Probability and Its Applications

Published in association with the Applied Probability Trust

Probability and Its Applications

Anderson: Continuous-Time Markov Chains.
Azencott/Dacunha-Castelle: Series of Irregular Observations.
Bass: Diffusions and Elliptic Operators.
Bass: Probabilistic Techniques in Analysis.
Chen: Eigenvalues, Inequalities, and Ergodic Theory
Choi: ARMA Model Identification.
Daley/Vere-Jones: An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods, Second Edition.
de la Peña/Giné: Decoupling: From Dependence to Independence.
Del Moral: Feynman-Kac Formulae: Genealogical and Interacting Particle Systems with Applications.
Durrett: Probability Models for DNA Sequence Evolution.
Galambos/Simonelli: Bonferroni-type Inequalities with Applications.
Gani (Editor): The Craft of Probabilistic Modelling.
Grandell: Aspects of Risk Theory.
Gut: Stopped Random Walks.
Guyon: Random Fields on a Network.
Kallenberg: Foundations of Modern Probability, Second Edition.
Last/Brandt: Marked Point Processes on the Real Line.
Leadbetter/Lindgren/Rootzén: Extremes and Related Properties of Random Sequences and Processes.
Molchanov: Theory of Random Sets.
Nualart: The Malliavin Calculus and Related Topics.
Rachev/Rüschendorf: Mass Transportation Problems. Volume I: Theory.
Rachev/Rüschendofr: Mass Transportation Problems. Volume II: Applications.
Resnick: Extreme Values, Regular Variation and Point Processes.
Shedler: Regeneration and Networks of Queues.
Silvestrov: Limit Theorems for Randomly Stopped Stochastic Processes.
Thorisson: Coupling, Stationarity, and Regeneration.
Todorovic: An Introduction to Stochastic Processes and Their Applications.

Ilya Molchanov

Theory of Random Sets

With 33 Figures

Springer

Ilya Molchanov
Department of Mathematical Statistics and Actuarial Science, University of Berne,
CH-3012 Berne, Switzerland

Mathematics Subject Classification (2000): 60-02, 60D05, 06B35, 26A15, 26A24, 28B20, 31C15, 43A05, 43A35, 49J53, 52A22, 52A30, 54C60, 54C65, 54F05, 60E07, 60G70, 60F05, 60H25, 60J99, 62M30, 90C15, 91B72, 93E20

British Library Cataloguing in Publication Data
Molchanov, Ilya S., 1962–
Theory of random sets. — (Probability and its applications)
1. Random sets 2. Stochastic geometry
I. Title
519.2
ISBN 185233892X

Library of Congress Cataloging-in-Publication Data
CIP data available.

ISBN-10: 1-85233-892-X
ISBN-13: 978-185223-892-3
Springer Science+Business Media
springeronline.com

Printed in the United States of America

Typesetting: Camera-ready by author.
12/3830-543210 Printed on acid-free paper SPIN 10997451

To my mother

Preface

History

The studies of random geometrical objects go back to the famous Buffon needle problem. Similar to the ideas of Geometric Probabilities that can be traced back to the first results in probability theory, the concept of a random set was mentioned for the first time together with the mathematical foundations of Probability Theory. A.N. Kolmogorov [321, p. 46] wrote in 1933:

> Let G be a measurable region of the plane whose shape depends on chance; in other words, let us assign to every elementary event ξ of a field of probability a definite measurable plane region G. We shall denote by J the area of the region G and by $\mathbf{P}(x, y)$ the probability that the point (x, y) belongs to the region G. Then
>
> $$\mathbf{E}(J) = \iint \mathbf{P}(x, y) \mathrm{d}x \mathrm{d}y .$$

One can notice that this is the formulation of Robbins' theorem and $\mathbf{P}(x, y)$ is the coverage function of the random set G.

The further progress in the theory of random sets relied on the developments in the following areas:

- studies of random elements in abstract spaces, for example groups and algebras, see Grenander [210];
- the general theory of stochastic processes, see Dellacherie [131];
- advances in image analysis and microscopy that required a satisfactory mathematical theory of distributions for binary images (or random sets), see Serra [532].

The mathematical theory of random sets can be traced back to the book by Matheron [381]. G. Matheron formulated the exact definition of a random closed set and developed the relevant techniques that enriched the convex geometry and laid out the foundations of mathematical morphology. Broadly speaking, the convex geometry contribution concerned properties of functionals of random sets, while the morphological part concentrated on operations with the sets themselves.

The relationship between random sets and convex geometry later on has been thoroughly explored within the stochastic geometry literature, see, e.g. Weil and Wieacker [607]. Within the stochastic geometry, random sets represent one type of objects along with point processes, random tessellations, etc., see Stoyan, Kendall and Mecke [544]. The main techniques stem from convex and integral geometry, see Schneider [520] and Schneider and Weil[523].

The mathematical morphology part of G. Matheron's book gave rise to numerous applications in image processing (Dougherty [146]) and abstract studies of operations with sets, often in the framework of the lattice theory (Heijmans [228]).

Since 1975 when G. Matheron's book [381] was published, the theory of random sets has enjoyed substantial developments. D.G. Kendall's seminal paper [295] already contained the first steps into many directions such as lattices, weak convergence, spectral representation, infinite divisibility. Many of these concepts have been elaborated later on in connection to the relevant ideas in pure mathematics. This made many of the concepts and notation used in [295] obsolete, so that we will follow the modern terminology that fits better into the system developed by G. Matheron; most of his notation was taken as the basis for the current text.

The modern directions in random sets theory concern

- relationships to the theories of semigroups and continuous lattices;
- properties of capacities;
- limit theorems for Minkowski sums and relevant techniques from probabilities in Banach spaces;
- limit theorems for unions of random sets, which are related to the theory of extreme values;
- stochastic optimisation ideas in relation to random sets that appear as epigraphs of random functions;
- studies of properties of level sets and excursions of stochastic processes.

These directions constitute the main core of this book which aims to cast the random sets theory in the conventional probabilistic framework that involves distributional properties, limit theorems and the relevant analytical tools.

Central topics of the book

The whole story told in this book concentrates on several important concepts in the theory of random sets.

The first concept is the *capacity functional* that determines the distribution of a random closed set in a locally compact Hausdorff separable space. It is related to positive definite functions on semigroups and lattices. Unlike probability measures, the capacity functional is non-additive. The studies of non-additive measures are abundant, especially, in view of applications to game theory, where the non-additive measure determines the gain attained by a coalition of players. The capacity functional can be used to characterise the weak convergence of random sets and some properties of their distributions. In particular, this concerns unions of random closed sets, where the regular variation property of the capacity functional is of primary importance. It is possible to consider random capacities that unify the concepts of a random closed set and a random upper semicontinuous function. However, the capacity functional does not help to deal with a number of other issues, for instance to define the expectation of a random closed set.

Here the leading role is taken over by the concept of a *selection*, which is a (single-valued) random element that almost surely belongs to a random set. In this framework it is convenient to view a random closed set as a multifunction (or set-valued function) on a probability space and use the well-developed machinery of set-valued analysis. It is possible to find a countable family of selections that completely fills the random closed set and is called its *Castaing representation*. By taking expectations of integrable selections, one defines the *selection expectation* of a random closed set. However, the families of all selections are very rich even for simple random sets.

Fortunately, it is possible to overcome this difficulty by using the concept of the *support function*. The selection expectation of a random set defined of a non-atomic probability space is always convex and can be alternatively defined by taking the expectation of the support function. The *Minkowski sum* of random sets is defined as the set of sums of all their points or all their selections and can be equivalently formalised using the arithmetic sum of the support functions. Therefore, limit theorems for Minkowski sums of random sets can be derived from the existing results in Banach spaces, since the family of support functions can be embedded into a Banach space. The support function concept establishes numerous links to convex geometry ideas. It also makes it possible to study set-valued processes, e.g. set-valued martingales and set-valued shot-noise.

Important examples of random closed sets appear as *epigraphs* of random lower semicontinuous functions. Viewing the epigraphs as random closed sets makes it possible to obtain results for lower semicontinuous functions under the weakest possible conditions. In particular, this concerns the convergence of minimum values and minimisers, which is the subject of stochastic optimisation theory.

