

$$\mathbf{A}\,\dot{\mathbf{a}} + \mathbf{B}\,\mathbf{a} = \mathbf{c}\,\dot{v}(t)\,, \tag{13b}$$

with initial conditions $v(0) = 0$ and $\mathbf{a}(0) = 0$. Recall that, for any value of α, the matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{c}$ are obtained once and for all using the method described in [4].

Equation (13) can be rewritten as a first-order system of ODEs, and solved using Matlab's standard ODE solver, "ode45". However, the equations are stiff and the solution takes time. Two or more orders of magnitude less effort seem to be needed if we use Matlab's stiff system and/or index one DAE solver, "ode23t". We will present numerical results later.

3.2 Series solution using Laplace transforms

The Laplace transform of the solution to Eq. (12) is given by

$$V(s) = \frac{1}{s(1+s+s^{\alpha})} = \frac{[1-(-s^{-1}-s^{\alpha-1})]^{-1}}{s^2}\,.$$

We can expand the numerator above in a Binomial series for $|(s^{-1} + s^{\alpha-1})| < 1$, because $\alpha < 1$ and we are prepared to let s be as large needed (in particular, suppose we consider s values on a vertical line in the complex plane, we are prepared to choose that line as far into the right half plane as needed). The series we obtain is

$$V(s) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{n} \binom{n}{r} \frac{1}{s^{n+2-r\alpha}}\,.$$

Taking the inverse Laplace transform of the above,

$$v(t) = \sum_{n=0}^{\infty} (-1)^n \sum_{r=0}^{n} \binom{n}{r} \frac{t^{n+1-r\alpha}}{\Gamma(n+2-r\alpha)}\,. \tag{14}$$

3.3 Results

Results for the above problem are shown in Fig. 3. The Galerkin approximation matches well with the series solutions of Eq. (12) for $\alpha = 1/2$ and $1/3$. The sum in Eq. (14) was taken upto the $\mathcal{O}(t^{150})$ term for both cases, using MAPLE (fewer than 150 terms may have worked; more were surely not needed).

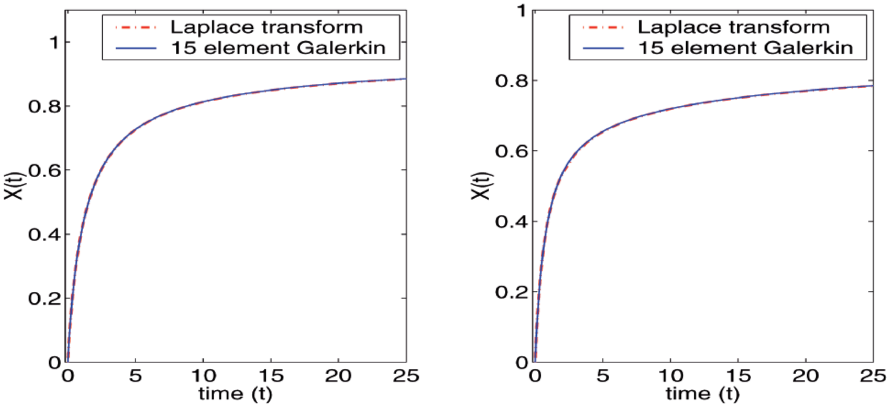

Fig. 3. Comparison between Laplace transform and 15-element Galerkin approximation solutions: Left: $\alpha = 1/2$ and sum in Eq. (14) upto $\mathcal{O}(t^{150})$ term. Right: $\alpha = 1/3$ and sum in Eq. (14) upto $\mathcal{O}(t^{150})$ term.

4 FDEs With Highest Derivative Fractional

Consider

$$\mathrm{D}^{\alpha} x(t) + x(t) = f(t)\,, \quad x(0) = 0. \tag{15}$$

Equations of this form are called relaxation fractional Eq. [8]. These equations have relevance to, e.g., mechanical systems with fractional-order damping and under slow loading (where inertia plays a negligible role), such as in creep tests. Here, we concentrate on demonstrating the use of our Galerkin technique for this class of problems.

