Statistical Image Processing and Multidimensional Modeling

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Statistical Image Processing and Multidimensional Modeling

Springer

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ISSN 1613-9011
ISBN 978-1-4419-7293-4 e-ISBN 978-1-4419-7294-1
DOI 10.1007/978-1-4419-7294-1
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2010938436

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

As a young professor in 1997 I taught my graduate course in *Stochastic Image Processing* for the first time. Looking back on my rough notes from that time, the course must have been a near impenetrable disaster for the graduate students enrolled, with a long list of errors, confusions, and bad notation.

With every repetition the course improved, with significant changes to notation, content, and flow. However, at the same time that a cohesive, large-scale form of the course took shape, the absence of any textbook covering this material became increasingly apparent. There are countless texts on the subjects of image processing, Kalman filtering, and signal processing, however precious little for random fields or spatial statistics. The few texts that *do* cover Gibbs models or Markov random fields tend to be highly mathematical research monographs, not well suited as a textbook for a graduate course.

More than just a graduate course textbook, this text was developed with the goal of being a useful reference for graduate students working in the areas of image processing, spatial statistics, and random fields. In particular, there are many concepts which are known and documented in the research literature, which are useful for students to understand, but which do not appear in many textbooks. This perception is driven by my own experience as a PhD student, which would have been considerably simplified if I had had a text accessible to me addressing some of the following gaps:

- FFT-based estimation (Section 8.3)
- A nice, simple, clear description of multigrid (Section 9.2.5)
- The inference of dynamic models from cross-statistics (Chapter 10)
- A clear distinction and relationship between squared and unsquared kernels (Chapter 5)

- A graphical summary relating Gibbs and Markov models (Figure 6.11)

To facilitate the use of this textbook and the methods described within it, I am making available online (see page XV) much of the code which I developed for this text. This code, some colour figures, and (hopefully few) errata can be found from this book's home page:

http://ocho.uwaterloo.ca/book

This text has benefited from the work, support, and ideas of a great many people. I owe a debt of gratitude to the countless researchers upon whose work this book is built, and who are listed in the bibliography. Please accept my apologies for any omissions.

The contents of this book are closely aligned with my research interests over the past ten years. Consequently the work of a number of my former graduate students appears in some form in this book, and I would like to recognize the contributions of Simon Alexander, Wesley Campaigne, Gabriel Carballo, Michale Jamieson, Fu Jin, Fakhry Khellah, Ying Liu, and Azadeh Mohebi.

I would like to thank my Springer editor, John Kimmel, who was an enthusiastic supporter of this text, and highly tolerant of my slow pace in writing. Thanks also to copy editor Valerie Greco for her careful examination of grammar and punctuation (and where any remaining errors are mine, not hers). I would like to thank the anonymous reviewers, who read the text thoroughly and who provided exceptionally helpful constructive criticism. I would also like to thank the *non*-anonymous reviewers, friends and students who gave the text another look: Werner Fieguth, Betty Pries, Akshaya Mishra, Alexander Wong, Li Liu, and Gerald Mwangi.

I would like to thank Christoph Garbe and Michael Winckler, the two people who coordinated my stay at the University of Heidelberg, where this text was completed. My thanks to the Deutscher Akademischer Austausch Dienst, the Heidelberg Graduate School, and to the Heidelberg Collaboratory for Image Processing for supporting my visit.

Many thanks and appreciation to Betty for encouraging this project, and to the kids in Appendix C for just being who they are.

Waterloo, Ontario

Paul Fieguth
July, 2010

Contents

List of Examples

List of Code Samples

To make the methods and algorithms described in this book as useful and accessible as possible, a variety of MATLAB scripts, written as part of the development of this textbook, are being made available.

The intent of the scripts is certainly not to be directly applicable to large multidimensional problems. Instead, the code is provided to give the reader some simple examples which actually work, which illustrate the concepts documented in the text, and which hopefully more rapidly lead the reader to be able to do implementations of his or her own.

The code can be found from the text website at

http://ocho.uwaterloo.ca/book

Nomenclature

The following tables of nomenclature are designed to assist the reader in understanding the mathematical language used throughout this text. In the author's opinion this is of considerable value particularly for readers who seek to use the book as a reference and need to be able to understand individual equations or sections without reading an entire chapter for context. Four sets of definitions follow:

1. Basic syntax
2. Mathematical functions
3. Definitions of commonly-used variables
4. Notation for spatial models

Page references are given to provide a few examples of use and some context to the notation, but are in no way intended to be exhaustive.

