# Advanced Methods of Structural Analysis

Igor A. Karnovsky · Olga Lebed

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*Dedicated to*
*Tamara L'vovna Gorodetsky*

# Preface

Theory of the engineering structures is a fundamental science. Statements and methods of this science are widely used in different fields of engineering. Among them are the civil engineering, ship-building, aircraft, robotics, space structures, as well as numerous structures of special types and purposes – bridges, towers, etc. In recent years, even micromechanical devices become objects of structural analysis.

Theory of the engineering structures is alive and is a very vigorous science. This theory offers an engineer-designer a vast collection of classical methods of analysis of various types of structures. These methods contain in-depth fundamental ideas and, at the present time, they are developed with sufficient completeness and commonness, aligned in a well-composed system of conceptions, procedures, and algorithms, use modern mathematical techniques and are brought to elegant simplicity and perfection.

We now live in a computerized world. A role and influence of modern engineering software for analysis of structures cannot be overestimated. The modern computer programs allow providing different types of analysis for any sophisticated structure. As this takes place, what is the role of classical theory of structures with its in-depth ideas, prominent conceptions, methods, theorems, and principles? Knowing classical methods of Structural Analysis is necessary for any practical engineer. An engineer cannot rely only on the results provided by a computer. Computer is a great help in modeling different situations and speeding up the process of calculations, but it is the sole responsibility of an engineer to check the results obtained by a computer. If users of computer engineering software do not have sufficient knowledge of fundamentals of structural analysis and of understanding of physical theories and principal properties of structures, then he/she cannot check obtained numerical results and their correspondence to an adopted design diagram, as well as explain results obtained by a computer. Computer programs "... can make a good engineer better, but it can make a poor engineer more dangerous" (Cook R.D, Malkus D.S, Plesha M.E (1989) Concepts and applications of finite element analysis, 3rd edn. Wiley, New York). Only the knowledge of fundamental theory of structures allows to estimate and analyze numerical data obtained from a computer; predict the behavior of a structure as a result of changing a design diagram and parameters; design a structure which satisfies certain requirements; perform serious scientific analysis; and make valid theoretical generalizations. No matter how sophisticated the structural model is, no matter how effective the numerical algorithms are, no matter how powerful the computers are that implement these algorithms, it is the engineer who analyzes the end result produced from these algorithms. Only an individual who has a deep knowledge and understanding of the structural model and analysis techniques can produce a qualitative analysis.

In 1970, one of the authors of this book was a professor at a structural engineering university in Ukraine. At that time computers were started to be implemented in all fields of science, structural analysis being one of them. We, the professors and instructors, were facing a serious methodical dilemma: given the new technologies, how to properly teach the students? Would we first give students a strong basis in classical structural analysis and then introduce them to the related software, or would we directly dive into the software after giving the student a relatively insignificant introduction to classical analysis. We did not know the optimal way for solving this problem. On this subject we have conducted seminars and discussions on a regular basis. We have used these two main teaching models, and many different variations of them. The result was somewhat surprising. The students who were first given a strong foundation in structural analysis quickly learned how to use the computer software, and were able to give a good qualitative analysis of the results. The students who were given a brief introduction to structural analysis and a strong emphasis on the computer software, at the end were not able to provide qualitative results of the analysis. The interesting thing is that the students themselves were criticizing the later teaching strategy.

Therefore, our vision of teaching structural analysis is as follows: on the first step, it is necessary to learn analytical methods, perform detailed analysis of different structures *by hand* in order to feel the behavior of structures, and correlate their behavior with obtained results; the second step is a computer application of engineering software.

Authors wrote the book on the basis of their many years of experience of teaching the Structural Analysis at the universities for graduate and postgraduate students as well as on the basis of their experience in consulting companies.

This book is written for students of universities and colleges pursuing Civil or Structural Engineering Programs, instructors of Structural Analysis, and engineers and designers of different structures of modern engineering.

The objective of the book is to help a reader to develop an understanding of the ideas and methods of structural analysis and to teach a reader to estimate and explain numerical results obtained by hand; this is a fundamental stone for preparation of reader for numerical analysis of structures and for use of engineering software with full understanding.