4.1 Adaptation of the Galerkin approximation

Our usual Galerkin approximation strategy will not work here directly, because it requires $\dot{x}(t)$ as an input (see Eqs. (2) and (3)). We could introduce $\dot{x}(t)$ by taking a $1-\alpha$ order derivative, but such differentiation requires the forcing function $f(t)$ to have such a derivative, and we avoid such differentiation here. Instead, we adopt the Galerkin approximation through constraints that lead to DAEs, which are then easily solved using standard available routines.

Observe that $\dot{x}(t)$ forcing in Eq. (2) results in an α order derivative of $x(t)$ in equation (3). We interpret the above as follows. If the forcing was some general function $h(t)$ instead of $\dot{x}(t)$; and if $h(t)$ was integrable, i.e., $h(t) = \dot{g}(t)$ for some function $g(t)$; and if, in addition, $g(t)$ was continuous at $t = 0$, then by adding a constant to $g(t)$ we could ensure that $g(0) = 0$ while still satisfying $h(t) = \dot{g}(t)$. Further, the forcing of $h(t)$ (in place of $\dot{x}(t)$) in Eq. (2) would result in an α order derivative of $g(t)$ (in place of $x(t)$) in Eq. (3). In other words, if

$$h(t) = \dot{g}(t)\,,\ g(0) = 0 \tag{16a}$$

and

$$\mathbf{A}\,\dot{\mathbf{a}} + \mathbf{B}\,\mathbf{a} = \mathbf{c}\,\dot{g}(t) \tag{16b}$$

then (within our Galerkin approximation)

$$\mathrm{D}^{\alpha}[g(t)] = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\,\mathbf{c}^T\mathbf{a}\,.$$

But, by definition,

$$\mathrm{D}^{\alpha}[g(t)] = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\dot{g}(\tau)}{(t-\tau)^{\alpha}}d\tau = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{h(\tau)}{(t-\tau)^{\alpha}}d\tau = \mathrm{D}^{\alpha-1}[h(t)]\,,$$

hence

$$\mathrm{D}^{\alpha-1}[h(t)] = \frac{1}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\,\mathbf{c}^T\mathbf{a}\,. \tag{17}$$

Keeping this in mind, we adopt the following strategy:

1. Compute matrices $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{c}$ for $1-\alpha$ order derivatives instead of α order derivatives. To emphasize this crucial distinction, we write $\mathbf{A}_{1-\alpha}$, $\mathbf{B}_{1-\alpha}$ and $\mathbf{c}_{1-\alpha}$ respectively.
2. Replace Eq. (15) by the following system:

$$x(t) + y(t) = f(t)\,, \tag{18a}$$

$$\mathbf{A}_{1-\alpha}\dot{\mathbf{a}} + \mathbf{B}_{1-\alpha}\mathbf{a} = \mathbf{c}_{1-\alpha}\,y(t) \tag{18b}$$

and

$$x(t) - \frac{1}{\Gamma(\alpha)\,\Gamma(2-\alpha)}\mathbf{c}_{1-\alpha}^T\mathbf{a} = 0\,. \tag{18c}$$

Here, Eq. (18) is a set of differential algebraic equations (DAEs). By Eqs. (16) and (17), Eq. (18c) can be rewritten as

$$x(t) - \mathrm{D}^{-\alpha} y(t) = 0$$

or

$$\mathrm{D}^{\alpha} x(t) = y(t)\,, \quad \text{provided} \quad \mathrm{D}^{\alpha}\mathrm{D}^{-\alpha} y(t) = y(t)\,. \tag{19}$$

It happens that $\mathrm{D}^{\alpha}\mathrm{D}^{-\alpha} y(t) = y(t)$ (see [1] for details).

We used $\alpha = 1/2$ and $1/3$ for numerical simulations. The index of the DAEs here (see [9] for details) is one. For both values of α, DAEs (18) are solved using Matlab's built-in function "ode23t" for $f(t) = 1$. Consistent initial conditions are calculated as $x(0) = 0\,,\ \mathbf{a}(0) = \mathbf{0}$ and $y(0) = 1$; a guess for corresponding initial slopes, which is an optional input to "ode23t," is $\dot{x}(0) = 0\,,\ \dot{\mathbf{a}}(0) = \mathbf{A}_{1-\alpha}^{-1}\mathbf{c}_{1-\alpha}$ and $\dot{y}(0) = 0$. Results obtained will be presented later.

