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# The Theory of the Moiré Phenomenon

## Volume I: Periodic Layers

Second Edition

by

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*To my parents*

*No one admires Fourier more than I do. It is the only entertaining mathematical work I ever saw. Its lucidity has always been admired. But it was more than lucid. It was luminous. Its light showed a crowd of followers the way to a heap of new physical problems.*

**Oliver Heaviside** [Heaviside71 p. 32]

**Front cover image:** A heart-shaped moiré which is generated in the off-centered superposition of two circular gratings with slightly different radial periods. See Problem 11-8 and Fig. 11.4(c).

**Back cover images:** Interesting moiré effects in the superposition of two bell-shaped curvilinear gratings. See Figs. 10.34(c),(d).

# Contents

## Appendices

# Preface to the Second Edition

Since the first edition of this book was published several new developments have been made in the field of the moiré theory. The most important of these concern new results that have recently been obtained on moiré effects between correlated aperiodic (or random) structures, a subject that was completely absent in the first edition, and which appears now for the first time in a second, separate volume.

This also explains the change in the title of the present volume, which now includes the subtitle “Volume I: Periodic Layers”. This subtitle has been added to clearly distinguish the present volume from its new companion, which is subtitled “Volume II: Aperiodic Layers”. It should be noted, however, that the new subtitle of the present volume may be somewhat misleading, since this book also treats (in Chapters 10 and 11) moiré effects between *repetitive* layers, which are, in fact, geometric transformations of periodic layers, that are generally no longer periodic in themselves. The most suitable subtitle for the present volume would therefore have been “Periodic or Repetitive Layers”, but in the end we have decided on the shorter version.

Although this revised edition maintains the general structure of the original book, it also includes some important improvements. It provides additional topics that were not explicitly treated in the first edition, such as the hybrid (1,-1)-moiré effects with 2D intensity profiles (now in Sec. C.14 of Appendix C), the moiré effects between hexagonal screens (now in Sec. C.15 of Appendix C) or the extension of the indicial equations method to the case of 2D screens (in Sec. 11.2). The present edition of the book also includes several new figures and some new or revised problems. New references have been added throughout the book, and all the Internet references have been verified and updated. And finally, cross-references have been added wherever appropriate to the second volume, and in particular to those of its appendices which may be of interest to readers of the present book. Note, however, that the two volumes are basically independent of each other. Each volume thus contains its own Glossary, List of notations and symbols, References and Index.

In preparing this second edition, we have also taken the opportunity to correct errors and typos that crept into the original edition of the book. However, some errors may have been overlooked, and some may have been inadvertently added in this new edition. Such errors, when detected, will be listed along with their corrections in the Internet site of the book, and we therefore encourage readers to inform us of any errors they may find.

The material in this book is based on the author's personal research at the Swiss Federal Institute of Technology of Lausanne (EPFL: *Ecole Polytechnique Fédérale de Lausanne*), and on his Ph.D. thesis (thesis No. 1341: "Analysis of Moiré Patterns in Multi-Layer Superpositions") which won the best EPFL thesis award in 1995.

This work would not have been possible without the support and the excellent research environment provided by the EPFL. In particular, the author wishes to express his gratitude to Prof. Roger D. Hersch, the head of the Peripheral Systems Laboratory of the EPFL, for his encouragement throughout the different stages of this project. Many thanks are also due to the publishers for their helpfulness and availability throughout the publishing cycle.

# From the Preface to the First Edition

Who has not noticed, on one occasion or another, those intriguing geometric patterns which appear at the intersection of repetitive structures such as two far picket fences on a hill, the railings on both sides of a bridge, superposed layers of fabric, or folds of a nylon curtain? This fascinating phenomenon, known as the *moiré effect*, has found useful applications in several fields of science and technology, such as metrology, strain analysis or even document authentication and anti-counterfeiting. However, in other situations moiré patterns may have an unwanted, adverse effect. This is the case in the printing world, and, in particular, in the field of colour reproduction: moiré patterns which may be caused by the dot-screens used for colour printing may severely deteriorate the image quality and turn into a real printer's nightmare.