We limit ourselves here to just defining the notation. For an explanation of algebraic concepts (matrix transpose, eigendecomposition, inverse, etc.) the reader is referred to Appendix A. For an explanation of related statistical concepts (expectation, covariance, etc.), see Appendix B. A brief overview of image processing can be found in Appendix C. Most of the spatial models are explained in Chapters 5 and 6.

Syntax	Definition	Page References
a	scalar, random variable	16 411
$\underline{a}$	column vector, random vector	13 414
a_i	ith element of vector $\underline{a}$	16 414
$\underline{a}_i$	ith vector in a sequence	15 294
a_{ij}	i, jth element of matrix A	299 385
A	matrix	13 383
A^T	matrix transpose	31 385
A^H	matrix Hermitian (complex transpose)	392
A^{-1}	matrix inverse	20 386
$\|A\|$	matrix determinant	63 386 418
$\mathcal{A}$	kernel corresponding to stationary matrix A	142 143 151
$\mathcal{A}^{-1}$	kernel corresponding to A^{-1}, but $\mathcal{A}^{-1} \neq (\mathcal{A})^{-1}$	146
$\mathcal{A}^T$	kernel corresponding to A^T, but $\mathcal{A}^T \neq (\mathcal{A})^T$	146 152 166
$\mathbb{R}^n$	real vector of length n	19 253 384
$\mathbb{R}^{k\times n}$	real $k \times n$ array	143 383
$[A]_:$ $[A]_{n\times 1}$	reordering of matrix to column vector	133 141 166
$[\underline{a}]_{n\times m}$	reordering of column vector to $n \times m$ matrix	133 146
$[\underline{a}]_{\underline{n}}$	reordering to $n_1 \times n_2 \times \ldots$ multidimensional array	265
$\hat{a}, \hat{\underline{a}}, \hat{A}$	estimate of $a, \underline{a}, A$	22 58
$\tilde{\underline{a}}$	estimation error in $\hat{\underline{a}}$	64 108
$\bar{a}, \bar{\underline{a}}, \bar{A}$	transformation of $a, \underline{a}, A$	41 241
$\check{\underline{a}}$	given sample data of $\underline{a}$	201 412
$\tilde{P}$	estimation error covariance	68 72 294
$\Pr(Q)$	probability of some event Q	119 181 411
$p(x)$	probability density of x	42 65 411
$p(x\|y)$	conditional probability density	45 179 413
$\{\ldots\}$	a set	15 415
$\|S\|$	number of elements in set S	15 121 203
$\|\underline{x}\|$, $\|A\|$	vector norm for $\underline{x}$, matrix norm for A	22 59
$\|\underline{x}\|_P = \underline{x}^T P \underline{x}$	vector squared-norm for $\underline{x}$ with respect to covariance P	31 63 73
ⓐ	convolution kernel origin	145 152 268
$\sim$	is distributed as ...	49 355
$\underline{x} \sim P$	$\underline{x}$ has covariance P; the mean is zero or not of interest	28 141 241
$\underline{x} \sim (\underline{\mu}, P)$	$\underline{x}$ has mean $\underline{\mu}$ and covariance P, distribution unknown	37 69 74
$\underline{x} \sim \mathcal{N}(\underline{\mu}, P)$	$\underline{x}$ is Gaussian with mean $\underline{\mu}$ and covariance P	28 63 417