4.2 Analytical solutions

The solution of Eq. (15) can be obtained using Laplace transforms. For $\alpha = 1/2$, MAPLE gives

$$x(t) = -e^{t}\left(\operatorname{erfc}\left(\sqrt{t}\right) - e^{-t}\right)\,. \tag{20}$$

Since we were unable to analytically invert the Laplace transform using MAPLE for $\alpha = 1/3$, we present a series solution below, along the lines of our previous series solutions (this solution is not new, and will be familiar to readers who know about Mittag-Leffler functions).

The Laplace transform of the solution to Eq. (15) for $\alpha = 1/3$ is given by

$$X(s) = \frac{1}{s(1+s^{1/3})} = \frac{[1-(-s^{-1/3})]^{-1}}{s^{4/3}}\,. \tag{21}$$

On expanding the numerator above (assuming $|s| > 1$) and simplifying, we get

$$X(s) = \sum_{n=4}^{\infty} \frac{(-1)^n}{s^{n/3}}\,. \tag{22}$$

The above series is absolutely convergent for $|s| > 1$. Inverting gives

$$x(t) = \sum_{n=4}^{\infty} \frac{(-1)^n\, t^{n/3-1}}{\Gamma(n/3)}\,. \tag{23}$$

4.3 Results

Numerical results are shown in Fig. 4. The Galerkin approximation matches the exact solutions well in both cases. The sum in Eq. (23) is taken upto the $\mathcal{O}(t^{150})$ term (fewer may have sufficed).

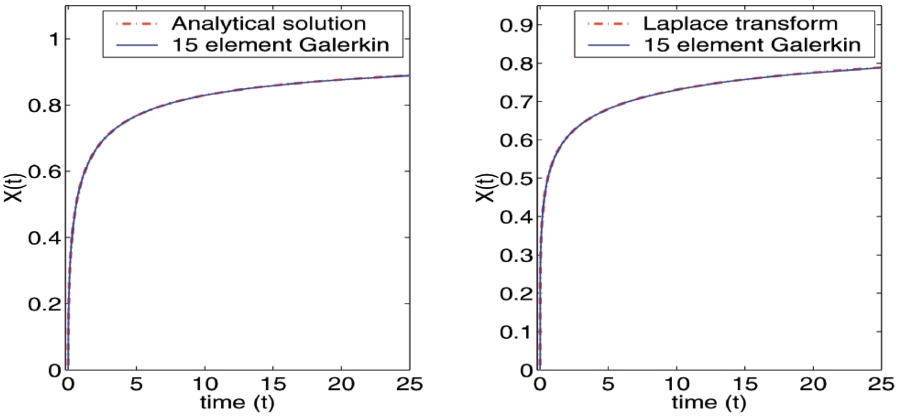

Fig. 4. Comparison between analytical and 15-element Galerkin approximation solutions. Left: $\alpha = 1/2$. Right: $\alpha = 1/3$. For $\alpha = 1/3$, the series is summed up to $\mathcal{O}(t^{150})$.

5 Discussion and Conclusions

We have identified three classes of FDEs that are amenable to solution using a new Galerkin approximation for the fractional-order derivative, that was developed recently in other work [4]. To showcase the effectiveness of the approximation technique, we have used linear FDEs, which could also be solved analytically (if only in the form of power series). However, more general and nonlinear problems which are impossible to solve analytically are also expected to be equally effectively solved using this approximation technique.

The approximation technique used here, as discussed in [4], involves numerical evaluation of certain matrices. For approximation of a derivative of a given fractional order between 0 and 1, and with a given number of shape functions in the Galerkin approximation, these matrices need be calculated only once. They can then be used in any problem where a derivative of the same order appears. A MAPLE file which calculates these matrices is available on the web. We hope that this technique will serve to provide a simple, reliable, and routine method of numerically solving FDEs in a wide range of applications.