It is possible to consider the family of closed sets as both a semigroup and a lattice. Therefore, random closed sets are simply a special case of general lattice- or semigroup-valued random elements. The concept of probability measure on a *lattice* is indispensable in the modern theory of random sets.

It is convenient to work with random *closed* sets, which is the typical setting in this book, although in some places we mention random open sets and random Borel sets.

Plan

Since the concept of a set is central for mathematics, the book is highly interdisciplinary and aims to unite a number of mathematical theories and concepts: capacities, convex geometry, set-valued analysis, topology, harmonic analysis on semigroups, continuous lattices, non-additive measures and upper/lower probabilities, limit theorems in Banach spaces, general theory of stochastic processes, extreme values, stochastic optimisation, point processes and random measures.

The book starts with a definition of random closed sets. The space $\mathbb{E}$ which random sets belong to, is very often assumed to be locally compact Hausdorff with a countable base. The Euclidean space $\mathbb{R}^d$ is a generic example (apart from rare moments when $\mathbb{E}$ is a line). Often we switch to the more general case of $\mathbb{E}$ being a Polish space or Banach space (if a linear structure is essential). Then the Choquet theorem concerning the existence of random sets distributions is proved and relationships with set-valued analysis (or multifunctions) and lattices are explained. The rest of Chapter 1 relies on the concept of the capacity functional. First it highlights relationships between capacity functionals and properties of random sets, then develops some analytic theory, convergence concepts, applications to point processes and random capacities and finally explains various interpretations for capacities that stem from game theory, imprecise probabilities and robust statistics.

Chapter 2 concerns expectation concepts for random closed sets. The main part is devoted to the selection (or Aumann) expectation that is based on the idea of the selection. Chapter 3 continues this topic by dealing with Minkowski sums of random sets. The dual representation of the selection expectation – as a set of expectations of all selections and as the expectation of the support function – makes it possible to refer to limit theorems in Banach spaces in order to prove the corresponding results for random closed sets. The generality of presentation varies in order to explain which properties of the carrier space $\mathbb{E}$ are essential for particular results.

The scheme of unions for random sets is closely related to extremes of random variables and further generalisations for pointwise extremes of stochastic processes. Chapter 4 describes the main results for the union scheme and explains the background ideas that mostly stem from the studies of lattice-valued random elements.

Chapter 5 is devoted to links between random sets and stochastic processes. On the one hand, this concerns set-valued processes that develop in time, in particular, set-valued martingales. On the other hand, the subject matter concerns random sets interpretations of conventional stochastic processes, where random sets appear as graphs, level sets or epigraphs (hypographs).

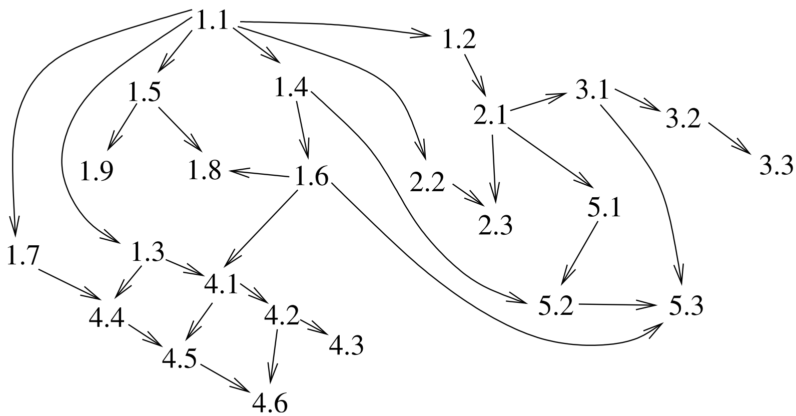
The Appendices summarise the necessary mathematical background that is normally scattered between various texts. There is an extensive bibliography and a detailed subject index.

Several areas that are related to random sets are only mentioned in brief. For instance, these areas include the theory of set-indexed processes, where random sets appear as stopping times (or stopping sets), excursions of random fields and potential theory for Markov processes that provides further examples of capacities related to hitting times and paths of stochastic processes.

It is planned that a companion volume to this book will concern models of random sets (germ-grain models, random fractals, growth processes, etc), convex geometry techniques, statistical inference for stationary and compact random sets and related modelling issues in image analysis.

Conventions

The numbering follows a two-digit pattern, where the first digit is the section number of the current chapter. When referring to results from other chapters, we add the chapter number using the three digit numbering scheme. When referring to the Appendices, the first digit is a letter that designates the particular appendix. The statements in theorems and propositions are mostly numbered by Roman numbers, while the conditions usually follow Arabic numeration.



A rough dependence guide between the sections.

Although the main concepts in this book are used throughout the whole presentation, it is anticipated that a reader will be able to read the book from the middle. The concepts are often restated and notation is set to be as consistent as possible taking into account various conventions within a number of mathematical areas that build up this book.

The problems scattered through the text are essentially open, meaning that their solutions are currently not known to the author.

The supporting information (e.g. bibliographies) for this book is available through Springer WEB site or from

http://www.cx.unibe.ch/∼ilya/rsbook/index.html

Acknowledgements

G. Matheron's book *Random Sets and Integral Geometry* [381] accompanied me throughout my whole life in mathematics since 1981 where I first saw its Russian translation (published in 1978). Then I became fascinated in this cocktail of techniques from topology, convex geometry and probability theory that essentially makes up the theory of random sets.

This book project has spanned my work and life in four different countries: Germany, the Netherlands, Scotland and Switzerland. I would like to thank people of

all these and many other countries who supported me at various stages of my work and from whom I had a chance to learn. In particular, I would like to thank Dietrich Stoyan who, a while ago, encouraged me to start writing this book and to my colleagues in Bern for a wonderful working and living environment that helped me to finish this project in a final spurt.

I am grateful to the creators of XEmacs software which was absolutely indispensable during my work on this large LaTeX project and to the staff of Springer who helped me to complete this work.

Finally, I would like to thank my family and close friends for being always ready to help whatever happens.

Bern, Switzerland *Ilya Molchanov*
April 2004

Contents

1

Random Closed Sets and Capacity Functionals

1 The Choquet theorem

1.1 Set-valued random elements

As the name suggests, a random set is an object with values being sets, so that the corresponding record space is the space of subsets of a given carrier space. At this stage, a mere definition of a general random element like a random set presents little difficulty as soon as a σ-algebra on the record space is specified. The principal new feature is that random sets may have something inside (different to random variables and random vectors) and the development of this idea is crucial in the studies of random sets. Because the family of all sets is too large, it is usual to consider random closed sets defined as random elements in the space of closed subsets of a certain topological space $\mathbb{E}$. The family of closed subsets of $\mathbb{E}$ is denoted by $\mathcal{F}$, while $\mathcal{K}$ and $\mathcal{G}$ denote respectively the family of all compact and open subsets of $\mathbb{E}$. It is often assumed that $\mathbb{E}$ is a *locally compact Hausdorff second countable* topological space (LCHS space). The Euclidean space $\mathbb{R}^d$ is a generic example of such space $\mathbb{E}$.

Let us fix a complete probability space $(\Omega, \mathfrak{F}, \mathbf{P})$ which will be used throughout to define random elements. It is natural to call an $\mathcal{F}$-valued random element a random closed set. However, one should be more specific about measurability issues, which acquire considerably more importance when studying random elements in complex spaces. In other words, when defining a random element it is necessary to specify which information is available in terms of the observable events from the σ-algebra $\mathfrak{F}$ in Ω. It is essential to ensure that the measurability requirement is restrictive enough to ensure that all functionals of interest become random variables. At the same time, the measurability condition must not be too strict in order to include as many random elements as possible. The following definition describes a rather flexible and useful concept of a random closed set.