Tab. Notation.1. Basic vector, matrix, and statistical syntax

Function	Definition	Page References
$\operatorname{sign}(a)$	the sign of scalar a, $\operatorname{sign}(a) = \begin{cases} -1 & a < 0 \\ 0 & a = 0 \\ 1 & a > 0 \end{cases}$	144 400
a mod b	division modulus (remainder)	155 262 392
$\min(\cdot)$	the minimum value in a set	198 385 385
$\min_{x \in X}(\cdot)$	the minimum value of a function over range X	401
$\arg_x \min(\cdot)$	the value of x which minimizes the function	30 63 409
$\operatorname{Ra}(A)$	the range space of A	19 53 385
$\operatorname{Nu}(A)$	the null space of A	19 53 384
$\operatorname{rank}(A)$	the rank of A	53 385
$\dim(\cdot)$	the dimension of a space	384
$\operatorname{tr}(A)$	the trace, the sum of the diagonal elements of A	247 387 395
$\det(A)$	the determinant of A	24 386 387
$\kappa(A)$	matrix condition number of A	26 104 246
$\operatorname{diag}(A)$	a vector containing the diagonal elements of A	54 140 266
$\operatorname{Diag}(\underline{x})$	a diagonal matrix with $\underline{x}$ along the diagonal	165 264 294
I	the identity matrix	37 277 357
FFT_d	the d-dimensional fast Fourier transform	267 361 428
FFT_d^{-1}	the d-dimensional inverse fast Fourier transform	265 361
WT	the wavelet transform	273 347 428
WT^{-1}	the inverse wavelet transform	275 290
$\odot$	element-by-element matrix multiplication	146 165 266
$\oslash$	element-by-element matrix division	146 254 265
$*$	convolution	142 146 424
$\circledast$	circular convolution	427 428
$\operatorname{var}(\cdot)$	variance	164 389
$\operatorname{cov}(\cdot)$	covariance	42 67 87
$E[\cdot]$	expectation	42 64 412
$E_a[\cdot]$	expectation over variable a, if otherwise ambiguous	232 363
$\equiv$	is equivalent to, identical to	31 61
$\triangleq$	is defined as	15 58
$< \; > \; \geq \; \leq$	inequalities, in positive-definite sense for matrices	81 100 388

Tab. Notation.2. Mathematical functions and operations (see Appendix A)

Symbol	Definition	Page References		
b	linear system target	245	293	403
b	estimator bias		65	66
c	random field clique		193	368
d	spatial dimensionality		262	267
e	error	54	58	298
$\underline{e}_i$	the ith unit vector: all zeros with a one in the ith position			384
f	forward problem	13	15	30
g	Markov random field model coefficient		186	202
i, j	general indices		17	37
k, n, q	matrix and vector dimensions	19	140	148
m	measurement	13	40	58
p	probability density	42	65	411
r	linear system residual	298	306	314
s, t	time		42	86
v	measurement noise	13	40	58
v	eigenvector	249	304	396
w	dynamic process noise	42	86	325
x, y	spatial location or indices	35	150	221
z	system state	13	40	58
A	linear system, normal equations	245	293	403
A	dynamic model predictor	86	143	325
B	dynamic model stochastic weight	42	86	325
C	measurement model	13	40	58
E	expectation	42	64	412
F	Fourier transform		257	263
F	change of basis (forwards)	241	247	314
G	Markov random field model		186	201
H	energy function	192	222	371
I	identity matrix	37	277	357
J	optimization criterion		198	249
K	estimator gain	97	108	330
L	constraints matrix	150	157	293
M	measurements field (multidimensional)	170	215	267
N	image or patch size	134	214	329
P	state covariance	40	143	160
Q	squared system constraints		152	152
R	measurement noise covariance	40	63	87
S	change of basis (backwards)	246	247	314
T	annealing temperature		192	368
U, V	orthogonal matrices	245	250	400
W	wavelet transform		277	348
Z	random field (multidimensional)	133	141	327

Tab. Notation.3. Symbol definitions

Symbol	Definition	Page References		
α, β, γ	constants	41	98	164
β	Gibbs inverse temperature	192	228	355
δ	Dirac delta	87	197	200
δ	small offset or perturbation	26	160	184
ϵ	small amount		21	305
κ	matrix condition number	26	104	246
θ	model parameters		50	170
λ	regularization parameter	31	35	63
λ	eigenvalue	304	396	420
μ	mean	37	67	412
ν	estimator innovations	95	96	110
ρ	correlation coefficient		390	413
ρ	spectral radius		300	398
σ	standard deviation	69	102	412
σ	singular value	27	256	400
τ	time offset or period		44	111
ξ	correlation length		28	124
ζ	threshold	198	300	432
$\varGamma$	covariance square root	104	166	401
$\varLambda, \varSigma$	covariance	66	93	389
$\varPsi$	state space or alphabet	119	122	181
$\varOmega$	problem space (multidimensional lattice)	184	193	415
$\varXi$	region subset operator		167	342
$\mathcal{B}$	matrix banding structure	144	146	167
$\mathcal{C}$	clique set		193	194
$\mathcal{N}$	neighbourhood	184	189	226
$\mathcal{N}(\mu, P)$	Gaussian distribution with mean $\underline{\mu}$, covariance P	28	63	417
$\mathcal{O}(\cdot)$	complexity order		143	326
$\mathbb{R}$	real			374
$\mathbb{R}^n$	real vector of length n	19	253	384
$\mathbb{R}^{k\times n}$	real $k \times n$ array		143	383
$\mathbb{Z}$	Gibbs partition function		192	355
$0,\ 1$	scalar constant zero, one		24	59
$\underline{0},\ \underline{1}$	vector constants of all zeros, all ones	19	45	72
$\mathbf{0},\ \mathbf{1}$	matrix constants of all zeros, all ones		66	76

