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Stochastic Control in Insurance

Springer

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Für Monika, Eliane und Stefan.

Für all die Zeit, die meine Familie auf mich verzichten musste.

Preface

In *operations research* and *physics*, *optimisation* was introduced a long time ago. A physical system always tries to reach a state with maximal entropy. Thus, a physicist needs to solve an optimisation problem in order to find these states. In operations research one looks for business strategies leading to minimal costs or maximal profit. For example, one has to decide how many items one has to produce. On the one hand, there are storage costs if items cannot be sold immediately; on the other hand, a penalty has to be paid if an order cannot be fulfilled because the non-served customer is lost. Another problem is the construction of a computer or telecommunication network. Such a network preferably is small, but at the same time the probability of a buffer overflow should be small, too.

The problems considered in this book start with some stochastic process $\{X_t^u\}$ whose dynamics can be changed via a *control process* $\{u_t\}$. To each initial value x and each admissible control process $\{u_t\}$ we associate a value $V^u(x)$. We are interested in determining the maximal value $V(x) = \sup_u V^u(x)$ called the *value function*. Two questions then arise: What is the value function $V(x)$ and — if it exists at all — what is the optimal control process $\{u_t^*\}$, i.e., the control process leading to the value function $V(x) = V^{u^*}(x)$? In many problems the optimal control process is of feedback form $u_t^* = u^*(X_t^*)$, where $\{X_t^*\}$ is the process following the dynamics if the optimal control is applied.

A well-known approach to these problems — particularly popular in theoretical physics — is the variational approach. One writes the function to optimise as a functional of the strategy applied. By disturbing the "optimal" control function $u^*(x)$ by another function $g(x)$, that is using the control $u(x) = u^*(x) + \varepsilon g(x)$, the functional has a maximum at $\varepsilon = 0$. Holding for all $g(x)$, this usually leads to an integro-differential equation for the value function or for an equation for the optimal control $u^*(x)$.

In this book we will use a complementary approach. Our approach is stochastic and based on *martingale arguments*. The martingale formulation of dynamic programming goes back to Striebel [177]. The method had been available in 1974 but was published 10 years later.

As with the variational approach, we will obtain a *Hamilton–Jacobi–Bellman equation*. However, the derivation of the equation is heuristic only. In order to find the equation, one has to make several assumptions on the unknown value function. Hence, one has to prove that a possible solution to the equation really is the value function. This *verification theorem* often is not difficult to prove using martingale arguments.

A second problem — and usually the hard problem — is to show that the value function really is a solution to the Hamilton–Jacobi–Bellman equation. The most convenient situation is the case where an explicit solution can be found. Then, of course, one undertakes the verification argument for the explicit solution and does not have to bother with possible further solutions to the equation. If an explicit solution cannot be found, one has to solve the equation numerically. But a numerical solution only makes sense if one first verifies that a solution really exists. In nice cases a (local) *contraction argument* yields this property. A contraction argument automatically gives uniqueness, and one will not have to worry about conditions guaranteeing the correct solution. If no contraction argument is at hand, one may have to show directly that the value function solves the Hamilton–Jacobi–Bellman equation. The problem then is usually to verify that the value function is (twice continuously) differentiable. In case the solution to the Hamilton–Jacobi–Bellman equation is not unique one will have to find further properties of the value function in order to obtain the correct solution.

Sometimes further problems may occur. So the solution may not be differentiable. In this case how to solve the equation numerically is not straightforward. One then needs to determine the points where the derivative jumps. Or if the process $\{X_t^u\}$ contains a diffusion term and the solution is not twice differentiable, the martingale arguments do not apply directly to the value function. A possibility to circumvent the problem is to consider *viscosity solutions*. We will only shortly discuss this more advanced tool in the present book.

The theory of martingales is standard in probability theory. The strength of the tool comes from the *martingale convergence theorem* A.1 and the *martingale stopping theorem* A.2. Martingales also occur naturally in financial mathematics where discounted price processes should be martingales under some measure. Thus, today's financial mathematicians are familiar with martingales. In actuarial mathematics Gerber [74] introduced martingale methods for the estimation of the ruin probability. Therefore, many actuaries are also familiar with the concept of martingales.

In this book we do not want to prove general results under quite restrictive conditions. The reader can find such general results, for example, in the monographs [20], [40], [60], [61], [144], [182], or [187]. We will here consider some optimisation problems and then discuss possible methods to approach these problems. In this way we will use different techniques that have been successful for the corresponding situation. The basic approach always will be to find a Hamilton–Jacobi–Bellman equation. However, for different problems we will use different methods to show that the value function really solves the equation. From this point of view this is a practical approach to the problems. In some situations the solution of the equation is quite difficult. We therefore just prove a verification theorem that shows that a solution with the correct boundary conditions is the value function. Often the general theory of differential equations shows that a solution really exists. However, the reader has to be aware that the main problem is to solve the equation.

A natural field of the application of control techniques is *insurance mathematics*. On the one hand, economic problems usually are optimisation problems. On the other hand, an actuary is educated to make decisions. These decisions should be taken in an optimal way. In the field of *mathematical finance* it was observed quite early that there is a need for the application of optimisation techniques; see, for instance, [114], [118], [121], [131], [132], and [182]. The optimisation problems considered in insurance were mainly utility maximisation problems or the determination of Pareto-optimal risk exchanges (see, for instance, [178]) or linear programming (see [27]) where no stochastic control problem is present. A stochastic control problem was formulated by de Finetti [59] and solved by Gerber [72]. We will consider this problem in Sections 1.2 and 2.4. The corresponding problem for a diffusion approximation has been solved by Shreve et al. [168]. A similar problem was treated by Frisque [63]. Another early work on the topic is a series of lectures given by Martin-Löf [130] for the Swedish society of actuaries. The author is not aware of other early work in the application of stochastic control techniques in insurance.

A conference on the *Interplay between Insurance, Finance and Control* (see [8]) initiated a lot of work on stochastic control applied to insurance. The present book is a summary of some of the problems that have been considered since or shortly before this conference. An alternative summary article on stochastic control problems in insurance is Hipp [96], where some of the problems considered in this book also have been treated.

The above-mentioned conference was also my starting point for research in the area. Here I saw the stochastic control approach for the first time and realised that this was the tool to solve the optimal reinsurance problem that we consider in Section 2.3.1. I had this problem in my mind for quite a long time but no clue how to attack it.

The prerequisite knowledge for this book is basic probability theory with a basic knowledge of Brownian motion, Markov processes, martingales, and stochastic calculus. These topics are covered in Appendices A and B. It is recommended that a reader not familiar with these tools also has a look at some of the references given in the appendix in order to obtain some experience. In order to understand the technical details, measure theory is also needed. However, a reader only interested in the application of the optimisation techniques may skip these theoretical aspects. But it is possible to understand intuitively many of the concepts only with the knowledge given in the appendices.