Definition 1.1 (Definition of a random closed set). A map $X : \Omega \mapsto \mathcal{F}$ is called a *random closed set* if, for every compact set K in $\mathbb{E}$,

$$\{\omega : \; X \cap K \neq \emptyset\} \in \mathfrak{F} . \tag{1.1}$$

Condition (1.1) simply means that observing X one can always say if X hits or misses any given compact set K. In more abstract language, (1.1) says that the map $X: \Omega \mapsto \mathcal{F}$ is measurable as a map between the underlying probability space and the space $\mathcal{F}$ equipped with the σ-algebra $\mathfrak{B}(\mathcal{F})$ generated by $\{F \in \mathcal{F}: F \cap K \neq \emptyset\}$ for K running through the family $\mathcal{K}$ of compact subsets of $\mathbb{E}$. Note that $\mathfrak{B}(\mathcal{F})$ is called the Effros σ-algebra, which is discussed in greater detail in Section 2.1 for the case of a general Polish space $\mathbb{E}$. As in Appendix B, we write

$$\mathcal{F}_K = \{F \in \mathcal{F}: F \cap K \neq \emptyset\}.$$

The σ-algebra generated by $\mathcal{F}_K$ for all K from $\mathcal{K}$ clearly contains

$$\mathcal{F}^K = \{F \in \mathcal{F}: F \cap K = \emptyset\}.$$

Furthermore, for every G from the family $\mathcal{G}$ of open sets,

$$\mathcal{F}_G = \{F \in \mathcal{F}: F \cap G \neq \emptyset\} = \bigcap_n \mathcal{F}_{K_n},$$

where $\{K_n, n \geq 1\}$ is a sequence of compact sets such that $K_n \uparrow G$ (here the local compactness of $\mathbb{E}$ is essential). Therefore, $\mathcal{F}_G \in \mathfrak{B}(\mathcal{F})$ for all $G \in \mathcal{G}$. It should be noted that the *Fell topology* on $\mathcal{F}$ (discussed in Appendix B) is generated by open sets $\mathcal{F}_G$ for $G \in \mathcal{G}$ and $\mathcal{F}^K$ for $K \in \mathcal{K}$. Therefore, the σ-algebra generated by $\mathcal{F}_K$ for $K \in \mathcal{K}$ coincides with the Borel σ-algebra generated by the Fell topology on $\mathcal{F}$. It is possible to reformulate Definition 1.1 as follows.

Definition 1.1′. A map $X: \Omega \mapsto \mathcal{F}$ is called a random closed set if X is measurable with respect to the Borel σ-algebra on $\mathcal{F}$ with respect to the Fell topology, i.e.

$$X^{-1}(\mathcal{X}) = \{\omega: X(\omega) \in \mathcal{X}\} \in \mathfrak{F}$$

for each $\mathcal{X} \in \mathfrak{B}(\mathcal{F})$.

Then (1.1) can be formulated as

$$X^{-1}(\mathcal{F}_K) = \{\omega: X(\omega) \in \mathcal{F}_K\} \in \mathfrak{F}. \tag{1.2}$$

As in Appendix D, we often write $X^{-}(K)$ instead of $X^{-1}(\mathcal{F}_K)$. It is easy to see that (1.2) implies the measurability of a number of further events, e.g. $\{X \cap G \neq \emptyset\}$ for every $G \in \mathcal{G}$, $\{X \cap F \neq \emptyset\}$ and $\{X \subset F\}$ for every $F \in \mathcal{F}$.

Since σ-algebra $\mathfrak{B}(\mathcal{F})$ is the Borel σ-algebra with respect to a topology on $\mathcal{F}$, this often leads to the conclusion that $f(X)$ is a random closed set if X is a random closed set and the map $f: \mathcal{F} \mapsto \mathcal{F}$ is continuous or semicontinuous (and therefore measurable).

Example 1.2 (Simple examples of random closed sets).
(i) If ξ is a random element in $\mathbb{E}$ (measurable with respect to the Borel σ-algebra on $\mathbb{E}$), then the singleton $X = \{\xi\}$ is a random closed set.

(ii) If ξ is a random variable, then $X = (-\infty, \xi]$ is a random closed set on the line $\mathbb{E} = \mathbb{R}^1$. Indeed, $\{X \cap K \neq \emptyset\} = \{\xi \geq \inf K\}$ is a measurable event for every $K \subset \mathbb{E}$. Along the same line, $X = (-\infty, \xi_1] \times \cdots \times (-\infty, \xi_d]$ is a random closed subset of $\mathbb{R}^d$ if $(\xi_1, \ldots, \xi_n)$ is a d-dimensional random vector.
(iii) If ξ_1, ξ_2, ξ_3 are three random vectors in $\mathbb{R}^d$, then the triangle with vertices ξ_1, ξ_2 and ξ_3 is a random closed set. If ξ is a random vector in $\mathbb{R}^d$ and η is a non-negative random variable, then random ball $B_\eta(\xi)$ of radius η centred at ξ is a random closed set. While it is possible to deduce this directly from Definition 1.1, it is easier to refer to general results established later on in Theorem 2.25.
(iv) Let ζ_x, $x \in \mathbb{E}$, be a real-valued stochastic process on $\mathbb{E}$ with continuous sample paths. Then its level set $X = \{x : \zeta_x = t\}$ is a random closed set for every $t \in \mathbb{R}$. Indeed, $\{X \cap K = \emptyset\} = \{\inf_{x \in K} \zeta_x > t\} \cup \{\sup_{x \in K} \zeta_x < t\}$ is measurable. Similarly, $\{x : \zeta_x \leq t\}$ and $\{x : \zeta_x \geq t\}$ are random closed sets.

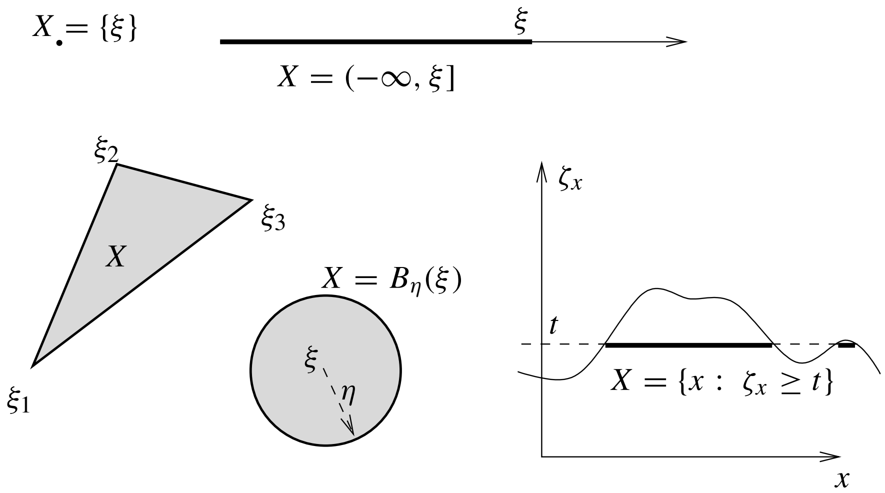


Figure 1.1. Simple examples of random closed sets.

Example 1.3 (Random variables associated with random closed sets).
(i) It is easy to see that the norm $\|X\| = \sup\{\|x\| : x \in X\}$ for a random closed set X in $\mathbb{E} = \mathbb{R}^d$ is a random variable (with possibly infinite values). The event $\{\|X\| > t\}$ means that X hits an open set G being the complement to the closed ball of radius t centred at the origin.
(ii) For every $x \in \mathbb{E}$ the indicator $\mathbf{1}_X(x)$ (equal to 1 if $x \in X$ and to zero otherwise) is a random variable.
(iii) If μ is a locally finite Borel measure on $\mathbb{E}$, then $\mu(X)$ is a random variable. This follows directly from Fubini's theorem since $\mu(X) = \int \mathbf{1}_X(x)\mu(\mathrm{d}x)$, see Section 4.4.

If two random closed sets X and Y share the same distribution, then we write $X \overset{d}{\sim} Y$. This means that $\mathbf{P}\{X \in \mathcal{X}\} = \mathbf{P}\{Y \in \mathcal{X}\}$ for every measurable family of closed sets $\mathcal{X} \in \mathfrak{B}(\mathcal{F})$.

1.2 Capacity functionals

Definition

The distribution of a random closed set X is determined by $\mathbf{P}(\mathcal{X}) = \mathbf{P}\{X \in \mathcal{X}\}$ for all $\mathcal{X} \in \mathfrak{B}(\mathcal{F})$. The particular choice of $\mathcal{X} = \mathcal{F}_K$ and $\mathbf{P}\{X \in \mathcal{F}_K\} = \mathbf{P}\{X \cap K \neq \emptyset\}$ is useful since the families $\mathcal{F}_K$, $K \in \mathcal{K}$, generate the Borel σ-algebra $\mathfrak{B}(\mathcal{F})$.