Tab. Notation.3. Symbol definitions (cont'd)

Nonstationary Model (Dense)	Stationary Model (Kernel)	Model Type	Page References
A, B	$\mathcal{A}, \mathcal{B}$	Square root model (dynamic)	86 143 325
Γ	$\mathbf{\Gamma}$	Square root model (static)	104 166 401
P	$\mathcal{P}$	Squared model (covariance)	40 143 160
V	$\mathcal{V}$	Square root inverse model (Gibbs field)	193
G	$\mathcal{G}$	Squared inverse model (Markov field)	186 201
L	$\mathcal{L}$	Square root deterministic model (constraints)	150 157 293
Q	$\mathcal{Q}$	Squared deterministic model	152 152
C, R	$\mathcal{C}, \mathcal{R}$	Measurement model	13 40 58

Tab. Notation.4. Spatial model definitions

1

Introduction

Images are all around us! Inexpensive digital cameras, video cameras, computer webcams, satellite imagery, and images off the Internet give us access to spatial imagery of all sorts. The vast majority of these images will be of scenes at human scales — pictures of animals / houses / people / faces and so on — relatively complex images which are not well described statistically or mathematically. Many algorithms have been developed to process / denoise / compress / segment such images, described in innumerable textbooks on image processing [36, 54, 143, 174, 210], and briefly reviewed in Appendix C.

Somewhat less common, but of great research interest, are images which do allow some sort of mathematical characterization, and to which standard image-processing algorithms may not apply. In most cases we do not necessarily have *images* here, per se, but rather spatial datasets, with one or more measurements taken over a two- or higher-dimensional space.

There are many important problems falling into this latter group of scientific images, and where this text seeks to make a contribution. Examples abound throughout remote sensing (satellite data mapping, data assimilation, sea-ice / climate-change studies, land use), medical imaging (denoising, organ segmentation, anomaly detection), computer vision (textures, image classification, segmentation), and other 2D / 3D problems (groundwater, biological imaging, porous media, etc.).

Although a great deal of research has been applied to scientific images, in most cases the resulting methods are not well documented in common textbooks, such that many experienced researchers will be unfamiliar with the use of the FFT method (Section 8.3) or of posterior sampling (Chapter 11), for example.

The goal, then, of this text is to address methods for solving multidimensional inverse problems. In particular, the text seeks to avoid the pitfall of being entirely mathematical / theoretical at one extreme, or primarily applied / algorithmic on the other, by deliberately developing the basic theory (Part I), the mathematical mod-

P. Fieguth, *Statistical Image Processing and Multidimensional Modeling*, Information Science and Statistics, DOI 10.1007/978-1-4419-7294-1_1, © Springer Science+Business Media, LLC 2011

elling (Part II), and the algorithmic / numerical methods (Part III) of solving a given problem.

Inverse Problems

So, to begin, why would we want to solve an inverse problem?

There are a great many spatial phenomena that a person might want to study ...

- The salinity of the ocean surface as a function of position;
- The temperature of the atmosphere as a function of position;
- The height of the grass growing in your back yard, as a function of location;
- The proportions of oil and water in an oil reservoir.

In each of these situations, you aren't just handed a map of the spatial process you wish to study, rather you have to *infer* such a map from given measurements. These measurements might be a simple function of the spatial process (such as measuring the height of the grass using a ruler) or might be complicated nonlinear functions (such as microwave spectra for inferring temperature).

The process by which measurements are generated from the spatial process is normally relatively straightforward, and is referred to as a *forward problem*. More difficult, then, is the *inverse problem*, discussed in detail in Chapter 2, which represents the mathematical inverting of the *forward problem*, allowing you to infer the process of interest from the measurements. A simple illustration is shown in Figure 1.1.