The textbook offers the reader the fundamental theoretical concepts of Structural Analysis, classical analytical methods, algorithms of their application, comparison of different methods, and a vast collection of distinctive problems with their detailed solution, explanation, analysis, and discussion of results; many of the problems have a complex character. Considered examples demonstrate features of structures, their behavior, and peculiarities of applied methods. Solution of all the problems is brought to final formula or number.

Analyses of the following structures are considered: statically determinate and indeterminate multispan beams, arches, trusses, and frames. These structures are subjected to fixed and moving loads, changes of temperature, settlement of supports, and errors of fabrication. Also the cables are considered in detail.

In many cases, same structure under different external actions is analyzed. It allows the reader to be concentrated on one design diagram and perform complex analysis of behavior of a structure.

In many cases, same structure is analyzed by different methods or by one method in different forms (for example, Displacement method in canonical, and matrix forms). It allows to perform comparison analysis of applied methods and see advantages and disadvantages of different methods.

## Distribution of Material in the Book

This book contains introduction, three parts (14 chapters), and appendix.

*Introduction* provides the subject and purposes of Structural Analysis, principal concepts, assumptions, and fundamental approaches.

*Part 1* (Chaps. 1–6) is devoted to analysis of statically determinate structures. Among them are multispan beams, arches, trusses, cables, and frames. Construction of influence lines and their application are discussed with great details. Also this part contains analytical methods of computation of displacement of deformable structures, subjected to different actions. Among them are variety loads, change of temperature, and settlements of supports.

*Part 2* (Chaps. 7–11) is focused on analysis of statically indeterminate structures using the fundamental methods. Among them are the force and displacement methods (both methods are presented in canonical form), as well as the mixed method. Also the influence line method (on the basis of force and displacement methods) is presented. Analysis of continuous beams, arches, trusses, and frames is considered in detail.

Chapter 11 is devoted to matrix stiffness method which is realized in the modern engineering software. Usually, the physical meaning of all matrix procedures presents serious difficulties for students. Comparison of numerical procedures obtained by canonical equations and their matrix presentations, which are applied to the same structure, allows trace and understands meaning of each stage of matrix analysis. This method is applied for fixed loads, settlement of supports, temperature changes, and construction of influence lines.

*Part 3* (Chaps. 12–14) contains three important topics of structural analysis. They are plastic behavior of structures, stability of elastic structures with finite and infinite number of degrees of freedom, including analysis of structures on the basis of the deformable design diagram ($P$–$\Delta$ analysis), and the free vibration analysis.

Each chapter contains problems for self-study. Answers are presented to all problems.

*Appendix* contains the fundamental tabulated data.

Authors will appreciate comments and suggestions to improve the current edition. All constructive criticism will be accepted with gratitude.

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# Contents

# Introduction

The subject and purposes of the Theory of Structures in the broad sense is the branch of applied engineering that deals with the methods of analysis of structures of different types and purpose subjected to arbitrary types of external exposures. Analysis of a structure implies its investigation from the viewpoint of its strength, stiffness, stability, and vibration.

The purpose of analysis of a structure from a viewpoint of its *strength* is determining internal forces, which arise in all members of a structure as a result of external exposures. These internal forces produce stresses; the *strength* of each member of a structure will be provided if their stresses are less than or equal to permissible ones.

The purpose of analysis of a structure from a viewpoint of its *stiffness* is determination of the displacements of specified points of a structure as a result of external exposures. The *stiffness* of a structure will be provided if its *displacements* are less than or equal to permissible ones.

The purpose of analysis of *stability* of a structure is to determine the *loads* on a structure, which leads to the appearance of *new forms of equilibrium*. These forms of equilibrium usually lead to collapse of a structure and corresponding loads are referred as *critical* ones. The stability of a structure will be provided if acting loads are less than the critical ones.

The purpose of analysis of a structure from a viewpoint of its vibration is to determine the frequencies and corresponding shapes of the vibration. These data are necessary for analysis of the forced vibration caused by arbitrary loads.