Part of this book was used as material for an optional course at the Laboratory of Actuarial Mathematics in Copenhagen. At the end of the course the students were able to find the Hamilton–Jacobi–Bellman equations and to perform the verification arguments without clearly knowing what the generator of a Markov process is. A reader interested in a broader understanding will find material to deepen the required knowledge in the references given in the bibliographical remarks at the end of the sections and the appendices.

The book is organised as follows. Chapter 1 gives an introduction to stochastic control in discrete time. In this case the results can be stated quite generally. Discrete-time dynamic programming was the starting point for stochastic control and was initiated in operations research a long time ago. The continuous time case is treated in Chapter 2. After the presentation of the Hamilton–Jacobi–Bellman approach, several optimisation problems are solved. Chapter 3 also deals with optimisation in continuous time, but the problems originate from life insurance. Finally, Chapter 4 considers the problem of how asymptotic properties of the value function can be obtained from the Hamilton–Jacobi–Bellman equation. The problem is that the solutions and the optimal controls are not known explicitly but via the Hamilton–Jacobi–Bellman equation only, that is, via a highly nonlinear equation. Several appendices give a short introduction to the theory the book is based on, such as stochastic processes, Markov processes, risk theory, or life insurance mathematics.

Finally, we make some conventions. Throughout the book we work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is large enough to carry all the stochastic objects defined. We assume that $\mathcal{F}$ is *complete*, i.e., that it contains all $\mathbb{P}$-null sets. The filtrations $\{\mathcal{F}_t\}$ are assumed to be right-continuous, that is, $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. But we do *not*, as usual in books, assume that $\{\mathcal{F}_t\}$ is complete in the continuous-time case. That is, $\mathcal{F}_t$ does not necessarily contain all the $\mathbb{P}$-null sets. If we completed the filtration, in Chapter 4 we would not be able to change the measure on $\mathcal{F}$, because the measure $\mathbb{P}^*$ could not be extended to $\mathcal{F}$.

Unless stated otherwise, the stochastic processes in continuous time are assumed to be *cadlag* (right-continuous with left limits existing). This simplifies some technical problems. For example, the martingale convergence theorem and the optional stopping theorem hold quite generally. In particular, we will choose controls that are cadlag. Instead of the left-continuity that usually is used in books, we will need the left limit of the control for the development of the controlled process. The disadvantage is that the controlled process is observed after the effect of the control and the information has to be taken from the filtration instead of from the controlled process. For example, in the optimal dividend problem the post-dividend process and not the pre-dividend process is observed. The controlled process then has a jump of the size of the claim plus the dividend. Hence, we cannot get the size of the dividend from the surplus. We need additionally to know the claim size to determine the dividend payment. However, because we lose the cadlag property when considering the pre-dividend process, I prefer the presentation with cadlag stochastic processes.

If one of the basic processes is Brownian motion, we often will deal with stochastic differential equations. Then we have the problem of the existence of a unique solution. Sometimes it may even happen that no solution exists on the given probability space, but that there is a probability space on which a solution exists. This is called a weak solution. In order to avoid the problem, we assume that we have chosen a probability space on which at least the stochastic differential equation for the optimal process has a (strong) solution. The reader, however, should be aware that some technical difficulties may arise. Anyway, an insurer has a surplus process and not a probability space. Since the law of the process and not the underlying probability space is important it is no problem to choose the "right" probability space.

To simplify the notation we will omit the expressions *almost surely* or *with probability one*. Unless otherwise stated, we consider all statements to hold almost surely. For example, we say that a stochastic process is "cadlag" rather than saying it is "cadlag a.s." Of course, we could consider a probability space on which all the paths are cadlag. But sometimes it is more convenient also to allow for paths in the probability space that are not cadlag. The reader should always be aware that there might be elements in the sample space Ω for which an assertion does not hold. Because this is more a technical problem. No confusion should arise with this simplification.

A book could never be written without the help and encouragement of many other people. I therefore conclude this preface by thanking Natalie Kulenko and Julia Eisenberg for finding many misprints. I further thank Hansjörg Albrecher, Christian Hipp, and Stefan Thonhauser for pointing out some misprints and some useful references. Many very helpful remarks from an unknown reviewer are acknowledged. The reviewer, which spent a lot of time giving me detailed comments, led to a considerable improvement of the presentation and removed several mistakes present in an earlier version of the book. Last but not least, the biggest thanks go to my family, Monika, Eliane, and Stefan, for accepting that their husband/father was busy writing a book instead of enjoying more time with them.

Hanspeter Schmidli
Cologne, June 2007

Contents

1

Stochastic Control in Discrete Time

We start by considering stochastic processes in discrete time. Optimisation is simpler in discrete time than in continuous time because we can give quite general results like the dynamic programming principle (Lemma 1.1) or the optimal strategy (Corollaries 1.2 and 1.3). We will show in some simple examples how the theory can be applied.

In this chapter we consider processes in discrete time, i.e., the set of possible time points is $I = \mathbb{N}$. We will work on some Polish measurable space $(E, \mathcal{E})$, with $\mathcal{E}$ denoting the Borel-σ-algebra on E. The Borel-σ-algebra is the smallest σ-algebra containing all the open sets. A reader not familiar with metric spaces can just replace E by $\mathbb{N}^d$, $\mathbb{Z}^d$, or $\mathbb{R}^d$ endowed with the Euclidean distance. By $\mathbb{N}^*$ we denote the strictly positive integers.

1.1 Dynamic Programming

1.1.1 Introduction

Let $\{Y_n : n \in \mathbb{N}^*\}$ be an iid sequence of random variables on some Polish space $(E_Y, \mathcal{E}_Y)$. These random variables model the stochastic changes over time. We work with the natural filtration $\{\mathcal{F}_n\} = \{\mathcal{F}_n^Y\}$. At each time point $n \in \mathbb{N}$ a decision is made. We model this decision as a variable U_n from some space $\mathcal{U}$. $\mathcal{U}$ is endowed with some topology we do not mention explicitly here. The stochastic process $U = \{U_n : n \in \mathbb{N}\}$ must be adapted, because the decision can only be based on the present and not on future information. We therefore only allow controls U that are adapted. We may make some restriction to the possible strategies U. Let $\mathfrak{U}$ denote the set of *admissible* strategies, i.e., the adapted strategies $U = \{U_n\}$ that are allowed.

The controlled stochastic process is now constructed in the following way. Let $(E, \mathcal{E})$ be a Polish space, the state space of the stochastic process, and

$x \in E$ be the initial state. We let $X_0 = x$ be the starting value of the process. Note that the initial value is not stochastic. The process at time $n+1$ is

$$X_{n+1} = f(X_n, U_n, Y_{n+1}) ,$$

where $f : E \times \mathcal{U} \times E_Y \to E$ is a measurable function. The interpretation is the following. The next state of the process X only depends on the present state and the present decision. The decisions made at earlier times and the path up to the present state do not matter. The process X therefore looks similar to a Markov process. Note that we do not have a Markov process unless the decision U_n depends on X_n only.