Large Multidimensional Problems

So why is it that we wish to study large multidimensional problems?

The solution to linear inverse problems (see Chapter 3) is easily formulated analytically, and even a nonlinear inverse problem can be reformulated as an optimization problem and solved. The challenge, then, is not the solving of inverse problems *in principle*, but rather actually solving them *in practice*.

For example, the solution to a linear inverse problem involves a matrix inversion. As the problem is made larger and larger, eventually the matrix becomes computationally or numerically impossible to invert. However, this is not just an abstract limit — even a modest two-dimensional problem at a resolution of 1000×1000 pixels contains one million unknowns, which would require the inversion of a one-million by one-million matrix: completely unfeasible.

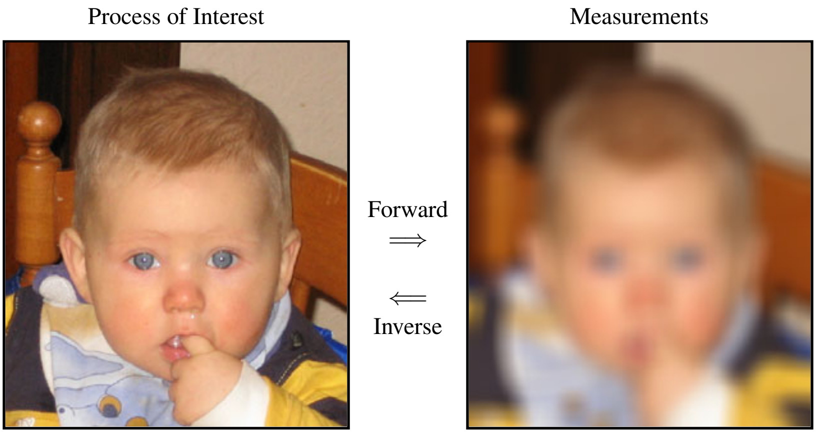


Fig. 1.1. An inverse problem: You want a nice clear photo of a face, however your camera yields blurry measurements. To solve this inverse problem requires us to mathematically invert the forward process of blurring.

Therefore even rather modestly sized two- and higher-dimensional problems become impossible to solve using straightforward techniques, yet these problems are very common. Problems having one million or more unknowns are littered throughout the fields of remote sensing, oceanography, medical imaging, and seismology, to name a few.

To be clear, a problem is considered to be multidimensional if it is a function of two or more independent variables. These variables could be spatial (as in a two-dimensional image or a three-dimensional volume), spatio-temporal (such as a video, a sequence of two-dimensional images over time), or a function of other variables under our control.

Multidimensional Methods versus Image Processing

What is it that the great diversity of algorithms in the image processing literature cannot solve?

The majority of images which are examined and processed in image processing are "real" images, pictures and scenes at human scales, where the images are not well described mathematically. Therefore the focus of image processing is on making relatively few explicit, mathematical assumptions about the image, and instead focusing on the development of algorithms that perform image-related tasks (such as compression, segmentation, edge detection, etc.).

Face Grass Clouds

Fig. 1.2. Which of these might be best characterized mathematically? Many natural phenomena, when viewed at an appropriate scale, have a behaviour which is sufficiently varied or irregular that it can be modelled via relatively simple equations, as opposed to a human face, which would need a rather complex model to be represented accurately.

In contrast, of great research interest are images taken at microscopic scales (cells in a Petri dish, the crystal structure of stone or metal) or at macroscopic scales (the temperature distribution of the ocean or of the atmosphere, satellite imagery of the earth) which do, in general, allow some sort of mathematical characterization, as explored in Figure 1.2. That is, the focus of this text is on the assumption or inference of rather *explicit* mathematical models of the unknown process.

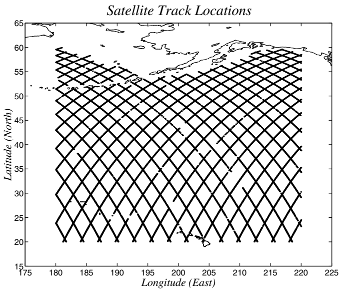
Next, in order to be able to say something about a problem, we need measurements of it. These measurements normally suffer from one of three issues, any one of which would preclude the use of standard image-processing techniques:

1. For measurements produced by a scientific instrument, acquiring a measurement normally requires time and/or money, therefore the number of measurements is constrained. Frequently this implies that the multidimensional problem of interest is only sparsely sampled, as illustrated in Figure 1.3.