References

1. Samko SG, Kilbas AA, Marichev OI (1993) *Fractional Integrals and Derivatives: Theory and Applications*. Gordon and Breach, Amsterdam.
2. Oldham KB (1974) *The Fractional Calculus*. Academic Press, New York.
3. Koh CG, Kelly JM (1990) Earthquake Eng. Struc. Dyn., 19:229–241.
4. Singh SJ, Chatterjee A (2005) *Nonlinear Dynamics* (in press).
5. http://www.geocities.com/dynamics_iisc/SystemMatrices.zip
6. Basset AB (1910) Quart. J. Math. 41:369–381.
7. Mainardi F, Pironi P, Tampieri F (1995) On a Generalization of Basset Problem via Fractional Calculus, in: Proceedings CANCAM 95.
8. Mainardi F (1996) Chaos, Solitons Fractals, 7(9):1461–1477.
9. Hairer E, Wanner G (1991) *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems*. Springer, Berlin.

ENUMERATION OF THE REAL ZEROS OF THE MITTAG-LEFFLER FUNCTION $E_\alpha(z)$, $1 < \alpha < 2$

John W. Hanneken[1], David M. Vaught[2], and B. N. Narahari Achar[3]

[1] University of Memphis, Physics Department, Memphis, TN 38152; Tel: 901.678.2417, Fax: 901.678.4733, E-mail: jhannekn@memphis.edu

[2] University of Memphis, Physics Department, Memphis, TN 38152; DHawkeye10@aol.com

[3] University of Memphis, Physics Department, Memphis, TN 38152; Tel: 901.678.3122, Fax: 901.678.4733, E-mail: nachar@memphis.edu

Abstract

The Mittag-Leffler function $E_\alpha(z)$, which is a generalization of the exponential function, arises frequently in the solutions of physical problems described by differential and/or integral equations of fractional order. Consequently, the zeros of $E_\alpha(z)$ and their distribution are of fundamental importance and play a significant role in the dynamic solutions. The Mittag- Leffler function $E_\alpha(z)$ is known to have a finite number of real zeros in the range $1 < \alpha < 2$ which is applicable for many physical problems. What has not been known is the exact number of real zeros of $E_\alpha(z)$ for a given value of α in this range. An iteration formula is derived for calculating the number of real zeros of $E_\alpha(z)$ for any value of α in the range $1 < \alpha < 2$ and some specific results are tabulated.

Keywords

Mittag-Leffler functions, zeros, fractional calculus.

1 Introduction

The single parameter Mittag-Leffler function $E_\alpha(z)$ is defined over the entire complex plane by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)} \qquad \alpha > 0,\ z \in C \tag{1}$$

J. Sabatier et al. (eds.), Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, 15–26.

and is named after Mittag-Leffler who introduced it in 1903 [1,2]. The two parameter generalized Mittag-Leffler function, which was introduced later [3,4], is also defined over the entire complex plane, and is given by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \qquad \alpha, \beta > 0, z \in C \qquad (2)$$

It may be noted that when $\beta = 1$, $E_{\alpha,1}(z) = E_\alpha(z)$. Properties of the Mittag-Leffler functions have been summarized in several references [5–7]. Although others have considered complex α [8,9] and complex β [10], the present work is restricted to real α and β. The Mittag-Leffler functions are natural extensions of the exponential function and solutions of fractional-order differential equations are often expressed in terms of Mittag-Leffer functions in much the same way that solutions of many integer order differential equations may be expressed in terms of exponential functions. Consequently, the zeros of $E_{\alpha,1}(z)$, which play a significant role in the dynamic solutions, are of intrinsic interest.