At each time point there is a reward, $r(X_n, U_n)$. A negative value of $r(X_n, U_n)$ can be regarded as a cost. The value connected to some strategy U is then

$$V_T^U(x) = \mathbb{E}\Bigl[\sum_{n=0}^{T} r(X_n, U_n)\mathrm{e}^{-\delta n}\Bigr] .$$

The time horizon T can be finite or infinite. The parameter $\delta \ge 0$ is a discounting parameter. If $T = \infty$, we often will have to assume that $\delta > 0$ in order for $V_\infty^U(x)$ to be finite for all $U \in \mathfrak{U}$.

Our goal will be to maximise $V_T^U(x)$. We therefore define the *value function*

$$V_T(x) = \sup_{U \in \mathfrak{U}} V_T^U(x) .$$

In the case $T = \infty$, we just write $V(x)$ and $V^U(x)$ instead of $V_\infty(x)$ and $V_\infty^U(x)$, respectively. We now assume that $V_T(x) \in \mathbb{R}$ for all x. It is clear that if there is a strategy U such that $V_T^U(x) \in \mathbb{R}$, then $V_T(x) > -\infty$. The property $V_T(x) < \infty$ has to be proved for every problem separately. Another (technical) problem is to show that $V_T(x)$ is a measurable function. In many problems it can be shown that $V_T(x)$ is increasing or continuous, and hence measurable.

In the following considerations we will need measurability in several steps. We will just assume that we can always make our choices in a measurable way. General conditions for measurability can be found in more advanced textbooks on control in discrete time. We will not worry about this point because the examples we consider later have continuous value functions. Then measurability is granted. In any case, this is a technical issue, and readers not familiar with measure theory should just accept that measurability is not a problem.

1.1.2 Dynamic Programming

It is not feasible to find $V(x)$ by calculating the value function $V_T^U(x)$ for each possible strategy U, particularly not if E and T are infinite. One therefore has to find a different way to characterise the value function $V_T(x)$. In our setup it turns out that the problem can be simplified. We next prove the *dynamic programming principle*, also called *Bellman's equation*. We allow all controls $\{U_n\}$ that are adapted. With $V_t(x)$ and $V_t^U(x)$ we denote the remaining value if t time units are left. For instance, $V_{T-1}(x)$ is the value if we stand at time 1 and $X_1 = x$. We let $V_{-1}(x) = 0$.

Lemma 1.1. *Suppose that $V_T(x)$ is finite. The function $V_T(x)$ fulfils the* dynamic programming principle

$$V_T(x) = \sup_{u \in \mathcal{U}} \big\{ r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}(f(x,u,Y))] \big\}\;, \tag{1.1}$$

where Y is a generic random variable with the same distribution as Y_n. If $T = \infty$, the dynamic programming principle *becomes*

$$V(x) = \sup_{u \in \mathcal{U}} \{ r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V(f(x,u,Y))] \}\;. \tag{1.2}$$

Proof. Let U be an arbitrary strategy. Then $X_1 = f(x, U_0, Y_1)$ and

$$V_T^U(x) = \mathbb{E}[r(x,U_0)] + \mathrm{e}^{-\delta}\, \mathbb{E}\Big[\sum_{n=0}^{T-1} r(X_{n+1}, U_{n+1}) \mathrm{e}^{-\delta n} \Big]\;.$$

Condition on X_1, U_0 (we allow random decisions) and let $\tilde{X}_n = X_{n+1}$, $\tilde{U}_n = U_{n+1}$, and $\tilde{Y}_n = Y_{n+1}$. Then

$$\tilde{X}_{n+1} = f(\tilde{X}_n, \tilde{U}_n, \tilde{Y}_{n+1})$$

and

$$\begin{aligned} \mathbb{E}\Big[\sum_{n=0}^{T-1} r(X_{n+1}, U_{n+1}) \mathrm{e}^{-\delta n} \;\Big|\; X_1, U_0 \Big] &= \mathbb{E}\Big[\sum_{n=0}^{T-1} r(\tilde{X}_n, \tilde{U}_n) \mathrm{e}^{-\delta n} \;\Big|\; X_1, U_0 \Big] \\ &= V_{T-1}^{\tilde{U}}(X_1) \le V_{T-1}(X_1)\;. \end{aligned}$$

Thus,

$$\begin{aligned} V_T^U(x) &\le \mathbb{E}\Big[r(x,U_0) + \mathrm{e}^{-\delta} V_{T-1}(X_1) \Big] \\ &= \mathbb{E}\Big[r(x,U_0) + \mathrm{e}^{-\delta} V_{T-1}(f(x,U_0,Y_1)) \Big] \\ &\le \sup_{u \in \mathcal{U}} \{ r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}(f(x,u,Y))] \}\;. \end{aligned}$$

Because U is arbitrary, this shows that

$$V_T(x) \le \sup_{u \in \mathcal{U}} \{ r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}(f(x,u,Y))] \}\;.$$

Fix $\varepsilon > 0$ and $u \in \mathcal{U}$. Let us now consider a strategy $\tilde{U}$ such that, conditioned on $X_1 = f(x, u, Y_1)$, $V_{T-1}(X_1) < V_{T-1}^{\tilde{U}}(X_1) + \varepsilon$. Here, we do not address here the problem of whether we can do that in a measurable way because this point usually is clear in the examples, particularly the examples treated in this book. For conditions on measurability, see, for example, [20]. Let $U_0 = u$ and $U_n = \tilde{U}_{n-1}$. Then

$$\begin{aligned} r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}(f(x,u,Y_1))] &< r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}^{\tilde{U}}(X_1)] + \varepsilon \\ &= V_T^U(x) + \varepsilon \le V_T(x) + \varepsilon \;. \end{aligned}$$

Thus,

$$\sup_{u \in \mathcal{U}} \{r(x,u) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_{T-1}(f(x,u,Y))]\} \le V_T(x) + \varepsilon \;.$$

Because ε is arbitrary, the result follows.

The proof does not explicitly use the finiteness of T. Thus, we can replace T and $T-1$ by ∞, and (1.2) is proved in the same way. □

The result says that we have to maximise the present reward plus the value of the future rewards. If we do that at each time point, we end up with the optimal value. Equation (1.1) can be solved recursively. We will discuss later how to solve Equation (1.2) numerically.

1.1.3 The Optimal Strategy

We next characterise the optimal strategy.