Definition 1.4 (Capacity functional). A functional $T_X : \mathcal{K} \mapsto [0, 1]$ given by

$$T_X(K) = \mathbf{P}\{X \cap K \neq \emptyset\}\,, \quad K \in \mathcal{K}\,, \tag{1.3}$$

is said to be the *capacity functional* of X. We write $T(K)$ instead of $T_X(K)$ where no ambiguity occurs.

Example 1.5 (Capacity functionals of simple random sets).
(i) If $X = \{\xi\}$ is a random singleton, then $T_X(K) = \mathbf{P}\{\xi \in K\}$, so that the capacity functional is the probability distribution of ξ.
(ii) Let $X = \{\xi_1, \xi_2\}$ be the set formed by two independent identically distributed random elements in $\mathbb{E}$. Then $T_X(K) = 1 - (1 - \mathbf{P}\{\xi_1 \in K\})^2$. For instance if ξ_1 and ξ_2 are the numbers shown by two dice, then $T_X(\{6\})$ is the probability that at least one dice shows six.
(iii) Let $X = (-\infty, \xi]$ be a random closed set in $\mathbb{R}$, where ξ is a random variable. Then $T_X(K) = \mathbf{P}\{\xi > \inf K\}$ for all $K \in \mathcal{K}$.
(iv) If $X = \{x \in \mathbb{E} : \zeta_x \geq t\}$ for a real-valued sample continuous stochastic process ζ_x, $x \in \mathbb{E}$, then $T_X(K) = \mathbf{P}\left\{\sup_{x\in K} \zeta_x \geq t\right\}$.

It follows immediately from the definition of $T = T_X$ that

$$T(\emptyset) = 0\,, \tag{1.4}$$

and

$$0 \leq T(K) \leq 1\,, \quad K \in \mathcal{K}\,. \tag{1.5}$$

Since $\mathcal{F}_{K_n} \downarrow \mathcal{F}_K$ as $K_n \downarrow K$, the continuity property of the probability measure $\mathbf{P}$ implies that T is *upper semicontinuous* (see Proposition D.7), i.e.

$$T(K_n) \downarrow T(K) \quad \text{as} \quad K_n \downarrow K \text{ in } \mathcal{K}\,. \tag{1.6}$$

Properties (1.4) and (1.6) mean that T is a (topological) precapacity that can be extended to the family of all subsets of $\mathbb{E}$ as described in Appendix E.

It is easy to see that the capacity functional T is *monotone*, i.e.

$$T(K_1) \leq T(K_2) \quad \text{if } K_1 \subset K_2\,.$$

Moreover, T satisfies a stronger monotonicity property described below. With every functional T defined on a family of (compact) sets we can associate the following *successive differences*:

$$\Delta_{K_1} T(K) = T(K) - T(K \cup K_1)\,, \tag{1.7}$$

$$\Delta_{K_n} \cdots \Delta_{K_1} T(K) = \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K) - \Delta_{K_{n-1}} \cdots \Delta_{K_1} T(K \cup K_n)\,, \quad n \geq 2\,. \tag{1.8}$$

If T from (1.3) is a capacity functional of X, then

$$\begin{aligned}\Delta_{K_1} T(K) &= \mathbf{P}\{X \cap K \neq \emptyset\} - \mathbf{P}\{X \cap (K \cup K_1) \neq \emptyset\}\\ &= -\mathbf{P}\{X \cap K_1 \neq \emptyset,\ X \cap K = \emptyset\}\,.\end{aligned}$$

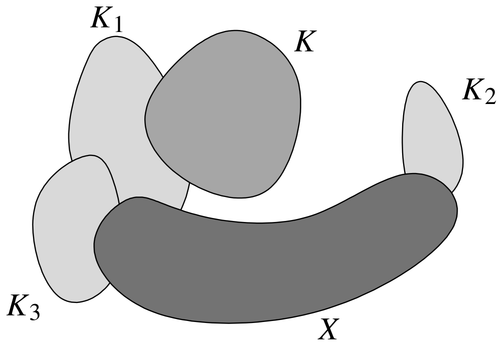


Figure 1.2. Set X from $\mathcal{F}^K_{K_1,K_2,K_3}$.

Applying this argument consecutively yields an important relationship between the higher-order successive differences and the distribution of X

$$\begin{aligned}-\Delta_{K_n} \cdots \Delta_{K_1} T(K) &= \mathbf{P}\{X \cap K = \emptyset,\ X \cap K_i \neq \emptyset,\ i = 1, \dots, n\}\\ &= \mathbf{P}\left\{X \in \mathcal{F}^K_{K_1,\dots,K_n}\right\}\,,\end{aligned} \tag{1.9}$$

where

$$\mathcal{F}^K_{K_1,\dots,K_n} = \{F \in \mathcal{F} :\ F \cap K = \emptyset,\ F \cap K_1 \neq \emptyset, \dots, F \cap K_n \neq \emptyset\}\,,$$

see Figure 1.2. In particular, (1.9) implies

$$\Delta_{K_n} \cdots \Delta_{K_1} T(K) \leq 0 \tag{1.10}$$

for all $n \geq 1$ and $K, K_1, \dots, K_n \in \mathcal{K}$.

Example 1.6 (Higher-order differences).
(i) Let $X = \{\xi\}$ be a random singleton with distribution $\mathbf{P}$. Then

$$-\Delta_{K_n} \cdots \Delta_{K_1} T(K) = \mathbf{P}\left\{\xi \in (K_1 \cap \cdots \cap K_n \cap K^{\mathsf{c}})\right\} .$$

(ii) Let $X = (-\infty, \xi_1] \times (-\infty, \xi_2]$ be a random closed set in the plane $\mathbb{R}^2$. Then $-\Delta_{\{x\}} T(\{y, z\})$ for $x = (a, c)$, $y = (b, c)$, $z = (a, d)$ is the probability that ξ lies in the rectangle $[a, b) \times [c, d)$, see Figure 1.3.
(iii) Let $X = \{x : \ \zeta_x \geq 0\}$ for a continuous random function ζ. Then

$$-\Delta_{K_n} \cdots \Delta_{K_1} T(K) = \mathbf{P}\left\{\sup_{x \in K} \zeta_x < 0, \ \sup_{x \in K_i} \zeta_x \geq 0, \ i = 1, \dots, n\right\} .$$

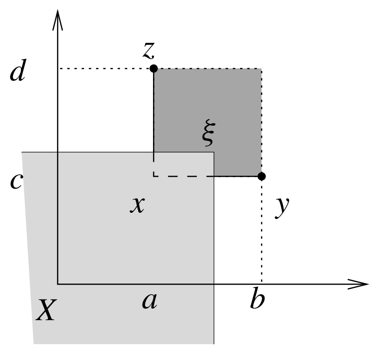


Figure 1.3. Random closed set from Example 1.6(ii).

The properties of the capacity functional T resemble those of the distribution function. The upper semicontinuity property (1.6) is similar to the right-continuity, and (1.10) generalises the monotonicity concept. However, in contrast to measures, functional T is not additive, but only *subadditive*, i.e.

$$T(K_1 \cup K_2) \leq T(K_1) + T(K_2) \tag{1.11}$$

for all compact sets K_1 and K_2.

Example 1.7 (Non-additive capacity functional). If $X = B_r(\xi)$ is the ball of radius r centred at a random point ξ in $\mathbb{R}^d$, then $T_X(K) = \mathbf{P}\{\xi \in K^r\}$, which is not a measure, since the r-envelopes K_1^r and K_2^r are not necessarily disjoint for disjoint K_1 and K_2.

Complete alternation and monotonicity

Because of the importance of properties (1.6) and (1.10) it is natural to consider general functionals on $\mathcal{K}$ that satisfy these properties without immediate reference to distributions of random closed sets. A real-valued functional φ on $\mathcal{K}$ which satisfies (1.4), (1.5), (1.6) and (1.10) is said to be a *capacity functional*. In other words, a capacity functional is a functional on $\mathcal{K}$ which takes values in [0, 1], equals 0 on the empty set and is upper semicontinuous and completely alternating on $\mathcal{K}$. The latter concept is addressed in the following definition.