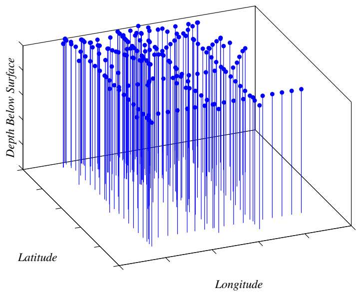
   There exist many standard methods to interpolate gaps in a sequence of data, however standard interpolation knows nothing about the underlying phenomenon being studied. That is, surely a grass-like texture should be interpolated differently from a map of ocean-surface temperature.

2. Most measurements are not exact, but suffer from some degree of noise. Ideally we would like to remove this noise, to infer a more precise version of the underlying multidimensional phenomenon.

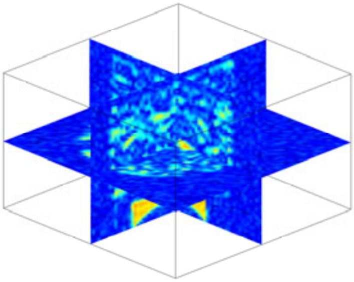
   There exist many algorithms for noise reduction in images, however these are necessarily heuristic, because they are designed to work on photographic images, which might contain images of faces / cars / trees and the like. Given a scientific dataset, surely we would wish to undertake denoising in a more systematic (ideally optimal) manner, somehow dependent on the behaviour of the underlying phenomenon.



Satellite Altimetry
(each point measures ocean height)



Ship-Based Measurements
(each point measures ocean temperature)



Slices from a 3D MRI
(each point represents the concentration of water in a block of concrete)

Fig. 1.3. Multidimensional measurements: Three examples of two- or three-dimensional measurements which could not be processed by conventional means of image processing. The altimetric measurements are sparse, following the orbital path of a satellite; the ship-based measurements are irregular and highly sparse, based on the paths that a ship followed in towing an instrument array; the MRI measurements are dense, but at poor resolution and with substantial noise.

3. In many cases of scientific imaging, the raw measurement produced by an instrument is *not* a direct measurement of the multidimensional field, but rather some function of it. For example, in Application 3 we wish to study atmospheric temperature based on radiometric measurements of microwave intensities: the air temperature and microwave intensity are indeed related, but are very different quantities.

   Standard methods in image processing normally assume that the measurements (possibly noisy, possibly blurred) form an image. However, having measurements being some complicated function of the field of interest (an inverse problem) is more subtle and requires a careful formulation.

Statistics and Random Fields

What is it that makes a problem statistical, and why do we choose to focus on statistical methods?

An interest in *spatial statistics* goes considerably beyond the modelling of phenomena which are inherently *random*. In particular, multidimensional random fields offer the following advantages:

1. Even if an underlying process is not random, in most cases measurements of the process are corrupted by noise, and therefore a statistical representation may be appropriate.

2. Many processes exhibit a degree of irregularity or complexity that would be extremely difficult to model deterministically. Two examples are shown in Figure 1.4; although there are physics which govern the behaviour of both of these examples (e.g., the Navier–Stokes differential equation for water flow) the models are typically highly complex, containing a great number of unknown parameters, and are computationally difficult to simulate.

   A random-fields approach, on the other hand, would implicitly approximate these complex models on the basis of observed statistics.

A random field[1] X is nothing but a large collection of random variables arranged on some set of points (possibly a two- or three-dimensional grid, perhaps on a sphere, or perhaps irregularly distributed in a high-dimensional space). The random field is characterized by the statistical interrelationships between its random variables.

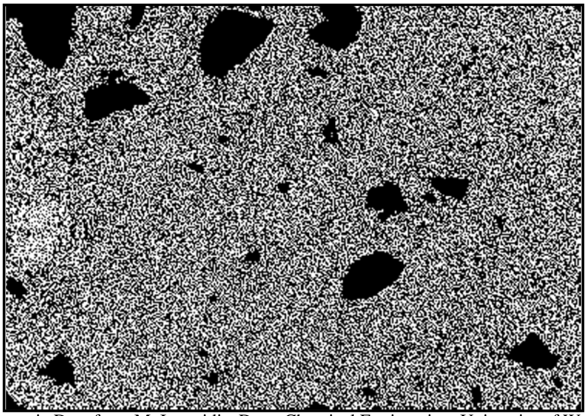
The main problem associated with a statistical formulation is the computational complexity of the resulting solution. However, as we shall see, there exists a comprehensive set of methods and algorithms for the manipulation and efficient solving of problems involving random fields. The development of this theory and of associated algorithms is the fundamental goal of this text.