Except for the special case of $\alpha = 1$, in general $E_{\alpha,1}(z)$ has an infinite number of zeros [11,12] and all complex zeros of $E_\alpha(z)$ appear as pairs of complex conjugates [13]. To facilitate the discussion of the zeros, the domain of α values can be conveniently divided into four ranges: $0 < \alpha < 1$, $\alpha = 1$, $1 < \alpha < 2$, and $\alpha \geq 2$ based on the nature of the zeros, but $E_{\alpha,1}(z)$ and its zeros exhibit similar properties within each range. For $0 < \alpha < 1$, $E_\alpha(z)$ has no real zeros [14] and thus must have an infinite number of complex zeros. For $\alpha = 1$, $E_{1,1}(z)$ can be written as $E_1(z) = \exp(z)$, which has no zeros real or complex. For $1 < \alpha < 2$, $E_\alpha(z)$ has a finite number of zeros on the negative real axis [5,8,9,11,14] and must in addition have an infinite number of complex zeros [11,15]. For $\alpha \geq 2$, $E_\alpha(z)$ has an infinite number of zeros that are real, negative, and simple and no complex zeros [8–10,16]. Note that regardless of the range of α, $E_{\alpha,1}(z)$ has no positive real zeros. Thus, for convenience, the variable x will be used to represent a positive real number so that $E_{\alpha,1}(-x)$ clearly has a negative real argument. Real zeros occur only in the ranges $1 < \alpha < 2$, and $\alpha \geq 2$. The range $1 < \alpha < 2$ is the range for which the least is known and yet is quite relevant for many physical problems [6,17]. The objective of this paper is to determine the exact number of real zeros for $E_{\alpha,1}(-x)$ for arbitrary α in the range $1 < \alpha < 2$. These results will be discussed later in connection with an asymptotic formula for the number of real zeros valid near $\alpha = 2$ [14]. The first requirement is a discussion of how to calculate $E_{\alpha,1}(-x)$ accurately.

2 Numerical Evaluation of $E_{\alpha,\beta}(-x)$

Numerical values of $E_{\alpha,\beta}(z)$ are easily calculated using the power series given in Eq. (2) when the argument z is not too large. However, for large arguments this method is impractical because of the extremely slow convergence of the series. Instead, use will be made of the representation of $E_{\alpha,\beta}(z)$ as a Laplace inversion integral [6]

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{Br} e^{s} \frac{s^{\alpha-\beta}}{s^{\alpha} - z} ds \tag{3}$$

where Br denotes the Bromwich path. Using standard techniques in the theory of calculus of residues [18], $E_{\alpha,\beta}(\lambda z^{\alpha})$ can be decomposed into two parts [14]. For the special case of a negative real argument, the result is given by:

$$E_{\alpha,\beta}(-x) = g_{\alpha,\beta}(-x) + f_{\alpha,\beta}(-x) \tag{4a}$$

$$g_{\alpha,\beta}(-x) = \frac{\frac{2}{\alpha} \exp\left[x^{1/\alpha} \cos\left(\frac{\pi}{\alpha}\right)\right] \cos\left[\frac{\pi(1-\beta)}{\alpha} + x^{1/\alpha} \sin\left(\frac{\pi}{\alpha}\right)\right]}{x^{(\beta-1)/\alpha}} \tag{4b}$$

$$f_{\alpha,\beta}(-x) = \frac{\frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\exp\left(-x^{1/\alpha} r\right) r^{\alpha-\beta} \left[r^{\alpha} \sin(\pi\beta) + \sin[\pi(\beta-\alpha)]\right]}{r^{2\alpha} + 2r^{\alpha}\cos(\pi\alpha) + 1} \right\} dr}{x^{(\beta-1)/\alpha}} \tag{4c}$$

where $\alpha + 1 > \beta$ and for $\alpha < 1$, $g_{\alpha,\beta}(-x) = 0$. For the special case of $\beta = 1$ Eqs. (4a–c) reduce to

$$E_{\alpha,1}(-x) = g_{\alpha,1}(-x) + f_{\alpha,1}(-x) \tag{5a}$$

$$g_{\alpha,1}(-x) = \frac{2}{\alpha} \exp\left[x^{\frac{1}{\alpha}} \cos\left(\frac{\pi}{\alpha}\right)\right] \cos\left[\frac{\pi(1-\beta)}{\alpha} + x^{\frac{1}{\alpha}} \sin\left(\frac{\pi}{\alpha}\right)\right] \tag{5b}$$

$$f_{\alpha,1}(-x) = \frac{1}{\pi} \int_0^{\infty} \left\{ \frac{\exp\left(-x^{1/\alpha} r\right) r^{\alpha-1} \sin(\pi\alpha)}{r^{2\alpha} + 2r^{\alpha}\cos(\pi\alpha) + 1} \right\} dr \tag{5c}$$