The Theory of Structures is fundamental science and presents the rigorous treatment for each group of analysis. In special cases, all results may be obtained in the close analytical form. In other cases, the required results may be obtained only numerically. However, in all cases algorithms for analysis are well defined.

The part of the Theory of Structures which allows obtaining the analytical results is called the classical Structural Analysis. In the narrow sense, the purpose of the classical Structural Analysis is to establish relationships between external exposures and corresponding internal forces and displacements.

## Types of Analysis

Analysis of any structure may be performed based on some assumptions. These assumptions reflect the purpose and features of the structure, type of loads and operating conditions, properties of materials, etc. In whole, structural analysis may be divided into three large principal groups. They are static analysis, stability, and vibration analysis.

Static analysis presumes that the loads act without any dynamical effects. Moving loads imply that only the position of the load is variable. Static analysis combines the analysis of a structure from a viewpoint of its strength and stiffness.

*Static linear analysis (SLA)*. The purpose of this analysis is to determine the internal forces and displacements due to time-independent loading conditions. This analysis is based on following conditions:

1. Material of a structure obeys Hook's law.
2. Displacements of a structure are small.
3. All constraints are two-sided – it means that if constraint prevents displacement in some direction then this constraint prevents displacement in the opposite direction as well.
4. Parameters of a structure do not change under loading.

*Nonlinear static analysis*. The purpose of this analysis is to determine the displacements and internal forces due to time-independent loading conditions, as if a structure is nonlinear. There are different types of nonlinearities. They are physical (material of a structure does not obey Hook's law), geometrical (displacements of a structure are large), structural (structure with gap or constraints are one-sided, etc.), and mixed nonlinearity.

Stability analysis deals with structures which are subjected to compressed time-independent forces.

*Buckling analysis*. The purpose of this analysis is to determine the critical load (or critical loads factor) and corresponding buckling mode shapes.

*P-delta analysis*. For tall and flexible structures, the transversal displacements may become significant. Therefore we should take into account the additional bending moments due by axial compressed loads $P$ on the displacements caused by the lateral loads. In this case, we say that a structural analysis is performed on the basis of the *deformed* design diagram.

Dynamical analysis means that the structures are subjected to time-dependent loads, the shock and seismic loads, as well as moving loads with taking into account the dynamical effects.

*Free-vibration analysis (FVA)*. The purpose of this analysis is to determine the natural frequencies (eigenvalues) and corresponding mode shapes (eigenfunctions) of vibration. This information is necessary for dynamical analysis of any structure subjected to arbitrary dynamic load, especially for seismic analysis. FVA may be considered for linear and nonlinear structures.

*Stressed free-vibration analysis*. The purpose of this analysis is to determine the eigenvalues and corresponding eigenfunctions of a structure, which is subjected to additional axial time-independent forces.

*Time-history analysis*. The purpose of this analysis is to determine the response of a structure, which is subjected to arbitrarily time-varying loads.

In this book, the primary emphasis will be done upon the static linear analysis of plane structures. Also the reader will be familiar with problems of stability in structural analysis and free-vibration analysis as well as some special cases of analysis will be briefly discussed.

## Fundamental Assumptions of Structural Analysis

Analysis of structures that is based on the following assumptions is called the elastic analysis.

1. Material of the structure is continuous and absolutely elastic.
2. Relationship between stress and strain is linear.
3. Deformations of a structure, caused by applied loads, are small and do not change original design diagram.
4. Superposition principle is applicable.

Superposition principle means that any factor, such as reaction, displacement, etc., caused by different loads which act simultaneously, are equal to the algebraic or geometrical sum of this factor due to each load separately. For example, reaction of a movable support under any loads has one fixed direction. So the reaction of this support due to different loads equals to the *algebraic* sum of reactions due to action of each load separately. Vector of total reaction for a pinned support in case of any loads has different directions, so the reaction of pinned support due to different loads equals to the *geometrical* sum of reactions, due to action of each load separately.