Definition 1.8 (Completely alternating and completely $\cup$-monotone functionals). Let $\mathcal{D}$ be a family of sets which is closed under finite unions (so that $M_1 \cup M_2 \in \mathcal{D}$ if $M_1, M_2 \in \mathcal{D}$). A real-valued functional φ defined on $\mathcal{D}$ is said to be

(i) *completely alternating* or completely $\cup$-alternating (notation $\varphi \in \mathsf{A}(\mathcal{D})$ or $\varphi \in \mathsf{A}_\cup(\mathcal{D})$) if

$$\Delta_{K_n} \cdots \Delta_{K_1}\varphi(K) \le 0\,, \quad n \ge 1\,,\ K, K_1, \dots, K_n \in \mathcal{D}\,. \tag{1.12}$$

If (1.12) holds for all $n \le m$, then φ is said to be alternating of degree m (or m-alternating).

(ii) *completely $\cup$-monotone* (notation $\varphi \in \mathsf{M}_\cup(\mathcal{D})$) if

$$\Delta_{K_n} \cdots \Delta_{K_1}\varphi(K) \ge 0\,, \quad n \ge 1\,,\ K, K_1, \dots, K_n \in \mathcal{D}\,.$$

As (1.10) shows, the capacity functional T is completely alternating. Definition 1.8 is usually applied to the case when $\mathcal{D} = \mathcal{K}$. It complies with Definition G.5 applied to the semigroup $\mathcal{D}$ with the union being the semigroup operation. Another natural semigroup operation is the intersection of sets, which leads to other (however closely related) concepts of alternating and monotone functionals. Similar to the definition of $\Delta_{K_n} \cdots \Delta_{K_1}\varphi(K)$, we introduce the following successive differences

$$\nabla_{K_1}\varphi(K) = \varphi(K) - \varphi(K \cap K_1)\,, \tag{1.13}$$

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) = \nabla_{K_{n-1}} \cdots \nabla_{K_1}\varphi(K) - \nabla_{K_{n-1}} \cdots \nabla_{K_1}\varphi(K \cap K_n)\,, \quad n \ge 2\,. \tag{1.14}$$

The following definition is a direct counterpart of Definition 1.8.

Definition 1.9 (Completely $\cap$-alternating and completely monotone functionals). Let $\mathcal{D}$ be a family of sets which is closed under finite intersections. A real-valued functional φ defined on $\mathcal{D}$ is said to be

(i) *completely $\cap$-alternating* (notation $\varphi \in \mathsf{A}_\cap(\mathcal{D})$) if

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) \le 0\,, \quad n \ge 1\,,\ K, K_1, \dots, K_n \in \mathcal{D}\,;$$

(ii) *completely monotone* or completely $\cap$-monotone (notation $\varphi \in \mathsf{M}(\mathcal{D})$ or $\varphi \in \mathsf{M}_\cap(\mathcal{D})$) if

$$\nabla_{K_n} \cdots \nabla_{K_1}\varphi(K) \ge 0\,, \quad n \ge 1\,,\ K, K_1, \dots, K_n \in \mathcal{D}\,.$$

When saying that φ is completely alternating we always mean that φ is completely $\cup$-alternating, while φ being completely monotone means that φ is completely $\cap$-monotone. For every functional φ on $\mathcal{D}$ with values in [0, 1], its *dual* functional

$$\tilde{\varphi}(K) = 1 - \varphi(K^{\mathsf{c}}), \quad K^{\mathsf{c}} \in \mathcal{D}, \tag{1.15}$$

is defined on the family $\mathcal{D}' = \{K^{\mathsf{c}} : K \in \mathcal{D}\}$ of complements to the sets from $\mathcal{D}$.

Proposition 1.10. *Let* $\varphi \colon \mathcal{D} \mapsto [0, 1]$. *Then*

(i) $\varphi \in \mathsf{A}_{\cup}(\mathcal{D})$ *if and only if, for any fixed* $L \in \mathcal{D}$,

$$-\Delta_L \varphi(K) = \varphi(K \cup L) - \varphi(K) \in \mathsf{M}_{\cup}(\mathcal{D});$$

(ii) $\varphi \in \mathsf{A}_{\cap}(\mathcal{D})$ *if and only if, for any fixed* $L \in \mathcal{D}$,

$$-\nabla_L \varphi(K) = \varphi(K \cap L) - \varphi(K) \in \mathsf{M}_{\cap}(\mathcal{D}).$$

(iii) *Let* $\varphi \colon \mathcal{D} \mapsto [0, 1]$. *Then* $\varphi \in \mathsf{A}_{\cup}(\mathcal{D})$ *(respectively* $\varphi \in \mathsf{A}_{\cap}(\mathcal{D})$*) if and only functional* $\tilde{\varphi}(K) \in \mathsf{M}_{\cap}(\mathcal{D}')$ *(respectively* $\tilde{\varphi}(K) \in \mathsf{M}_{\cup}(\mathcal{D}')$*) for the dual functional* $\tilde{\varphi}$ *on* $\mathcal{D}' = \{K^{\mathsf{c}} : K \in \mathcal{D}\}$.

Proof. (i) It suffices to note that

$$\Delta_{K_n} \dots \Delta_{K_1}(-\Delta_L \varphi(K)) = -\Delta_L \Delta_{K_n} \dots \Delta_{K_1} \varphi(K)$$

with a similar relationship valid for the successive differences based on intersections. Statement (ii) is proved similarly. The proof of (iii) is a matter of verification that

$$\Delta_{K_n} \cdots \Delta_{K_1} \tilde{\varphi}(K) = -\nabla_{K_n^{\mathsf{c}}} \cdots \nabla_{K_1^{\mathsf{c}}} \varphi(K^{\mathsf{c}}).$$ □

Alternation and monotonicity of capacity functionals

Every measure μ is a completely alternating functional, since

$$-\Delta_{K_n} \cdots \Delta_{K_1} \mu(K) = \mu((K_1 \cup \cdots \cup K_n) \setminus K) \geq 0.$$

In particular, $\Delta_{K_1} \mu(K) = -\mu(K_1)$ if K and K_1 are disjoint.

Note that φ is increasing if and only if

$$\Delta_{K_1} \varphi(K) = \varphi(K) - \varphi(K \cup K_1)$$

is non-positive. Furthermore, for $n = 2$,

$$\Delta_{K_2} \Delta_{K_1} \varphi(K) = \varphi(K) - \varphi(K \cup K_1) - \varphi(K \cup K_2) + \varphi(K \cup K_1 \cup K_2).$$

Therefore, (1.12) for $n = 2$ is equivalent to

$$\varphi(K) + \varphi(K \cup K_1 \cup K_2) \leq \varphi(K \cup K_1) + \varphi(K \cup K_2). \tag{1.16}$$

In particular, if $K = \emptyset$ and $\varphi(\emptyset) = 0$, then

$$\varphi(K_1 \cup K_2) \leq \varphi(K_1) + \varphi(K_2), \tag{1.17}$$

meaning that φ is subadditive. Clearly, if $\varphi = \mu$ is a measure, then (1.17) turns into an equality for disjoint K_1 and K_2. For an increasing φ, inequality (1.16) is equivalent to

$$\varphi(K_1 \cap K_2) + \varphi(K_1 \cup K_2) \le \varphi(K_1) + \varphi(K_2) \tag{1.18}$$

for all K_1 and K_2. A functional φ satisfying (1.18) is called *concave* or strongly subadditive. Functionals satisfying the reverse inequality in (1.18) are called *convex* or strongly superadditive. If only $\Delta_{K_1}\varphi(K)$ and $\Delta_{K_2}\Delta_{K_1}\varphi(K)$ are non-positive, then φ is called 2-alternating. Therefore, φ is 2-alternating if it is both concave and monotone.