Corollary 1.2. *Suppose that $T < \infty$, $V_T(x)$ is finite, and that for any $t \le T$ there exists $u_t(x)$ such that $u = u_t(x)$ is maximising the right-hand side of (1.1) for $T = t$. We assume that $u_t : E \to \mathcal{U}$ is measurable for each t. Let $U_n = u_{T-n}(X_n)$. Then*

$$V_T(x) = V_T^U(x) \;.$$

Proof. Clearly, $V_T^U(x) \le V_T(x)$. If $T = 0$, then for any strategy $U' = U_0'$

$$V_0^{U'}(x) = \mathbb{E}[r(x, U_0')] \le r(x, u_0(x)) = V_0^U(x) \;,$$

and $V_0(x) \le V_0^U(x)$ follows. We prove the assertion for $T < \infty$ by induction. Suppose that the assertion is proved for $T = n$. Let U' be an arbitrary strategy for $T = n+1$, and use the tilde sign as in the proof of Lemma 1.1. Then

$$\begin{aligned} V_{n+1}^{U'}(x) &= \mathbb{E}[r(x,U_0') + \mathrm{e}^{-\delta}\, \mathbb{E}[V_n^{\tilde{U}'}(f(x,U_0',Y_1)) \mid U_0']] \\ &\le \mathbb{E}[r(x,U_0') + \mathrm{e}^{-\delta}\, \mathbb{E}[V_n(f(x,U_0',Y_1)) \mid U_0']] \\ &\le r(x, u_{n+1}(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_n(f(x,u_{n+1}(x),Y_1))] \\ &= r(x, u_{n+1}(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[V_n^{\tilde{U}}(f(x,u_{n+1}(x),Y_1))] = V_{n+1}^U(x) \;. \end{aligned}$$

This proves that $V_{n+1}(x) \le V_{n+1}^U(x)$. □

We can easily see from the proof that if U_n does not maximise the Bellman equation, then it cannot be optimal. In particular, if $u_n(x)$ does not exist for all $n \le T$, then an optimal strategy cannot exist.

If the time horizon is infinite, the proof of the existence of an optimal strategy is slightly more complicated. But the optimal strategy does not explicitly depend on time and is therefore simpler.

Corollary 1.3. *Suppose that $T = \infty$, $V(x) < \infty$, and that for every x there is a $u(x)$ maximising the right-hand side of* (1.2)*. Suppose further that $u(x)$ is measurable and that*

$$\lim_{n\to\infty} \sup_{U'\in\mathfrak{U}} \mathbb{E}\Bigl[\sum_{k=n}^{\infty} |r(X_k', U_k')|\mathrm{e}^{-\delta k}\Bigr] = 0\;, \tag{1.3}$$

where $X_{n+1}' = f(X_n', U_n', Y_{n+1})$. Let $U_n = u(X_n)$. Then $V^U(x) = V(x)$.

Proof. We first show that for any strategy U' with a value U_0' that does not maximise the right-hand side of (1.2) there exists a strategy U'' with $U_0'' = u(x)$ that yields a larger value. Choose $\varepsilon > 0$. For each initial value $\tilde{x}$ there exists a strategy $\tilde{U}''$ such that $V(\tilde{x}) < V^{\tilde{U}''}(\tilde{x}) + \varepsilon$. Also here we refrain from the technical problem of showing that $\tilde{U}''$ can be chosen in a measurable way, because it is simpler to address this problem for the specific examples. Let U'' be the strategy with $U_0'' = u(x)$ and $U_{n+1}'' = \tilde{U}_n''$, where the initial capital is $\tilde{x} = f(x, u(x), Y_1)$. Thus,

$$\begin{aligned} V^{U'}(x) &= \mathbb{E}[r(x, U_0') + \mathrm{e}^{-\delta}\, \mathbb{E}[V^{\tilde{U}'}(f(x, U_0', Y_1)) \mid U_0']] \\ &\le \mathbb{E}[r(x, U_0') + \mathrm{e}^{-\delta}\, \mathbb{E}[V(f(x, U_0', Y_1)) \mid U_0']] \\ &< r(x, u(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[V(f(x, u(x), Y_1))] = V(x) \\ &< r(x, u(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[V^{\tilde{U}''}(f(x, u(x), Y_1))] + \varepsilon = V^{U''}(x) + \varepsilon\;. \end{aligned}$$

If $\varepsilon < V(x) - V^{U'}(x)$, we have that $V^{U'}(x) < V^{U''}(x)$.

Let $\mathfrak{U}_n$ be the set of all strategies U' with $U_k' = u(X_k)$ for $0 \le k \le n$. We just have shown that $V(x) = \sup_{U'\in\mathfrak{U}_0} V^{U'}(x)$. Suppose that $V(x) = \sup_{U'\in\mathfrak{U}_n} V^{U'}(x)$. Let U' be a strategy such that $U_k' = u(X_k)$ for $k \le n$ and U_{n+1}' does not maximise the right-hand side of (1.2) for $x = X_{n+1}$. Let $\tilde{U}_k' = U_{n+1+k}'$. Then by the argument used for $n = 0$, there is a strategy $\tilde{U}''$ with $\tilde{U}_0'' = u(X_{n+1})$ such that $V^{\tilde{U}''}(X_{n+1}) > V^{\tilde{U}'}(X_{n+1})$. Let U'' be the strategy with $U_k'' = U_k'$ and $U_{n+1+k}'' = \tilde{U}_k''$. Because

$$\begin{aligned} V^{U'}(x) &= \mathbb{E}\Bigl[\sum_{k=0}^{n} r(X_k, u(X_k))\mathrm{e}^{-\delta k} + \mathrm{e}^{-\delta(n+1)} V^{\tilde{U}'}(X_{n+1})\Bigr] \\ &< \mathbb{E}\Bigl[\sum_{k=0}^{n} r(X_k, u(X_k))\mathrm{e}^{-\delta k} + \mathrm{e}^{-\delta(n+1)} V^{\tilde{U}''}(X_{n+1})\Bigr]\;, \end{aligned}$$

we get $V(x) = \sup_{U'\in\mathfrak{U}_{n+1}} V^{U'}(x)$.

Because for all n we have that $V(x) = \sup_{U' \in \mathfrak{U}_n} V^{U'}(x)$, we are now able to prove that $U_n = u(X_n)$ is optimal. Let $\varepsilon > 0$. There exists $n \in \mathbb{N}$ such that $\mathbb{E}[\sum_{k=n+1}^{\infty} |r(X'_k, U'_k)| \mathrm{e}^{-\delta k}] < \varepsilon$ for any strategy U'. Let U' be a strategy in $\mathfrak{U}_n$ such that $V(x) - V^{U'}(x) < \varepsilon$. Then

$$\begin{aligned} V(x) < V^{U'}(x) + \varepsilon &= \mathbb{E}\Bigl[\sum_{k=0}^{\infty} r(X'_k, U'_k)\mathrm{e}^{-\delta k}\Bigr] + \varepsilon \\ &< \mathbb{E}\Bigl[\sum_{k=0}^{n} r(X'_k, U'_k)\mathrm{e}^{-\delta k}\Bigr] + 2\varepsilon \\ &= \mathbb{E}\Bigl[\sum_{k=0}^{n} r(X_k, U_k)\mathrm{e}^{-\delta k}\Bigr] + 2\varepsilon \le V^U(x) + 3\varepsilon \;. \end{aligned}$$

Because ε is arbitrary, it follows that $V(x) \le V^U(x)$. □

The reader should note that the technical condition (1.3) is always fulfilled if the reward $r(x, u)$ is bounded and $\delta > 0$. Alternatively, if one knows the value function $V(x)$, one could just prove that $V^U(x) = V(x)$. Then condition (1.3) is not needed.