## Fundamental Approaches of Structural Analysis

There are two fundamental approaches to the analysis of any structure. The first approach is related to analysis of a structure subjected to given fixed loads and is called the fixed loads approach. The results of this analysis are diagrams, which show a distribution of internal forces (bending moment, shear, and axial forces) and deflection for the entire structure due to the given fixed loads. These diagrams indicate the most unfavorable point (or member) of a structure under the given fixed loads. The reader should be familiar with this approach from the course of mechanics of material.

The second approach assumes that a structure is subjected to unit concentrated moving load only. This load is not a real one but imaginary. The results of the second approach are graphs called the influence lines. Influence lines are plotted for reactions, internal forces, etc. Internal forces diagrams and influence lines have a fundamental difference. Each influence line shows distribution of internal forces in the *one specified section* of a structure due to location of imaginary unit moving load only. These influence lines indicate the point of a structure where a load should be placed in order to reach a maximum (or minimum) value of the function under consideration at the specified section. It is very important that the influence lines may be also used for analysis of structure subjected to any fixed loads. Moreover, in many cases they turn out to be a very effective tool of analysis.

Influence lines method presents the higher level of analysis of a structure, than the fixed load approach. Good knowledge of influence lines approaches an immeasurable increase in understanding of behavior of structure. Analyst, who combines both approaches for analysis of a structure in engineering practice, is capable to perform a complex analysis of its behavior.

Both approaches do not exclude each other. In contrast, in practical analysis both approaches complement each other. Therefore, learning these approaches to the analysis of a structure will be provided in parallel way. This textbook presents sufficiently full consideration of influence lines for different types of statically determinate and indeterminate structures, such as beams, arches, frames, and trusses.

# Part I
# Statically Determinate Structures

# Chapter 1
# Kinematical Analysis of Structures

Kinematical analysis of a structure is necessary for evaluation of ability of the structure to resist external load. Kinematical analysis is based on the concept of *rigid disc*, which is an unchangeable (or rigid) part of a structure. Rigid discs may be separate members of a structure, such as straight members, curvilinear, polygonal (Fig. 1.1), as well as their special combination.

Any structure consists of separate rigid discs. Two rigid discs may be connected by means of link, hinge, and fixed joint. These types of connections and their static and kinematical characteristics are presented in Table 1.1.

The members of a structure may be connected together by a hinge in various ways. Types of connection are chosen and justified by an engineer as follows:

1. *Simple hinge*. One hinge connects *two* elements in the joint.
2. *Multiple hinge*. One hinge connects *three or more* elements in the joint. The multiple hinge is equivalent to $n$–1 simple hinges, where $n$ is a number of members connected in the joint. Hinged joints can transmit axial and shear forces from one part of the structure to the other; the bending moment at the hinge joint is zero.

## 1.1 Classification of Structures by Kinematical Viewpoint

All structures may be classified as follows:

- *Geometrically unchangeable structure*. For this type of structure, any distortion of the structure occurs only with deformation of its members. It means that this type of structure with absolutely rigid members cannot change its form. The simplest geometrically unchangeable structure is triangle, which contains the pin-joined members (Fig. 1.2a).
- *Geometrically changeable structure*. For this type of structure, any finite distortion of the structure occurs without deformation of its members. The simplest geometrically changeable system is formed as hinged four-bar linkage (Fig. 1.2b, c). In Fig. 1.2c, the fourth bar is presented as ground. In both cases, even if the system would be made with absolutely rigid members, it still can change its form.

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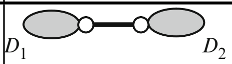
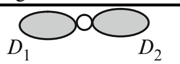
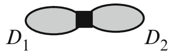
**Fig. 1.1** Types of the rigid discs

**Table 1.1** Types of connections of rigid discs and their characteristics

| Type of connection | Link | Hinge | Fixed joint |
|---|---|---|---|
| Presentation and description of connection |  Rigid discs $D_1$ and $D_2$ are connected by link (rod with hinges at the ends) |  Rigid discs $D_1$ and $D_2$ are connected by hinge |  Rigid discs $D_1$ and $D_2$ are connected by fixed joint |
| Kinematical characteristics | Mutual displacement of both discs *along the link* is zero | Mutual displacements of both discs in *both* horizontal and vertical directions are zeros | *All* mutual displacements of both discs (in horizontal, vertical, and angular directions) are zeros |
| Static characteristics | Connection transmits one force, which prevents mutual displacement along the link | Connection transmits two forces, which prevent mutual displacements in vertical and horizontal directions | Connection transmits two forces, which prevent mutual displacements in vertical and horizontal directions, and moment, which prevents mutual angular displacement |