Numerical values of the Mittag-Leffler function $E_{\alpha,1}(-x)$ were computed primarily from Eqs. (5a–c) using Mathematica[19] with the integration performed using the built-in function NIntegrate. The values computed using Eqs. (5a–c) were in agreement to better than 40 significant digits with the values calculated directly from Eq. (1) for small values of the argument. As an alternative to the numerical integration required in Eq. (5c), $f_{\alpha,1}(-x)$ can be written in an asymptotic infinite series as follows[14]

$$f_{\alpha,1}(-x) = \frac{1}{x\Gamma(1-\alpha)} - \frac{1}{x^2\Gamma(1-2\alpha)} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{x^n\Gamma(1-n\alpha)} \tag{6}$$

This series is particularly useful when both x and the gamma function are large and the series converges very quickly. The value of the gamma function approaches infinity as its argument approaches a negative integer. Thus, Eq. (6) is most useful for α close to 2 and x large.

3 Zeros of $E_{\alpha,1}(-x)$ of Multiplicity 2

Critical to the derivation of a formula for the number of real zeros is an understanding of the nature of the zeros and this is best done by examining the graphs of $E_{\alpha,1}(-x)$. For $1 < \alpha < 2$, $E_{\alpha,1}(0) = 1$ and for large x values $E_{\alpha,1}(-x)$ is negative and asymptotically approaches zero governed predominately by $f_{\alpha,1}(-x)$, Eq. (5c), with the exponentially decreasing oscillations of $g_{\alpha,1}(-x)$, Eq. (5b), superimposed. The fact that the curves of $E_{\alpha,1}(-x)$ are positive at $x = 0$ and ultimately become negative for large x implies that $E_{\alpha,1}(-x)$ can only cross the x-axis an odd number of times[5]. This is illustrated in the plot of $E_{\alpha,1}(-x)$ for $\alpha = 1.3$ shown in Fig. 1.

The curve exhibits only one zero at $x \cong 2.293$ and for larger x remains negative with the superimposed oscillation of $g_{\alpha,1}(-x)$ imperceptible on this scale. The rate of exponential decay of $g_{\alpha,1}(-x)$ is determined by the exponent $x^{1/\alpha}\cos(\pi/\alpha)$, the $\cos(\pi/\alpha)$ being negative in the range $1 < \alpha < 2$. As α increases this exponent decreases resulting in larger amplitude oscillations. This is illustrated in the graph of $E_{\alpha,1}(-x)$ for $\alpha = 1.5$ also shown in Fig. 1. The larger amplitude oscillations of $g_{\alpha,1}(-x)$ give rise to a relative maximum at $x \cong 17.472$ extending above the x-axis and yielding two more zeros at $x \cong 13.765$ and $x \cong 24.243$ in addition to the one at $x \cong 2.110$.

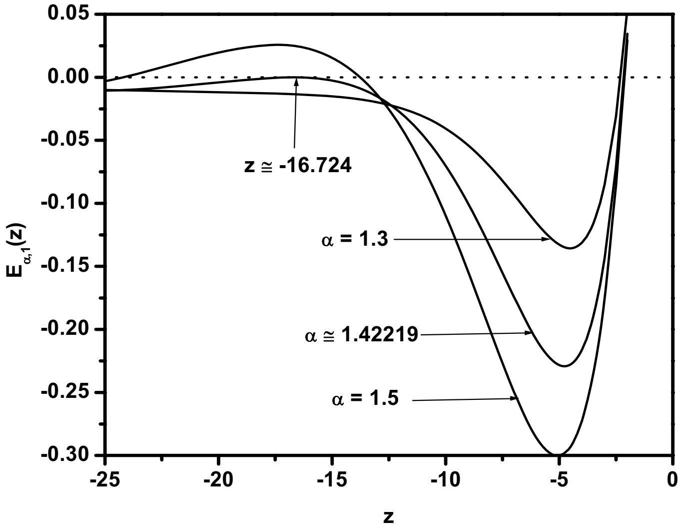

Fig. 1. Plots of $E_{\alpha,1}(-x)$ for various values of α.