1.1.4 Numerical Solutions for $T = \infty$

In general, if $T < \infty$, then the optimal strategy U_n will depend on the time point n. Therefore, the only way to calculate the value function (and the optimal strategy) is to calculate $V_n(x)$ recursively. The situation is different for $T = \infty$. A first idea is to consider the corresponding finite horizon problem and calculate $V_n(x)$. Letting $n \to \infty$ will yield the value function, at least under condition (1.3). The problem is only that we need some criteria when n is large enough. The result below uses this idea [let $v_0(x) = 0$] and shows that $V(x)$ is the fixed point of a contraction.

Lemma 1.4. *Suppose that* $\delta > 0$ *and* $\sup_{x \in E} |V(x)| < \infty$. *Then the operator*

$$\mathcal{V}(v)(x) = \sup_{u \in \mathcal{U}} \bigl\{ r(x, u) + \mathrm{e}^{-\delta}\, \mathbb{E}[v(f(x, u, Y))] \bigr\}$$

is a contraction. In particular, $V(x)$ *is the only bounded solution to* (1.2). *If* $v_0(x)$ *is an arbitrary function and* $v_{n+1}(x) = \mathcal{V}(v_n)(x)$, *then* $\lim_{n\to\infty} v_n(x) = V(x)$. *The convergence rate is geometric, i.e.,*

$$\sup_{x \in E} |V(x) - v_n(x)| \le \mathrm{e}^{-\delta n} \sup_{x \in E} |V(x) - v_0(x)| \;.$$

Remark 1.5. The assumption that $\sup_{x \in E} |V(x)| < \infty$ is quite strong. In some cases it may happen that it is enough to find the value function on a bounded interval and that outside this interval the value function can be calculated as a function of the values inside the interval. Such an example is given in Section 1.2. Another situation may arise when $\mathcal{V}$ is locally a contraction, and some value $v(x_0)$ is known. In this case $v_n(x)$ will first converge close to x_0. If the value is close enough on some interval around x_0, then it will also converge close to this interval. In this way it also is possible to calculate $V(x)$ even though $\mathcal{V}$ is not globally a contraction. An example in continuous time is given in Section 2.3. ■

Proof. Note that by (1.2), the reward $r(x, u)$ must be bounded from above. Define the norm $\|v\| = \sup_{x \in E} |v(x)|$. Let $v_1(x)$ and $v_2(x)$ be two functions. Suppose that $u(x)$ satisfies

$$r(x, u(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[v_1(f(x, u(x), Y))] > \mathcal{V}(v_1)(x) - \varepsilon\ .$$

Then $r(x, u(x))$ is finite and

$$\begin{aligned}\mathcal{V}(v_1)(x) - \mathcal{V}(v_2)(x) &< r(x, u(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[v_1(f(x, u(x), Y))] \\ &\qquad - \{r(x, u(x)) + \mathrm{e}^{-\delta}\, \mathbb{E}[v_2(f(x, u(x), Y))]\} + \varepsilon \\ &= \mathrm{e}^{-\delta}\mathbb{E}[v_1(f(x, u, Y)) - v_2(f(x, u, Y))] + \varepsilon \\ &\le \mathrm{e}^{-\delta}\|v_1 - v_2\| + \varepsilon\ .\end{aligned}$$

Because ε is arbitrary, we have $\mathcal{V}(v_1)(x) - \mathcal{V}(v_2)(x) \le \mathrm{e}^{-\delta}\|v_1 - v_2\|$. Interchanging the rôles of v_1 and v_2, we find that $\|\mathcal{V}(v_1)(x) - \mathcal{V}(v_2)(x)\| \le \mathrm{e}^{-\delta}\|v_1 - v_2\|$. Thus, $\mathcal{V}$ is a contraction. We have already proved in Lemma 1.1 that $\mathcal{V}(V) = V$. If v is a solution to (1.2), then $\|V - v\| = \|\mathcal{V}(V) - \mathcal{V}(v)\| \le \mathrm{e}^{-\delta}\|V - v\|$, and $\|V - v\| = 0$ follows. Finally, for $v_{n+1}(x) = \mathcal{V}(v_n)(x)$, we find that

$$\|v_{n+1} - V\| = \|\mathcal{V}(v_n) - \mathcal{V}(V)\| \le \mathrm{e}^{-\delta}\|v_n - V\| \le \mathrm{e}^{-\delta(n+1)}\|v_0 - V\|\ .$$

Thus, $\|v_n - V\|$ converges to zero. □

Lemma 1.4 provides a possibility to calculate the function V numerically.

An alternative way to find the solution works in the case where both E and $\mathcal{U}$ are finite. This alternative algorithm is often faster than the method of Lemma 1.4, because it solves the problem in finite time.

Let $n = 0$ and choose controls $u_0(x)$ for all $x \in E$. Then

i) Solve the equations

$$V_n(x) = r(x, u_n(x)) + \mathrm{e}^{-\delta}\mathbb{E}[V_n(f(x, u_n(x), Y))] \tag{1.4}$$

in order to find $V_n(x)$.

ii) Choose the largest $u_{n+1}(x)$ (with respect to some ordering of $\mathcal{U}$) maximising

$$r(x, u_{n+1}(x)) + \mathrm{e}^{-\delta}\mathbb{E}[V_n(f(x, u_{n+1}(x), Y))]\ . \tag{1.5}$$

iii) If $u_{n+1}(x) = u_n(x)$ for all $x \in E$, the algorithm terminates; otherwise, increase n by 1 and return to step i).

The next result shows that the procedure works.

Lemma 1.6. *Suppose that E and $\mathcal{U}$ are finite and $\delta > 0$. Then the algorithm described above terminates in finite time m, say. The function $V_m(x)$ is the value function and $u_m(x)$ is an optimal control.*

Proof. Because V_n solves (1.4) and $u_{n+1}(x)$ maximises (1.5), we have

$$\begin{aligned} V_{n+1}(x) &= r(x, u_{n+1}(x)) + \mathrm{e}^{-\delta}\mathbb{E}[V_{n+1}(f(x, u_{n+1}(x), Y))] \;, \\ V_n(x) &\le r(x, u_{n+1}(x)) + \mathrm{e}^{-\delta}\mathbb{E}[V_n(f(x, u_{n+1}(x), Y))] \;. \end{aligned}$$

Taking the difference yields

$$V_{n+1}(x) - V_n(x) \ge \mathrm{e}^{-\delta}\mathbb{E}[V_{n+1}(f(x, u_{n+1}(x), Y)) - V_n(f(x, u_{n+1}(x), Y))] \;.$$