According to Definition E.8, a function $\varphi\colon \mathcal{P} \mapsto [-\infty, +\infty]$ on the family $\mathcal{P}$ of all subsets of $\mathbb{E}$ is called a *capacity* (or $\mathcal{K}$-capacity) if it satisfies the following conditions:

(i) $M \subset M'$ implies $\varphi(M) \le \varphi(M')$;
(ii) $M_n \uparrow M$ implies $\varphi(M_n) \uparrow \varphi(M)$;
(iii) $K_n \downarrow K$ for compact sets K_n, K implies $\varphi(K_n) \downarrow \varphi(K)$.

Definition 1.8 singles out those capacities which are completely alternating or completely monotone. Since the family $\mathcal{K}$ forms a *semigroup* with union being the semigroup operation and the neutral element being the empty set, it is possible to use the results of Appendix G within this context. It follows from Theorem G.6 that each completely alternating capacity is *negative definite* on $\mathcal{K}$. Theorem G.8 states that $\varphi \in \mathsf{A}_\cup(\mathcal{K})$ (respectively $\varphi \in \mathsf{A}_\cap(\mathcal{K})$) if and only if $e^{-t\varphi} \in \mathsf{M}_\cup(\mathcal{K})$ (respectively $e^{-t\varphi} \in \mathsf{M}_\cap(\mathcal{K})$) for all $t > 0$. Let us formulate one particularly important corollary of this fact.

Proposition 1.11. *If φ is a completely alternating non-negative capacity with possibly infinite values, then $T(K) = 1 - e^{-\varphi(K)}$ is a completely alternating capacity with values in* $[0, 1]$.

Proposition 1.11 is often used to construct a capacity functional from a completely alternating upper semicontinuous capacity that may take values greater than 1. The random closed set with the capacity functional T from Proposition 1.11 is infinite divisible for unions, see Chapter 4.

Extension of capacity functional

As explained in Appendix E, a capacity φ defined on $\mathcal{K}$ can be naturally extended onto the family $\mathcal{P}$ of all subsets of $\mathbb{E}$ keeping alternation or the monotonicity properties enjoyed by φ. In its application to capacity functionals of random closed sets, put

$$T^*(G) = \sup\{T(K) :\ K \in \mathcal{K}, K \subset G\}, \quad G \in \mathcal{G}, \tag{1.19}$$

and

$$T^*(M) = \inf\{T^*(G) :\ G \in \mathcal{G}, G \supset M\}, \quad M \in \mathcal{P}. \tag{1.20}$$

Theorem 1.12 (Consistency of extension).

(i) *$T^*(K) = T(K)$ for each $K \in \mathcal{K}$.*

(ii) *For each Borel set B,*

$$T^*(B) = \sup\{T(K) : \ K \in \mathcal{K}, \ K \subset B\}.$$

Proof. The first statement follows from the upper semicontinuity of T. Note that $T^*(K)$ is a limit of $T^*(G_n)$ for a sequence of open sets $G_n \downarrow K$. By choosing $K_n \in \mathcal{K}$ such that $K \subset K_n \subset G_n$ we deduce that $T(K_n) \downarrow T^*(K)$, while at the same time $T(K_n) \downarrow T(K)$ since T is upper semicontinuous. The second statement is a corollary from the more intricate Choquet capacitability theorem, see Theorem E.9. □

Since the extension T^* coincides with T on $\mathcal{K}$, in the following we use the same notation T to denote the extension, i.e. $T(G)$ or $T(B)$ denotes the values of the extended T on arbitrary open set G and Borel set B. Theorem 1.12 and the continuity property of probability measures imply $T(B) = \mathbf{P}\{X \cap B \neq \emptyset\}$ for all Borel B.

The Choquet theorem

Since the σ-algebra $\mathfrak{B}(\mathcal{F})$ is rich, it is difficult to explicitly assign a measure to its elements. Nonetheless, since the σ-algebra $\mathfrak{B}(\mathcal{F})$ is generated by the families $\mathcal{F}_K$, $K \in \mathcal{K}$, it is quite natural to expect that a capacity functional on $\mathcal{K}$ determines uniquely the distribution of a random closed set. The following fundamental theorem singles out upper semicontinuous completely alternating capacities on $\mathcal{K}$ as those which correspond to distributions of random closed sets. The uniqueness part easily follows from the fact that σ-algebra $\mathfrak{B}(\mathcal{F})$ is generated by $\mathcal{F}_K$ for $K \in \mathcal{K}$. It is the existence part that is more complicated. The proof of the Choquet theorem is presented in Section 1.3.

Theorem 1.13 (Choquet theorem). *Let $\mathbb{E}$ be a LCHS space. A functional $T : \mathcal{K} \mapsto [0, 1]$ such that $T(\emptyset) = 0$ is the capacity functional of a (necessarily unique) random closed set in $\mathbb{E}$ if and only if T is upper semicontinuous and completely alternating.*

The following results follow from the uniqueness part of the Choquet theorem.

Proposition 1.14. *Let $\mathbb{E}$ be a LCHS space.*

(i) *The capacity functional T_X of a random closed set X is a probability measure if and only if X is a random singleton.*

(ii) *T_X is a sub-probability measure (i.e. a measure with the total mass not exceeding 1) if and only if X with probability 1 consists of at most a single point, i.e. $\mathbf{P}\{\mathrm{card}(X) > 1\} = 0$.*

(iii) *A random closed set X is deterministic if and only if $T_X(K)$ takes only values 0 or 1 for each $K \in \mathcal{K}$.*

Proposition 1.14(iii) (and the uniqueness part of the Choquet theorem) does not hold in an arbitrary (e.g. not locally compact) space $\mathbb{E}$. For instance, if $\mathbb{E} = \mathbb{R}$ with the discrete metric, then compact sets are necessarily finite, so that $T_X(K) = 0$ for each $K \in \mathcal{K}$ if $X = \{\xi\}$ is a random singleton with a non-atomic distribution.

Maxitive capacity functionals

A functional T is said to be *maxitive* if

$$T(K_1 \cup K_2) = \max(T(K_1), T(K_2)) \tag{1.21}$$

for all compact sets K_1 and K_2. Maxitive functionals arise naturally in the theory of extremal processes, see Norberg [430, 431]. Every sup-measure (defined in Appendix E) is maxitive, while the converse is false since the definition of sup-measures involves taking a supremum over an arbitrary family of sets on the right-hand side of (1.21). If T is maxitive on $\mathcal{K}$, then (1.21) also holds for the extension of T onto the family of open sets and all subsets of $\mathbb{E}$.

Example 1.15 (Maxitive capacity). Define a *maxitive capacity* T by

$$T(K) = \sup\{f(x) : x \in K\}, \tag{1.22}$$

where $f : \mathbb{E} \mapsto [0,1]$ is an upper semicontinuous function. Then $T = f^\vee$ is the sup-integral of f as defined in Appendix E. This capacity functional T describes the distribution of the random closed set $X = \{x \in \mathbb{E} : f(x) \geq \alpha\}$, where α is a random variable uniformly distributed on $[0, 1]$.

The following proposition shows that Example 1.15 actually describes all maxitive capacities that correspond to distributions of random closed sets. In a sense, the upper semicontinuity assumption makes it possible to move from finite maximum in (1.21) to a general supremum over all singletons.

Proposition 1.16 (Maxitive upper semicontinuous capacities). *If T is a maxitive upper semicontinuous functional with values in* $[0, 1]$, *then T is given by* (1.22) *for an upper semicontinuous function* $f : \mathbb{E} \mapsto [0, 1]$.

Proof. Since T is upper semicontinuous, $f(x) = T(\{x\})$ is an upper semicontinuous function and $T(K_n) \downarrow T(\{x\})$ if $K_n \downarrow \{x\}$. This implies that for each $x \in \mathbb{E}$ and any $\varepsilon > 0$ there exists a neighbourhood $G_\varepsilon(x)$ of x such that $T(G_\varepsilon(x)) < f(x) + \varepsilon$. Every $K \in \mathcal{K}$ is covered by $G_\varepsilon(x)$, $x \in K$, so that K has a finite subcover of $G_\varepsilon(x_1), \dots, G_\varepsilon(x_n)$. Then (1.21) implies

$$T(K) \leq \max(T(G_\varepsilon(x_1)), \dots, T(G_\varepsilon(x_n))) \leq \max(f(x_1), \dots, f(x_n)) + \varepsilon,$$

whence (1.22) immediately holds. □

Proposition 1.16 means that together with the upper semicontinuity assumption, (1.21) implies that T is a sup measure. If (1.21) holds for all K_1 and K_2 from a family of sets $\mathcal{D}$ closed under finite unions, then T is called maxitive on $\mathcal{D}$.