Clearly, there is a value of α between $\alpha = 1.3$ and $\alpha = 1.5$ for which the curve of $E_{\alpha,1}(-x)$ is exactly tangent to the x-axis. This is illustrated in the graph of $E_{\alpha,1}(-x)$ for $\alpha \cong 1.422190690801$ also shown in Fig. 1. This curve has a zero at $x \cong 2.145$ and is tangential at $x \cong 16.724$ where it has a zero of multiplicity of 2 still yielding an odd total number of zeros. It may be noted that for $\alpha = 1.3$ the curve crosses the x-axis only once yielding one zero and for $\alpha = 1.5$ the curve crosses the x-axis 3 times yielding 3 zeros. Thus, the value of $\alpha \cong 1.422190690801$ separates the range of α values where $E_{\alpha,1}(-x)$ has only one zero from the range where $E_{\alpha,1}(-x)$ has three zeros. The next larger value of α where the curve is tangent to the x-axis is at $\alpha \cong 1.5718839229424$ where $E_{\alpha,1}(-x)$ has five zeros. The iteration formula for the number of real zeros described in the next section depends essentially on the existence of these values of α where the curve of $E_{\alpha,1}(-x)$ is tangent to the x-axis and for which one of the zeros has a multiplicity of 2. The first 5,641 of these α values where the curve of $E_{\alpha,1}(-x)$ is tangent to the x-axis have been numerically determined. A few selected values are given in Table 1. These values will be most useful in section 5 to establish ranges of reliability for the iteration results for $\alpha < 1.999$. In reading Table 1, for example, α_5 is the lowest value of α for which $E_{\alpha,1}(-x)$ has 5 zeros and $E_{\alpha,1}(-x)$ is tangent to the x-axis. Thus, $E_{\alpha,1}(-x)$ has 5 zeros for $\alpha_5 \le \alpha < \alpha_7$, 7 zeros for $\alpha_7 \le \alpha < \alpha_9$, 9 zeros for $\alpha_9 \le \alpha < \alpha_{11}$, … , 11281 zeros for $\alpha_{11281} \le \alpha < \alpha_{11283}$.

Table 1. Values of α (truncated) at which $E_{\alpha,1}(-x)$ is tangent to the x-axis

n	α_n	n	α_n
3	1.422190690801	11217	1.998994787610
5	1.571883922942	11219	1.998994948054
7	1.649068237342	11221	1.998995108443
9	1.698516223760	11223	1.998995268780
11	1.733693032768	11225	1.998995429062
13	1.760338811725	11227	1.998995589290
15	1.781392651685	11229	1.998995749465
17	1.798543344750	11231	1.998995909586
19	1.812841949070	11233	1.998996069654
21	1.824982270661	11235	1.998996229667
23	1.835443517675	11237	1.998996389627
25	1.844568817828	11239	1.998996549534
27	1.852611186687	11241	1.998996709387
29	1.859761810886	11243	1.998996869186
31	1.866168176867	11245	1.998997028932
33	1.871946096560	11247	1.998997188625
35	1.877187921171	11249	1.998997348263
37	1.881968294552	11251	1.998997507849
39	1.886348272721	11253	1.998997667381
41	1.890378331112	11255	1.998997826860
43	1.894100597857	11257	1.998997986285
45	1.897550537931	11259	1.998998145657
47	1.900758240821	11261	1.998998304976
49	1.903749417395	11263	1.998998464241
51	1.906546180470	11265	1.998998623453
53	1.909167662339	11267	1.998998782612
55	1.911630507999	11269	1.998998941718
57	1.913949272538	11271	1.998999100770
59	1.916136743903	11273	1.998999259770
61	1.918204207029	11275	1.998999418716
63	1.920161661487	11277	1.998999577609
65	1.922018001994	11279	1.998999736450
67	1.923781169033	11281	1.998999895237
69	1.925458275243	11283	1.999000053971