Suppose that $V_{n+1}(x) < V_n(x)$ for some x. We can choose x such that $V_{n+1}(x) - V_n(x) = \inf_{y \in E} V_{n+1}(y) - V_n(y)$. But then

$$\begin{aligned} V_{n+1}(x) - V_n(x) &\ge \mathrm{e}^{-\delta}\mathbb{E}[V_{n+1}(f(x, u_{n+1}(x), Y)) - V_n(f(x, u_{n+1}(x), Y))] \\ &\ge \mathrm{e}^{-\delta}(V_{n+1}(x) - V_n(x)) \end{aligned}$$

would be a contradiction. Thus, $V_n(x)$ is increasing. Because there is only a finite number of controls $u(x)$, there exists m such that $u_m(x) = u_k(x)$ for some $k < m$ for all $x \in E$. Thus, as the solution to (1.4) we also have

$$V_m(x) - V_k(x) = \mathrm{e}^{-\delta}\mathbb{E}[V_m(f(x, u_m(x), Y)) - V_k(f(x, u_m(x), Y))] \;.$$

If we choose x such that $V_m(x) - V_k(x)$ is maximal, we see that $V_k(x) = V_m(x)$. Because $V_n(x)$ is increasing, the algorithm terminates. By the construction of $u_m(x)$, it follows that $V_m(x)$ solves (1.2). By Lemma 1.4, $V_m(x) = V(x)$. Because $r(x, u)$ necessarily is bounded as a function on a finite space, the conditions of Corollary 1.3 are fulfilled and $u_m(x)$ is the optimal strategy. □

In solving the problem numerically, it may happen that the maximiser of (1.5) is not unique. If in this case one chooses the maximal or the minimal u at which the maximum is taken, the algorithm will terminate as soon as the value function is reached.

Bibliographical Remarks

Introductions to discrete-time dynamic programming can be found in such textbooks as [17], [18], [19], [24], [81], [95], [106], or [111]. These textbooks give a more general introduction to the topic than considered in this section.

In our situation we only considered the Markov case. This is because usually a Markov process can be obtained by *Markovization*, that is, by adding more state variables $\{(X_k, J_k)\}$ to the existing process $\{X_k\}$. If the process J is not observable, it is possible to estimate the process by filtering techniques. If one then considers the process $\{(X_k, J'_k)\}$, where J' is the filtered process (for example, the parameters of the distribution of J_k), it is often possible to find the optimal control based on the observable information.

The algorithm in Lemma 1.6 is described by Howard [106].

Discrete-time optimisation in insurance was also considered by Martin-Löf [130].

1.2 Optimal Dividend Strategies in Risk Theory

1.2.1 The Model

Let us consider the following risk model. In a unit interval an insurance company earns some premia and has to pay possible claims. Premia minus payout are denoted by Y_n, where Y_n is an integer. We let $\{Y_n\}$ be an iid sequence, and $p_n = \mathbb{P}[Y_k = n]$ for all $n \in \mathbb{Z}$. As before we work with the natural filtration $\{\mathcal{F}_t = \mathcal{F}_t^Y\}$. We denote by Y a generic random variable and suppose that $\mathbb{E}[|Y|] < \infty$. In order not to deal with a trivial problem we assume that $\mathbb{P}[Y < 0] > 0$; otherwise, the optimal strategy defined below is $U_n = X_n$. At time n the insurer can pay a dividend U_n with $0 \le U_n \le X_n$. The (pre-dividend) surplus process is modelled as $X_0 = x$, and

$$X_{n+1} = X_n - U_n + Y_{n+1} \; .$$

The process is stopped at the time of ruin, $\tau = \inf\{n : X_n < 0\}$. Ruin is a technical expression and does not necessarily mean that the company is bankrupt, but that the capital reserved for the business was not sufficient; see also Appendix D. It will turn out that under the optimal strategy ruin will happen in finite time almost surely.

The goal is now to maximise the expected discounted dividend payments

$$V^U(x) = \mathbb{E}\Bigl[\sum_{n=0}^{\tau-1} \mathrm{e}^{-\delta n} U_n\Bigr] \; ,$$

where $\delta > 0$. The factor δ has to be considered as an additional discounting. Because the $\{Y_n\}$ are iid, the claim sizes and the premium should be measured in values at time 0, i.e., claim sizes and premia are already discounted. The additional discounting of the dividends can be seen as the investor's preference for dividends today to dividends tomorrow.

In order to fit into the setup of Section 1.1, we let $X_{n+1} = X_n$ if $X_n < 0$, $X_{n+1} = X_n - U_n$ if $U_n > X_n$, and $r(X_n, U_n) = U_n 1\!\!1_{X_n \ge 0} 1\!\!1_{U_n \le X_n}$. Thus, $E = \mathbb{R}$ and $\mathcal{U} = [0, \infty)$. We will soon see that we can work on smaller spaces E and $\mathcal{U}$.

Before we consider the problem in more detail, we prove a useful tool.

Lemma 1.7. *Let $\{U_n\}$ and $\{U'_n\}$ be some dividend strategies. We denote by $W_n = \sum_{k=0}^n U_k$ and $W'_n = \sum_{k=0}^n U'_k$ the accumulated dividend payments. If $W_n \ge W'_n$ for all n, then*

$$\sum_{n=1}^{\infty} \mathrm{e}^{-\delta n} U_n \ge \sum_{n=1}^{\infty} \mathrm{e}^{-\delta n} U'_n \; .$$

If $\mathbb{P}[U \ne U'] > 0$, then

$$\mathbb{E}\Bigl[\sum_{n=1}^{\infty} \mathrm{e}^{-\delta n} U_n\Bigr] > \mathbb{E}\Bigl[\sum_{n=1}^{\infty} \mathrm{e}^{-\delta n} U'_n\Bigr] \; .$$

Proof. The discounted dividend payments can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{e}^{-\delta n} U_n &= (1 - \mathrm{e}^{-\delta}) \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \mathrm{e}^{-\delta k} U_n = (1 - \mathrm{e}^{-\delta}) \sum_{k=0}^{\infty} \sum_{n=0}^{k} U_n \mathrm{e}^{-\delta k} \\ &= (1 - \mathrm{e}^{-\delta}) \sum_{k=0}^{\infty} W_k \mathrm{e}^{-\delta k} \; .\end{aligned}$$

The first inequality follows readily. If $\mathbb{P}[U \ne U'] > 0$, there must be an n such that $\mathbb{P}[W_n > W'_n] > 0$. In particular, the first inequality is strict with strictly positive probability. Taking expected values proves the result. □

The result suggests paying dividends as early as possible. One therefore has the trade-off between paying dividends early and getting ruined early, or paying dividends later and getting ruined later.

We first obtain some upper and lower bounds for $V(x) = \sup_{U \in \mathfrak{U}} V^U(x)$.