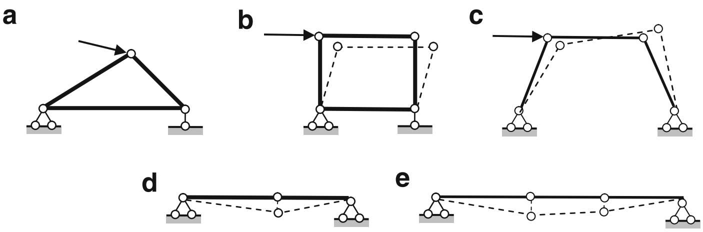



**Fig. 1.2** Types of structure by kinematical viewpoint

It is pertinent to do the following important remark related to terminology. Sometimes terms "stable" and "unstable" are applied for the above-mentioned types of structures. However, the commonly accepted term "stable/unstable" in classical theory of deformable systems is related to concept of critical load, while term "geometrically unchangeable/changeable" is related to way of connection of rigid disks. There is the fundamental difference between kinematical analysis of a structure on the one hand, and analysis of stability of a structure subjected to compressed load, on the other hand. Thus, in kinematical analysis of structures, we will use the term "unchangeable structure" instead of "stable structure," and term "changeable structure" instead of "unstable structure." Stability analysis is considered in Chap. 13.

- *Instantaneously changeable structure*. This system allows infinitesimal relative displacements of its members without their deformation and after that the structure becomes geometrically unchangeable (Fig. 1.2d).
- *Instantaneously rigid structure*. This system allows infinitesimal relative displacements of its members without their deformation and after that the structure becomes geometrically changeable (Fig. 1.2e). The term "instantaneously" is related to initial condition of the structure only.

In structural engineering only geometrically unchangeable structures may be accepted.

## 1.2 Generation of Geometrically Unchangeable Structures

In order to produce a rigid structure in a whole, the rigid discs should be connected in specific way. Let us consider general rules for formation of geometrically unchangeable structures from two and three rigid discs.

If a structure is formed from *two discs*, then their connections may be as follows:

1. Connection by fixed joint (Fig. 1.3a)
2. Connection by hinge $C$ and rod $AB$, if axis of $AB$ does not pass through the hinge $C$ (Fig. 1.3b)
3. Connection by three nonparallel rods. Point of intersection of any two rods presents a fictitious hinge $C'$. In this case, the other rod and fictitious hinge corresponds to Case 2 (Fig. 1.3c)

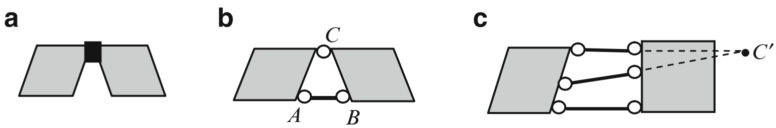



**Fig. 1.3** Geometrically unchangeable structures formed from two discs

If a structure is formed from *three discs*, then their connections may be as follows:

1. Connection in pairs by three hinges $A$, $B$, and $C$, which do not belong to one line (Fig. 1.4a).
2. Connection in pairs by two (or more) concurrent links, if points of their intersections $A$, $B$, and $C$ do not belong to one line (Fig. 1.4b).

Case 1.4a may be presented as shown in Fig. 1.4c: additional joint $A$ is attached to rigid disc $D$ by two links 1 and 2. This case leads to a rigid triangle (Fig. 1.2a), which is a simplest geometrically unchangeable structure. This is the main principle of formation of simplest trusses.