Theorem 1.17 (Complete alternation of a maxitive capacity). *Every functional φ maxitive on a family $\mathcal{D}$ closed under finite unions is completely alternating on $\mathcal{D}$.*

Proof. Consider arbitrary $K, K_1, K_2, \ldots \in \mathcal{D}$. Let us prove by induction that

$$\Delta_{K_n} \cdots \Delta_{K_1} \varphi(K) = \varphi(K) - \varphi(K \cup K_1) \tag{1.23}$$

given that $\varphi(K_1) = \min(\varphi(K_i),\ i = 1, \ldots, n)$. This fact is evident for $n = 1$. Assume that $\varphi(K_1) = \min(\varphi(K_i),\ i = 1, \ldots, n+1)$. Using the induction assumption, it is easy to see that

$$\begin{aligned}\Delta_{K_{n+1}} \cdots \Delta_{K_1} \varphi(K) &= \Delta_{K_n} \cdots \Delta_{K_1} \varphi(K) - \Delta_{K_n} \cdots \Delta_{K_1} \varphi(K \cup K_{n+1}) \\ &= [\varphi(K) - \varphi(K \cup K_1)] - [\varphi(K \cup K_{n+1}) - \varphi(K \cup K_{n+1} \cup K_1)]\,.\end{aligned}$$

By the maxitivity assumption and the choice of K_1,

$$\begin{aligned}&\varphi(K \cup K_{n+1}) - \varphi(K \cup K_{n+1} \cup K_1) \\ &\qquad = \max(\varphi(K), \varphi(K_{n+1})) - \max(\varphi(K), \varphi(K_{n+1}), \varphi(K_1)) = 0\,.\end{aligned}$$

Now the monotonicity of φ implies that the left-hand side of (1.23) is non-positive, i.e. φ is completely alternating. □

For example, the *Hausdorff dimension* is a maxitive functional on sets in $\mathbb{R}^d$, and so is completely alternating. However, it is not upper semicontinuous, whence there is no random closed set whose capacity functional is the Hausdorff dimension.

Independence and conditional distributions

Definition 1.18 (Independent random sets). Random closed sets $X_1, \ldots, X_n$ are said to be *independent* if

$$\mathbf{P}\{X_1 \in \mathcal{X}_1, \ldots, X_n \in \mathcal{X}_n\} = \mathbf{P}\{X_1 \in \mathcal{X}_1\} \cdots \mathbf{P}\{X_n \in \mathcal{X}_n\}$$

for all $\mathcal{X}_1, \ldots, \mathcal{X}_n \in \mathfrak{B}(\mathcal{F})$.

The Choquet theorem can be used to characterise independent random closed sets in a LCHS space.

Proposition 1.19. *Random closed sets $X_1, \ldots, X_n$ are independent if and only if*

$$\mathbf{P}\{X_1 \cap K_1 \neq \emptyset, \ldots, X_n \cap K_n \neq \emptyset\} = \prod_{i=1}^{n} T_{X_i}(K_i)$$

for all $K_1, \ldots, K_n \in \mathcal{K}$.

Conditional distributions of random sets can be derived in the same way as conditional distributions of random elements in an abstract measurable space. However, this is not the case for conditional expectation, as the latter refers to a linear structure on the space of sets, see Chapter 2.

If $\mathfrak{H}$ is a sub-σ-algebra of $\mathfrak{F}$, then the conditional probabilities $T_X(K|\mathfrak{H}) = \mathbf{P}\{X \cap K \neq \emptyset|\mathfrak{H}\}$ are defined in the usual way. As noticed in Section 1.4, it suffices to define the capacity functional on a countable family $\mathcal{A}$ of compact sets, which simplifies the measurability issues. The family $T_X(K|\mathfrak{H})$, $K \in \mathcal{A}$, is a random capacity functional that defines the conditional distribution X given $\mathfrak{H}$.

1.3 Proofs of the Choquet theorem

Measure-theoretic proof

The proof given by Matheron [381] is based on the routine application of the measure-theoretic arguments related to extension of measures from algebras to σ-algebras. In fact, the idea goes back to the fundamental paper by Choquet [98] and his theorem on characterisation of positive definite functionals on cones. Here we discuss only sufficiency, since the necessity is evident from the explanations provided in Section 1.2.

Let us start with several auxiliary lemmas. The first two are entirely non-topological and their proofs do not refer to any topological assumption on the carrier space $\mathbb{E}$.

Lemma 1.20. *Let $\mathcal{V}$ be a family of subsets of $\mathbb{E}$ which contains $\emptyset$ and is closed under finite unions. Let $\mathfrak{V}$ be the family which is closed under finite intersections and generated by $\mathcal{F}_V$ and $\mathcal{F}^V$ for $V \in \mathcal{V}$. Then $\mathfrak{V}$ is an algebra and each non-empty $\mathcal{Y} \in \mathfrak{V}$ can be represented as*

$$\mathcal{Y} = \mathcal{F}^{V}_{V_1,\dots,V_n} \tag{1.24}$$

for some $n \geq 0$ and $V, V_1, \dots, V_n \in \mathcal{V}$ with $V_i \not\subset V \cup V_j$ for $i \neq j$ (then (1.24) is said to be a reduced representation of $\mathcal{Y}$). If $\mathcal{Y} = \mathcal{F}^{V'}_{V'_1,\dots,V'_k}$ is another reduced representation of $\mathcal{Y}$, then $V = V'$, $n = k$, and for each $i \in \{1, \dots, n\}$ there exists $j_i \in \{1, \dots, n\}$ such that $V \cup V_i = V \cup V'_{j_i}$.

Proof. The family $\mathfrak{V}$ is closed under finite intersections and $\emptyset = \mathcal{F}_\emptyset \in \mathfrak{V}$. If $\mathcal{Y} \in \mathfrak{V}$, then the complement to $\mathcal{Y}$,

$$\mathcal{F} \setminus \mathcal{Y} = \mathcal{F}_V \cup \mathcal{F}^{V \cup V_1} \cup \mathcal{F}^{V \cup V_2}_{V_1} \cup \dots \cup \mathcal{F}^{V \cup V_n}_{V_1,\dots,V_{n-1}},$$

is a finite union of sets from $\mathfrak{V}$. Hence $\mathfrak{V}$ is an algebra.

If $\mathcal{Y}$ satisfies (1.24) with $V_i \subset V \cup V_j$ for some $i \neq j$, then the set V_j can be eliminated without changing $\mathcal{Y}$. Therefore, a reduced representation of $\mathcal{Y}$ exists. Consider two reduced representations of a non-empty $\mathcal{Y}$. Without loss of generality assume that there exists a point $x \in V' \setminus V$. Since $\mathcal{Y} \neq \emptyset$, there exist k points (some of them may be identical) $x_1, \dots, x_k$ such that $x_j \in V'_j \setminus V'$, $1 \leq j \leq k$ and

$$\{x_1, \dots, x_k\} \in \mathcal{F}^{V'}_{V'_1,\dots,V'_k} = \mathcal{Y} = \mathcal{F}^{V}_{V_1,\dots,V_n}.$$

Since $x \notin V$, we have $\{x, x_1, \dots, x_k\} \in \mathcal{F}^{V}_{V_1,\dots,V_n}$. At the same time, $x \in V'$, whence $\{x, x_1, \dots, x_k\} \notin \mathcal{F}^{V'}_{V'_1,\dots,V'_k}$. The obtained contradiction shows that $V = V'$.