Lemma 1.8. i) *The function $V(x)$ is bounded by*

$$x + \frac{\mathbb{E}[Y^+] \mathrm{e}^{-\delta}}{1 - p_+ \mathrm{e}^{-\delta}} \le V(x) \le x + \frac{\mathbb{E}[Y^+] \mathrm{e}^{-\delta}}{1 - \mathrm{e}^{-\delta}} \; , \qquad (1.6)$$

where $p_+ = \mathbb{P}[Y \ge 0]$ and $Y^+ = Y \vee 0$ is the positive part of Y.

ii) *$V(x)$ is strictly increasing and $V(x) - V(y) \ge x - y$ for any $x \ge y$.*

iii) *If $\mathbb{P}[Y > 0] = 0$, then $V(x) = x$, and the optimal strategy is $U_0 = x$, resulting in $U_n = 0$ for $n \ge 1$.*

Proof. Consider the following "pseudo strategy" $U_0 = x$ and $U_n = Y_n^+$ for $n \geq 1$. Suppose that we do not stop this process at ruin. Then $X_n - U_n \leq 0$ for all n. Moreover, for any strategy $\{U'_n\}$ with $U'_k = 0$ for $k \geq \tau'$, we have for $n < \tau'$

$$X'_n = x + \sum_{k=1}^{n}(Y_k - U'_{k-1}) \leq x - U'_0 + \sum_{k=1}^{n-1}(Y_k^+ - U'_k) + Y_n = \sum_{k=0}^{n-1}(U_k - U'_k) + Y_n \; .$$

Because $X'_n - Y_n = X'_{n-1} - U'_{n-1} \geq 0$, it follows that $\sum_{k=0}^{n} U'_k \leq \sum_{k=0}^{n} U_k$. By Lemma 1.7,

$$V^{U'}(x) \leq x + \mathbb{E}\Bigl[\sum_{n=1}^{\infty} \mathrm{e}^{-\delta n} Y_n^+\Bigr] = x + \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - \mathrm{e}^{-\delta}} \; ,$$

yielding the upper bound. Using the strategy U, ruin occurs the first time where $Y_n < 0$. Thus, $\mathbb{P}[\tau = n + 1] = (1 - p_+)p_+^n$. The value of the strategy U is

$$\begin{aligned} V^U(x) - x &= \mathbb{E}\Bigl[\sum_{n=1}^{\tau-1} \mathrm{e}^{-\delta n} Y_n\Bigr] = (1 - p_+) \sum_{n=1}^{\infty} p_+^n \sum_{k=1}^{n} \mathrm{e}^{-\delta k} \mathbb{E}[Y \mid Y \geq 0] \\ &= (1 - p_+) \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} p_+^n \mathrm{e}^{-\delta k} \mathbb{E}[Y \mid Y \geq 0] \\ &= \sum_{k=1}^{\infty} p_+^k \mathrm{e}^{-\delta k} \mathbb{E}[Y \mid Y \geq 0] = \frac{\mathbb{E}[Y \mid Y \geq 0] p_+ \mathrm{e}^{-\delta}}{1 - p_+ \mathrm{e}^{-\delta}} \\ &= \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - p_+ \mathrm{e}^{-\delta}} \; . \end{aligned}$$

This proves the lower bound.

Let U be a strategy used for initial capital y. Apply the strategy U' with $U'_0 = U_0 + x - y$ and $U'_n = U_n$ for $n \geq 1$ to the initial capital x. Then $X'_n = X_n$ for all $n \geq 1$, and $V(x) \geq V^{U'}(x) = V^U(y) + x - y$. Taking the supremum over all strategies U yields the second assertion.

Now if $\mathbb{P}[Y > 0] = 0$, then $\mathbb{E}[Y^+] = 0$ and $V(x) = x$ follows. One can readily see that the claimed strategy has the maximal value $V(x) = x$. □

We next prove that one can restrict to integer initial capital and to integer dividend payments. This will simplify our spaces E and $\mathcal{U}$. We denote by $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \leq x\}$ the integer part of x. In the rest of this section we only consider strategies with $U_n = 0$ for $n \geq \tau$.

Lemma 1.9. *Let U be a strategy and X be the corresponding surplus process. Then there exists a strategy U' such that the corresponding surplus process X' fulfils $X'_n = \lfloor X_n \rfloor$ for $n \geq 1$ and $V^{U'}(x) \geq V^U(x)$. If $\mathbb{P}[X \neq X'] > 0$, then the strict inequality holds.*

Proof. Define $U'_0 = x - \lfloor x - U_0 \rfloor$ and $U'_n = \lfloor X_n \rfloor - \lfloor X_n - U_n \rfloor$. By the definition, $0 \le U'_n \le \lfloor X_n \rfloor$. For the process X' we obtain $X'_0 = x$, $X'_1 = x + Y_1 - U'_0 = Y_1 + \lfloor x - U_0 \rfloor = \lfloor X_1 \rfloor$ because $Y_1 \in \mathbb{Z}$. By induction,

$$X'_{n+1} = \lfloor X_n \rfloor + Y_{n+1} - U'_n = \lfloor X_n - U_n \rfloor + Y_{n+1} = \lfloor X_{n+1} \rfloor \; .$$

By the definition of ruin, we get that $\tau' = \tau$, i.e., ruin occurs at the same time as for the original strategy. From

$$x + \sum_{k=0}^{n} (Y_{k+1} - U_k) = X_{n+1} \ge \lfloor X_{n+1} \rfloor = x + \sum_{k=0}^{n} (Y_{k+1} - U'_k) \; ,$$

we conclude that $\sum_{k=0}^{n} U'_n \ge \sum_{k=0}^{n} U_n$. The assertion now follows from Lemma 1.7, noting that the dividend stream is different if $X \ne X'$. □

If $x \notin \mathbb{N}$, the above result shows that $V(x) = V(\lfloor x \rfloor) + x - \lfloor x \rfloor$ because $x - U'_0 \in \mathbb{N}$. We therefore can restrict to $E = \mathbb{Z}$. If $x \in \mathbb{N}$, it is not optimal to choose a dividend with $U_n \notin \mathbb{N}$. We therefore can also restrict to $\mathcal{U} = \mathbb{N}$.

1.2.2 The Optimal Strategy

We proved in Lemma 1.1 that the function $V(x)$ fulfils the dynamic programming principle. Equation (1.2) reads

$$V(x) = \sup_{0 \le u \le x} \Bigl\{ u + \mathrm{e}^{-\delta} \sum_{j=-(x-u)}^{\infty} p_j V(x - u + j) \Bigr\} \; . \tag{1.7}$$

Here $V : \mathbb{N} \to \mathbb{R}_+$. Because there are only a finite number of values over which the supremum is taken, there exists for each $x \in \mathbb{N}$ a value $u(x)$ such that

$$V(x) = u(x) + \mathrm{e}^{-\delta} \sum_{j=-(x-u(x))}^{\infty} p_j V(x - u(x) + j) \; . \tag{1.8}$$

In order to define the optimal strategy in a unique way, we take the largest u fulfilling (1.8) if $u(x)$ is not uniquely defined.