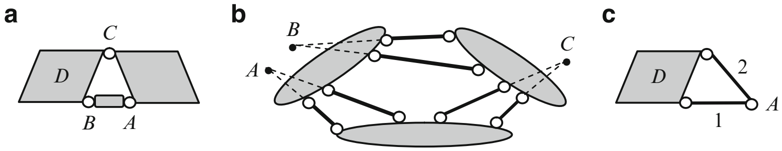



**Fig. 1.4** Geometrically unchangeable structures formed from three discs

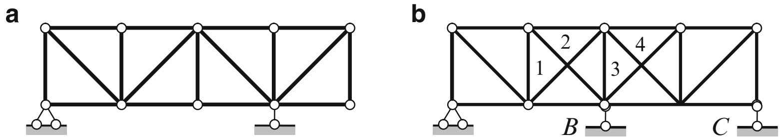



**Fig. 1.5** (**a**) Structure with required constraints; (**b**) structure with redundant constraints

## *Required and Redundant Constraints*

All constraints of any structure are divided into two groups.

*Required constraint* of a structure is such a constraint, eliminations of which change a kinematical characteristic of the structure. It means that the entire unchangeable structure transforms into changeable or instantaneously changeable one, instantaneously changeable transforms into changeable, and changeable transforms into changeable with mobility more by unity. Note that *constraint* assumes not only supports, but elements as well. For example, elimination of *any* member of truss in Fig. 1.5a transforms this structure into changeable one, so for this structure all the elements are required.

*Redundant constraint* of a structure is such a constraint, eliminations of which do not change a kinematical characteristic of the structure. It means that the entire unchangeable structure remains the unchangeable (Fig. 1.5b), changeable structure remains changeable one, and instantaneously changeable remains instantaneously changeable structure. The structure in Fig. 1.5a has no redundant constraints. For structure in Fig. 1.5b, the following constraints may be considered as redundant: 1 or 2, and 3 or 4, and one of the supports – $B$ or $C$, so the total number of redundant constraints is three.

*Constraint replacing*. In case of unchangeable structure, the constraints may be replaced. It means that the *required* constraint may be eliminated and instead of that *another* required constraint should be introduced, or the *redundant* constraint is replaced by *another* redundant constraint. This procedure allows from a given structure to create a lot of other structures.

## 1.3 Analytical Criteria of the Instantaneously Changeable Structures

Kinematical analysis of a structure can also be done using the static equations. The following criteria for instantaneously changeable systems may be used:

(a) Load has finite quantity but the internal forces have infinite values
(b) Load is absent and the internal forces are uncertain (type of 0/0)

Let us discuss these criteria for structure shown in Fig. 1.6a.

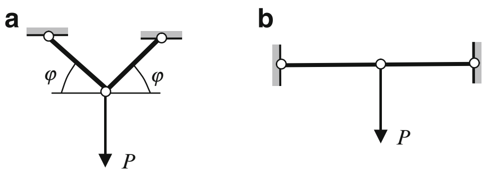



**Fig. 1.6** Kinematical analysis using the static equations

Internal forces in members of the structure are $N = P/(2\sin\varphi)$. If $\varphi = 0$ (Fig. 1.6b), then $N = \infty$. Thus, external load $P$ of finite quantity leads to the internal forces of infinite values. It happens because the system is instantaneously changeable. Indeed, two rigid discs are connected using three hinges located on the one line.

Figure 1.7 presents the design diagram of the truss. This structure is generated from simplest rigid triangle; each next joint is attached to previous rigid disc by two end-hinged links. The structure contains three support constraints, which are necessary minimum for plane structure. However, location of these supports may be wrong. Thorough kinematical analysis of this structure may be performed by static equations.

Reaction $R$ of support may be calculated using equilibrium condition

$$\sum M_A = 0 \rightarrow R \times a - P \times b = 0 \rightarrow R = \frac{Pb}{a}.$$

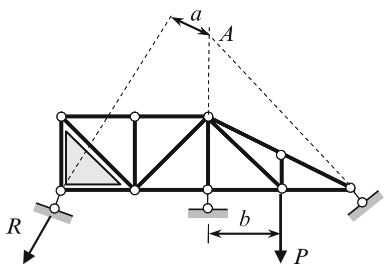



**Fig. 1.7** Kinematical analysis of the truss

1. If $a = 0$, then for any external load $P$ the reaction of the left support is infinitely large ($R = Pb/0$)
2. If $a = 0$ and $P = 0$, then reaction $R$ is uncertain ($R = 0/0$). Thus, if all lines of support constraints are concurrent at one point ($a = 0$), then this case leads to instantaneously changeable system.