The starting point of the work on which this book is based was, indeed, in the research of moiré phenomena in the context of the colour printing process. The initial aim of this research was to understand the nature and the causes of the superposition moiré patterns between regular screens in order to find how to avoid, or at least minimize, their adverse effect on colour printing. This interesting research led us, after all, to a much more far-reaching mathematical understanding of the moiré phenomenon, whose interest stands in its own right, independently of any particular application. Based on these results, the present book offers a profound insight into the moiré phenomenon and a solid theoretical basis for its full understanding. In addition to the question of moiré minimization between regular screens, the book covers many interesting and important subjects such as the navigation in the moiré parameter space, the intensity profile forms of the moiré, its singular states, its periodic or almost-periodic properties, the phase of the superposed layers and of each of the eventual moirés, the relations between macro- and micro-structures in the superposition, polychromatic moirés between colour layers, etc. All this is done in the most general way for any number of superposed layers having any desired forms (line-gratings, dot-screens with any dot shape, etc.). The main aim of this book is, therefore, to present all this material in the form of a single, unified and coherent text, starting from the basics of the theory, but also going in depth into recent research results and showing the new insight they offer in the understanding of the moiré phenomenon.

Fourier-based tools are but a natural choice when dealing with periodic phenomena; and, indeed, our approach is largely based on the Fourier theory. We consider each of the superposed layers as a function (*reflectance* or *transmittance* function) having values in the range between 0 and 1. We study the original layers, their superpositions, and their moiré effects by analyzing their properties both in the image domain and in the spectral, frequency domain using the Fourier theory. Further results are obtained by investigating the spectrum using concepts from geometry of numbers and linear algebra, and by interpreting the corresponding image-domain properties by means of the theories of periodic and almost-periodic functions. However, no prior knowledge of these fields of mathematics is assumed, and the required background is fully introduced in the text (in Chapter 5 and in Appendices A and B, respectively). The only prerequisite mathematical background is limited to undergraduate mathematics and an elementary familiarity with the Fourier theory (Fourier series, Fourier transforms, convolutions, Dirac impulses, etc.).

This book presents a comprehensive approach that provides a full explanation of the various phenomena which occur in the superposition, both in the image and in the spectral domains. This includes not only a quantitative and qualitative analysis of the moiré effect, but also the synthesis of moiré effects having any desired geometric forms and intensity profiles. In the first chapters we present the basic theory which covers the most fundamental case, namely: the superposition of monochrome, periodic layers. In later chapters of the book we extend the theory to the even more fascinating cases of polychromatic moirés and moirés between repetitive, non-periodic layers. Throughout the whole text we favour a pictorial, intuitive approach supported by mathematics, and the discussion is accompanied by a large number of figures and illustrative examples, some of which are visually striking and even spectacular.

This book is intended for students, scientists, and engineers wishing to widen their knowledge of the moiré effect; on the other hand it also offers a beautiful demonstration of the Fourier theory and its relationship with other fields of mathematics and science. Teachers and students of imaging science will find moiré phenomena to be an excellent didactic tool for illustrating the Fourier theory and its practical applications in one or more dimensions (Fourier transforms, Fourier series, convolutions, etc.). People interested in the various moiré applications and moiré-based technologies will find in this book a theoretical explanation of the moiré phenomenon and its properties. Readers interested in mathematics will find in the book a novel approach combining Fourier theory and geometry of numbers; physicists and crystallographers may be interested in the intricate relationship between the macro- and microstructures in the superposition and their relation to the theories of periodic and almost-periodic functions; and colour scientists and students will find in the polychromatic moirés a vivid demonstration of the additive and subtractive principles of colour theory. Finally, the occasional reader will enjoy the beauty of the effects demonstrated throughout this book, and — it is our hope — may be tempted to learn more about their nature and their properties.

# Chapter 1

# Introduction

## 1.1 The moiré effect

The moiré effect is a well known phenomenon which occurs when repetitive structures (such as screens, grids or gratings) are superposed or viewed against each other. It consists of a new pattern of alternating dark and bright areas which is clearly observed at the superposition, although it does not appear in any of the original structures.[1]

The moiré effect occurs due to an interaction between the overlaid structures. It results from the geometric distribution of dark and bright areas in the superposition: areas where dark elements of the original structures fall on top of each other appear brighter than areas in which dark elements fall between each other and fill the spaces better (see Fig. 1.1).

Because of its extreme sensitivity to the slightest displacements, variations, or distortions in the overlaid structures the moiré phenomenon has found a vast number of applications in many different fields. For example, in strain analysis moirés are used for the detection of slight deflections or object deformations, and in metrology moirés are used in the measurement of very small angles, displacements or movements [Patorski93; Kafri89; Shepherd79; Takasaki70; Durelli70; Theocaris69]. Among the numerous applications of the moiré one can mention fields as far apart as optical alignment [King72], crystallography [Oster63 p. 58], and document anti-counterfeiting [Renesse05 pp. 146–161]. Moiré effects have been used also in art [Oster65; Witschi86; Durelli70 pp. xiii–xxxviii], and even just for fun, enjoying their various intriguing shapes.