Specifically, the key problem explored in this text is representational and computational efficiency in the solving of large problems. The question of efficiency is easily motivated: even a very modestly sized 256×256 image has 65 536 elements, and the glass beads image in Figure 1.4 contains in excess of 100 million elements! It comes as no surprise that a great part of the research into random fields involves the discovery or definition of *implicit* statistical forms which lead to effective or faithful representations of the true statistics, while admitting computationally efficient algorithms.

Broadly speaking there are four typical problems associated with random fields [112]:

[1] Random variables, random vectors, and random fields are reviewed in Appendix B.1.

A Porous Medium of Packed Glass Beads



(Microscopic Data from M. Ioannidis, Dept. Chemical Engineering, University of Waterloo)

Global Ocean Surface Temperature

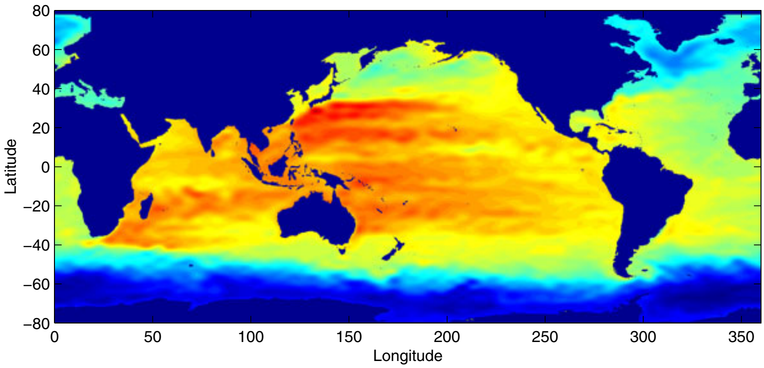


Fig. 1.4. Two examples of phenomena which may be modelled via random fields: packed glass beads (top), and the ocean surface temperature (bottom). Alternatives to random fields do exist to model these phenomena, such as ballistics methods for the glass beads, and coupled differential equations for the ocean, however such approaches would be greatly more complex than approximating the observed phenomena on the basis of inferred spatial statistics.

1. Representation: how is the random field represented and parametrized?

2. Synthesis: how can we generate "typical" realizations of the random field?

3. Parameter estimation: given a parametrized statistical model and sample image, how can we estimate the unknown parameters in the model?

4. Random fields estimation: given noisy observations of the random field, how can the unknown random field be estimated?

All four of these issues are of interest to us, and are developed throughout the text.

For each of these there are separate questions of formulation,

How do I write down the equations that need to be solved?

as opposed to those of solution,

How do I actually find a solution to these equations?

Part I of this text focuses mostly on the former question, establishing the mathematical fundamentals that are needed to express a solution, *in principle*. This gives us a solution which we might call

1. **Brute Force:** The direct implementation of the solution equations, irrespective of computational storage, complexity, and numerical robustness issues.

Parts II and III then examine the latter question, seeking practical, elegant, or indirect solutions to the problems of interest. However, *practical* should not be interpreted to mean that the material is only of dry interest to the specialist sitting at a computer, about to develop a computer program. Many of the most fundamental ideas expressed in this text are particularly in Part II, where deep insights into the nature of spatial random fields are explored.

A few kinds of efficient solutions, alternatives to the direct implementations from Part I, are summarized as follows:

2. **Dimensionality Reduction:** Transforming a problem into one or more lower-dimensional problems.
3. **Change of Basis:** A mathematical transformation of the problem which simplifies its computational or numerical complexity.
4. **Approximate Solution:** An approximation to the exact analytical solution.
5. **Approximated Problem:** Rather than solving the given problem, identifying a similar problem which can be solved exactly.
6. **Special Cases:** Circumstances in which the statistics or symmetry of the problem gives rise to special, efficient solutions.

These six points give a broad sense of what this text is about.