Instantaneously changeable systems may occur if two rigid discs of a structure join inappropriately. Two such connections of rigid discs are shown in Fig. 1.8. If a system may be separated into two rigid discs (shown by solid color) by a section cutting three elements, which are parallel (Fig. 1.8a, elements 1, 2, 3), or concurrent in one point (Fig. 1.8b, point $A$, elements $a$, $b$, $c$), then the system is instantaneously changeable one.

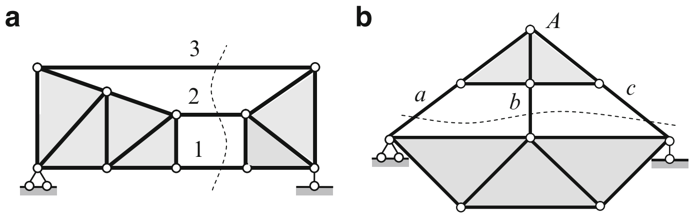



**Fig. 1.8** Instantaneously changeable systems

However, in practice, connection of two rigid discs by two (or more) parallel members may be used in special condition of loading. Figure 1.9 presents the rigid beam (disc $D_1$), which is supported by vertical hinged-end rods 1–3 (disc $D_2$ is a support part). The system may be used if the axial forces in members 1–3 are tensile (Fig. 1.9a). However, the system cannot be used if the axial forces in members 1–3 are compressive (Fig. 1.9b).

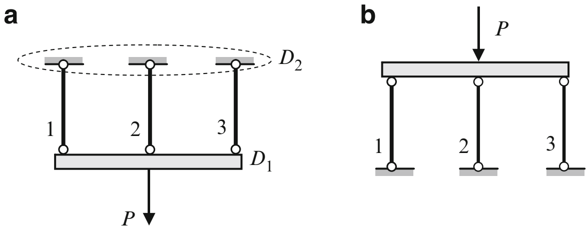



**Fig. 1.9** Geometrically changeable systems

Evolution of the structure caused by changing the type of supports is shown in Fig. 1.10. Constraint $A$ (Fig. 1.10a) prevents two displacements, in vertical and horizontal directions. If one element of the constraint $A$, which prevents horizontal displacement, will be removed, then the structure becomes *geometrically changeable* (Fig. 1.10b), so the removed constraint is the *required* one. In case of any horizontal displacement of the structure, all support constraints $A$, $B$, and $C$ remain parallel to each other.

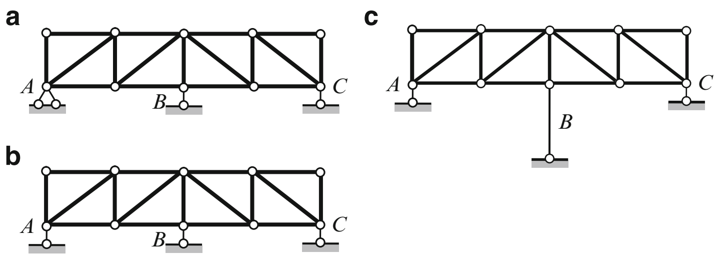



**Fig. 1.10** (**a**) Geometrically unchangeable structure; (**b**) geometrically changeable structure; (**c**) instantaneously changeable system

The next evolution is presented in Fig. 1.10c. If any support, for example supporting element *B* will be longer than other supports, then structure becomes *instantaneously changeable* system. Indeed, in case of any horizontal displacement of the structure, the support constraints will not be parallel any more.

It is worth to mention one more static criterion for instantaneously changeable and geometrically changeable structures: internal forces in *some element* obtained by two different ways are *different*, or in another words, analysis of a structure leads to *contradictory* results. This is shown in the example below.