Theorem 1.10. i) *The strategy $U_n = u(X_n)$ is an optimal strategy.*

ii) *For all $x \in \mathbb{N}$, the equality $u(x - u(x)) = 0$ holds and $u(y) = u(x) - (x - y)$ for all $x - u(x) \le y \le x$.*

iii) *For all $x \in \mathbb{N}$, one has $V(x) = V(x - u(x)) + u(x)$. In particular, $V(x) - V(y) = x - y$ for all $x - u(x) \le y \le x$.*

iv) *The number $x_0 := \sup\{x : u(x) = 0\}$ is finite, i.e., for x large enough a dividend should be paid immediately.*

v) *For all* $x \ge x_0$, $u(x) = x - x_0$, *and* $V(x) = V(x_0) + x - x_0$, *i.e., it is not optimal to have a capital larger than* x_0 *immediately after dividends are paid.*

vi) *Under the optimal dividend strategy ruin occurs almost surely.*

Proof. i) We need to show (1.3). As in the proof of Lemma 1.7, we find for any strategy that

$$\sum_{k=n}^{\infty} \mathrm{e}^{-\delta k} U_k = (1 - \mathrm{e}^{-\delta}) \sum_{m=n}^{\infty} \sum_{k=n}^{m} U_k \mathrm{e}^{-\delta m} .$$

The pseudo-strategy

$$U'_k = \begin{cases} 0, & \text{if } k < n, \\ {X'_n}^+, & \text{if } k = n, \\ Y_k^+, & \text{if } k > n, \end{cases}$$

majorises $\sum_{k=n}^{m} U_k$ for any strategy U. Therefore,

$$\mathbb{E}\Big[\sum_{k=n}^{\infty} \mathrm{e}^{-\delta k} U_k\Big] \le \mathbb{E}[(X'_n)^+]\mathrm{e}^{-\delta n} + \frac{\mathrm{e}^{-\delta(n+1)}\mathbb{E}[Y^+]}{1 - \mathrm{e}^{-\delta}} .$$

Because $\mathbb{E}[(X'_n)^+] \le x + n\mathbb{E}[Y^+]$, the assertion follows from Corollary 1.3.

ii) and iii) The assertion is trivial if $y = x$. Let $x - u(x) \le y < x$. For initial capital y, there is the possibility to pay a dividend $u(x) - (x - y)$; thus,

$$\begin{aligned} V(y) &\ge u(x) - (x - y) + \mathrm{e}^{-\delta} \sum_{j=-(y-(u(x)-(x-y)))}^{\infty} p_j V(y - (u(x) - (x - y)) + j) \\ &= V(x) - (x - y) . \end{aligned}$$

For initial capital x, paying the dividend $u(y) + x - y$ gives

$$\begin{aligned} V(x) &\ge u(y) + x - y + \mathrm{e}^{-\delta} \sum_{j=-\{x-(u(y)+(x-y))\}}^{\infty} p_j V(x - (u(y) + (x - y)) + j) \\ &= V(y) + x - y . \end{aligned}$$

Thus, equality must hold. By our convention to take the largest possible u, we find that $u(x) = u(y) + x - y$. In particular, $V(y) = V(x) - (x - y)$. That $u(x - u(x)) = 0$ and $V(x) = u(x) + V(x - u(x))$ follows for $y = x - u(x)$.

iv) Suppose that $u(x) = 0$. Applying (1.6) and (1.8), we find that

$$\begin{aligned} x + \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - p_+\mathrm{e}^{-\delta}} &\le V(x) = \mathrm{e}^{-\delta} \sum_{j=-x}^{\infty} p_j V(x + j) \\ &\le \mathrm{e}^{-\delta} \sum_{j=-x}^{\infty} p_j \Big(x + j + \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - \mathrm{e}^{-\delta}}\Big) \\ &\le \mathrm{e}^{-\delta} x + \mathrm{e}^{-\delta}\Big(\mathbb{E}[Y^+] + \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - \mathrm{e}^{-\delta}}\Big) = \mathrm{e}^{-\delta} x + \frac{\mathbb{E}[Y^+]\mathrm{e}^{-\delta}}{1 - \mathrm{e}^{-\delta}} . \end{aligned}$$

This yields

$$x \le \frac{\mathbb{E}[Y^+]\mathrm{e}^{-2\delta}(1-p_+)}{(1-\mathrm{e}^{-\delta})^2(1-p_+\mathrm{e}^{-\delta})}\;,$$

i.e., $x_0 < \infty$. Note that $u(0) = 0$, showing that $\{x : u(x) = 0\} \neq \emptyset$. Therefore, $u(x_0) = 0$.

v) From ii) we conclude that $u(x - u(x)) = 0$, and therefore $u(x) \ge x - x_0$ follows. We also obtain $0 = u(x_0) = u(x) - (x - x_0)$. From part iii) we conclude that $V(x) = V(x_0) + u(x) = V(x_0) + x - x_0$.

vi) We have

$$\mathbb{P}[Y_1 < 0, Y_2 < 0, \dots, Y_{x_0+1} < 0] = (1-p_+)^{x_0+1} > 0\;.$$

Thus, by the strong law of large numbers, $\{Y_{k(x_0+1)+1} < 0, Y_{k(x_0+1)+2} < 0, \dots, Y_{(k+1)(x_0+1)} < 0\}$ holds for some k. But then ruin must occur. □

The result gives us an alternative way to characterise the optimal strategy $u(x)$. Clearly, $u(0) = 0$. For $x \ge 1$ we obtain

$$u(x) = \sup\{n \in \mathbb{N} : V(x) = V(x-n) + n\}\;.$$

Indeed, by part iii) of the above theorem, $V(x) = V(x-n) + n$ for $n \le u(x)$. If $u(x) = x$, the statement is proved. If $V(x) = V(x - u(x) - 1) + u(x) + 1$, then we could conclude from (1.7) for $V(x)$ and for $V(x - u(x) - 1)$ that $u = u(x - u(x) - 1) + u(x) + 1$ maximises the right-hand side of (1.7) for $V(x)$. But this is not possible, because we assumed that $u(x)$ is the maximal value. Finally, for $2 \le n \le x - u(x)$ we have

$$\begin{aligned} V(x) &> V(x-u(x)-1) + u(x) + 1 \ge (V(x-u(x)-n) + n - 1) + u(x) + 1 \\ &= V(x-u(x)-n) + u(x) + n\;, \end{aligned}$$

by Lemma 1.8.

Let us now consider the numerical algorithm of Lemma 1.4. Choosing

$$x_1 = \left\lfloor \frac{\mathbb{E}[Y^+]\mathrm{e}^{-2\delta}(1-p_+)}{(1-\mathrm{e}^{-\delta})^2(1-p_+\mathrm{e}^{-\delta})} \right\rfloor, \tag{1.9}$$

we can restrict to functions $v(x) : \mathbb{N} \to \mathbb{R}_+$ with $v(x_1 + n) = v(x_1) + n$ for all $n \in \mathbb{N}$. The contraction operator then becomes

$$\mathcal{V}(v)(x) = \sup_{0 \le u \le x} u + \mathrm{e}^{-\delta} \sum_{j=-(x-u)}^{\infty} p_j v(x - u + j)\;,$$

for $x \le x_1$ and $\mathcal{V}(v)(x) = \mathcal{V}(v)(x_1) + x - x_1$ for $x > x_1$. In the algorithm it is not necessary to consider all possible u. From the proof of Theorem 1.10 parts ii) and iii), we can construct the values $u_v(x)$ maximising the right-hand