Choose $y \in V_n \setminus V$ and $y_i \in V_i \setminus (V \cup V_n)$, $i = 1, \dots, n-1$. Since $\{y_1, \dots, y_{n-1}\} \notin \mathcal{Y}$ and $\{y, y_1, \dots, y_{n-1}\} \in \mathcal{Y}$, there exists $j_n \in \{1, \dots, k\}$ such that $y \in V'_{j_n}$ and $y_i \notin V'_{j_n}$ for $i = 1, \dots, n-1$. For any other point $y' \in V_n \setminus V$ we similarly conclude that $y' \in V'_{j_n}$, whence $V_n \setminus V \subset V'_{j_n}$ and

$$V_n \subset V \cup V'_{j_n} .$$

Using identical arguments in the other direction we obtain $V'_{j_n} \setminus V \subset V_{i_n}$. If $i_n \neq n$, this leads to $V_n \subset V_{i_n} \cup V$ and so contradicts the assumption that $\mathcal{Y}$ has a reduced representation. Thus, $i_n = n$ and $V_n \setminus V = V_{j_n} \setminus V$. The proof is finished by repeating these arguments for every other set V_i, $i = 1, \dots, n-1$. □

Lemma 1.21. *In the notation of Lemma 1.20, let T be a completely alternating functional on $\mathcal{V}$ such that $T(\emptyset) = 0$, $0 \le T \le 1$. Then there exists a unique additive map $P : \mathfrak{V} \mapsto [0, 1]$ such that $\mathbf{P}(\emptyset) = 0$ and $\mathbf{P}(\mathcal{F}_V) = T(V)$ for all $V \in \mathcal{V}$. This map is given by*

$$\mathbf{P}(\mathcal{Y}) = -\Delta_{V_n} \cdots \Delta_{V_1} T(V) , \tag{1.25}$$

where $\mathcal{Y} = \mathcal{F}^V_{V_1,\dots,V_n}$ is any representation of $\mathcal{Y} \in \mathfrak{V}$.

Proof. By the additivity property, we get

$$\mathbf{P}(\mathcal{F}^V_{V_1,\dots,V_n}) = \mathbf{P}(\mathcal{F}^V_{V_1,\dots,V_{n-1}}) - \mathbf{P}(\mathcal{F}^{V\cup V_n}_{V_1,\dots,V_{n-1}}) , \tag{1.26}$$

which immediately shows that the only additive extension of $\mathbf{P}(\mathcal{F}_V) = T(V)$ is given by (1.25). It is easy to show that the right-hand side of (1.25) retains its value if any representation of $\mathcal{Y}$ is replaced by its reduced representation. Furthermore,

$$\Delta_{V_n} \cdots \Delta_{V_1} T(V) = \Delta_{V_n \cup V} \cdots \Delta_{V_1 \cup V} T(V) ,$$

which, together with Lemma 1.20, show that $\mathbf{P}(\mathcal{Y})$ is identical for any reduced representation of $\mathcal{Y}$. The function $\mathbf{P}$ is non-negative since T is completely alternating and $\mathbf{P}(\emptyset) = \mathbf{P}(\mathcal{F}_\emptyset) = T(\emptyset) = 0$. Furthermore, (1.26) implies

$$\mathbf{P}(\mathcal{F}^V_{V_1,\dots,V_n}) \le \mathbf{P}(\mathcal{F}^V_{V_1,\dots,V_{n-1}}) \le \cdots \le \mathbf{P}(\mathcal{F}^V) = 1 - T(V) \le 1 .$$

It remains to show that P is additive. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be two disjoint non-empty elements of $\mathfrak{V}$ with the reduced representations

$$\mathcal{Y} = \mathcal{F}^V_{V_1,\dots,V_n} , \quad \mathcal{Y}' = \mathcal{F}^{V'}_{V'_1,\dots,V'_k} ,$$

such that $\mathcal{Y} \cup \mathcal{Y}' \in \mathfrak{V}$. Since

$$\mathcal{Y} \cap \mathcal{Y}' = \mathcal{F}^{V\cup V'}_{V_1,\dots,V_n,V'_1,\dots,V'_k} = \emptyset ,$$

without loss of generality assume that $V_n \subset V \cup V'$. Since $\mathcal{Y} \cup \mathcal{Y}' \in \mathfrak{V}$, this union itself has a reduced representation

$$\mathcal{Y} \cup \mathcal{Y}' = \mathcal{F}^{V''}_{V''_1,\dots,V''_m} .$$

If $V = \mathbb{E}$, then $\mathcal{Y} = \{\emptyset\}$ if all subscripts in the representation of $\mathcal{Y}$ are empty, or $\mathcal{Y} = \emptyset$ otherwise, so that the additivity is trivial. Assume that there exists $x \notin V$

and $x_i \in V_i \setminus V$, $i = 1, \dots, n$. Then $F = \{x, x_1, \dots, x_n\} \in \mathcal{Y}$. Since $F \in \mathcal{Y} \cup \mathcal{Y}'$, we have $F \cap V'' = \emptyset$, i.e. $x \notin V''$. Therefore, $V'' \subset V$. Similar arguments lead to $V'' \subset V'$, whence

$$V'' \subset (V \cap V').$$

Let us show that $V'' = V$. Assume that there exist points $x \in V \setminus V''$ and $x' \in V' \setminus V''$. Choose points $x_i'' \in V_i'' \setminus V''$ for $i = 1, \dots, m$. Then $\{x, x', x_1'', \dots, x_m''\} \in \mathcal{Y} \cup \mathcal{Y}'$, so that $\{x, x'\} \cap V = \emptyset$ or $\{x, x'\} \cap V' = \emptyset$. Since both these statements lead to contradictions, we conclude that $V = V''$ or $V' = V''$. The latter is impossible, since then $V_n \subset V \cup V' = V$ leads to $\mathcal{Y} = \emptyset$. Therefore, $V = V''$, $V \subset V'$ and $V_n \subset V'$.

For each $F \in \mathcal{Y} \cup \mathcal{Y}'$, the condition $F \cap V_n \neq \emptyset$ yields $F \notin \mathcal{Y}'$, while $F \cap V_n = \emptyset$ implies $F \in \mathcal{Y}'$. Thus,

$$\mathcal{Y} = (\mathcal{Y} \cup \mathcal{Y}') \cap \mathcal{F}_{V_n} = \mathcal{F}^{V}_{V_1'', \dots, V_m'', V_n},$$
$$\mathcal{Y}' = (\mathcal{Y} \cup \mathcal{Y}') \cap \mathcal{F}^{V_n} = \mathcal{F}^{V \cup V_n}_{V_1'', \dots, V_m''}.$$

Then

$$\begin{aligned} -\mathbf{P}(\mathcal{Y}) &= \Delta_{V_n} \Delta_{V_m''} \cdots \Delta_{V_1''} T(V) \\ &= \Delta_{V_m''} \cdots \Delta_{V_1''} T(V) - \Delta_{V_m''} \cdots \Delta_{V_1''} T(V \cup V_n) \\ &= -\mathbf{P}(\mathcal{Y} \cup \mathcal{Y}') + \mathbf{P}(\mathcal{Y}'), \end{aligned}$$

which implies the additivity of $\mathbf{P}$ on $\mathfrak{V}$. □

The following lemma uses the upper semicontinuity assumption on T and the local compactness of $\mathbb{E}$.

Lemma 1.22. *Let T be a completely alternating upper semicontinuous functional on $\mathcal{K}$. By the same letter denote its extension defined by* (1.19) *and* (1.20). *Consider any two open sets G and G_0, any $K \in \mathcal{K}$, a sequence $\{K_n, n \geq 1\} \subset \mathcal{K}$ such that $K_n \uparrow G$ and a sequence $\{G_n, n \geq 1\} \subset \mathcal{G}$ such that $G_n \downarrow K$ and $G_n \supset \mathrm{cl}(G_{n+1}) \in \mathcal{K}$ for every $n \geq 1$. Then*

$$T(G_0 \cup K \cup G) = \lim_{n \to \infty} T(G_0 \cup G_n \cup K_n).$$

Proof. Since T is monotone,

$$T(G_0 \cup K \cup K_n) \leq T(G_0 \cup G_n \cup K_n) \leq T(G_0 \cup G_n \cup G).$$

For each open $G' \supset G_0 \cup G \cup K$ we have $G' \supset G_n$ for sufficiently large n. By (1.20), $T(G_0 \cup G_n \cup G) \downarrow T(G_0 \cup G \cup K)$. Similarly, $T(K \cup K_n \cup G_0)$ converges to $T(K \cup G \cup G_0)$, since T is continuous from below. □