Design diagram of the truss is presented in Fig. 1.11a: the constraint support *B* is directed along the element *BC*. The system has the necessary minimum number of elements and constraints to be geometrically unchangeable structure. Let us provide more detailed analysis of this structure. Free body diagrams and equilibrium conditions for joints *A*, 1, and *B* are shown in Fig. 1.11b.

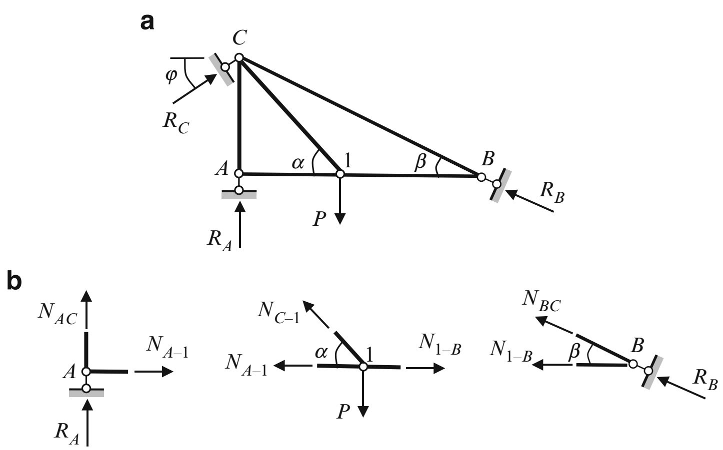



**Fig. 1.11** Kinematical analysis of instantaneously changeable system

Equilibrium conditions for joints $A$, 1, and $B$ and corresponding results are presented below.

Joint $A$. $\sum X = 0 \to N_{A-1} = 0,$

$$\sum Y = 0: \quad N_{C-1}\sin\alpha - P = 0 \to N_{C-1} = \frac{P}{\sin\alpha}$$

Joint 1. $\sum X = 0: \quad N_{1-B} - N_{C-1}\cos\alpha = 0 \to N_{1-B} = N_{C-1}\cos\alpha = P\cot\alpha$

$$\sum Y = 0: \quad N_{BC}\sin\beta + R_B\sin\beta = 0 \to N_{BC} = -R_B$$

Joint $B$. $\sum X = 0: \quad -N_{1-B} - N_{BC}\cos\beta - R_B\cos\beta = 0 \to N_{1-B} = 0$

Two different results for internal force $N_{1-B}$ have been obtained, i.e., $P\cot\alpha$ and zero. This indicates that the system is defective. From mathematical point of view, this happens because the set of equilibrium equations for different parts of the structure is incompatible. From physical point of view, this happens because three support constraints are concurrent in one point $C$ for any angle $\varphi$. Any variation of this angle $\varphi$ remains the system as instantaneously changeable. If constraint at support $C$ will be removed to point $A$, or angle of inclination of support $B$ will be different, then system becomes geometrically unchangeable structure.

Let us show this criterion for system presented in Fig. 1.12. Note that hinges $D$ and $E$ are multiple similarly to hinges $C$ and $F$.

Reaction

$$R_A \to \sum M_B = 0 \to R_A = \frac{Pb}{a+b}.$$

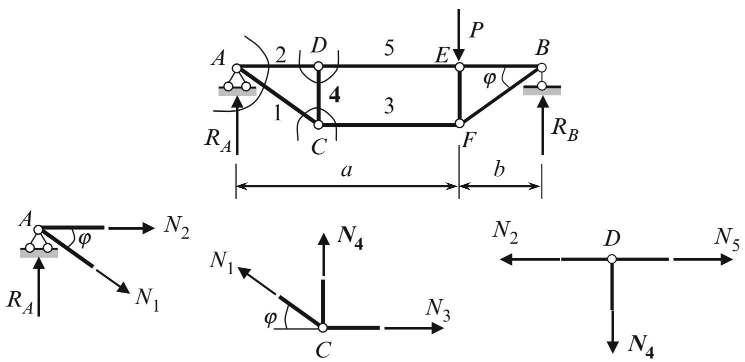



**Fig. 1.12** Kinematical analysis of geometrically changeable system