However, in other situations moiré patterns prove to be an undesired nuisance, and many efforts may be required to avoid or to eliminate them. This is the case, for example, in the printing world, and in particular in the field of colour image reproduction, where moiré patterns may appear between the dot-screens used for colour printing and severely corrupt the resulting image.

Clearly, mastering the moiré theory is essential for the proper use and control of moiré-based techniques, as well as for the elimination of unwanted moirés. It is the aim of this book, therefore, to provide the reader with a full theoretical understanding of the moiré phenomenon.

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[1] The term *moiré* comes from the French, where it originally referred to watered silk, a glossy cloth with wavy, alternating patterns which change form as the wearer moves, and which is obtained by a special technique of pressing two watered layers of cloth together. Note that the term moiré does *not* refer to a presumed French physicist who studied moiré patterns, as has sometimes been stated (either mistakenly or humorously; see, for example, [Coudray91] and [Weber73]). Therefore the term moiré should not be written with a capital letter.

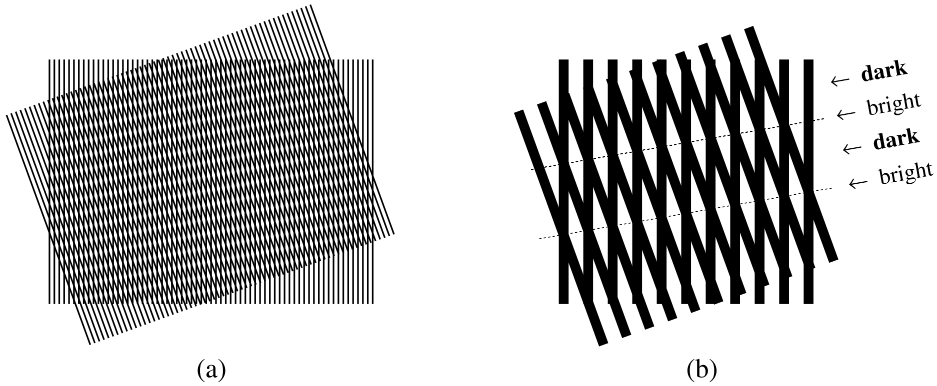



**Figure 1.1:** (a) Alternating dark and bright areas which form the moiré effect in the superposition of two identical, mutually rotated line-gratings. (b) Enlarged view.

## 1.2 A brief historical background

The moiré phenomenon has been known for a long time; it was already used by the Chinese in ancient times for creating an effect of dynamic patterns in silk cloth. However, modern scientific research into the moiré phenomenon and its application started only in the second half of the 19th century with pioneering works such as [Rayleigh74] and [Righi87]. During almost a full century since then the theoretical analysis of moiré phenomena has been based on purely geometric or algebraic approaches (see Sec. 2.1). Based on these approaches many special purpose mathematical developments have been devised for the needs of specific applications such as strain analysis, metrology, etc. More recently several new approaches have been proposed for studying moiré phenomena, based, respectively, on non-standard analysis [Harthong81], on elementary geometry and potential theory [Firby84], or on algebraic geometry [Kendig80]. It was, however, undoubtedly the Fourier-based approach that most significantly contributed to the theoretic investigation of the moiré phenomenon.

The first significant steps in the introduction of the Fourier theory to the study of moiré phenomena can be traced back to the 1960s and 1970s. This pioneering work can be divided into two distinct stages: First came the use of Fourier series decompositions, purely in the image domain, for representing the original repetitive structures, their superpositions and their moirés (see, for example, [Lohmann67]). Only then were introduced further elements of the Fourier theory, such as the dual role of the image and the spectral domains [Bryngdahl74; Bryngdahl75], and the interpretation of the moiré in spectral terms as an aliasing phenomenon [Legault73]. Since then the Fourier approach has been used occasionally for the needs of some particular applications [Steinbach82; Takeda82; Morimoto88], but no systematic effort has been made to explore the full possibilities it offers. A possible reason for this fact may be that, as we show in this book, any such systematic attempt inevitably leads one to some branches of mathematics that are not very widespread, notably the theory of almost-periodic functions (see Appendix B) and the theory of geometry of numbers (see Chapter 5). The present book offers, for the first time, a full scale theoretical exploration of the moiré phenomenon which is based on the Fourier approach, and it contains several new results, both qualitative and quantitative, which have been obtained thanks to this fruitful approach.

A more detailed historical account on the research of the moiré phenomenon can be found in [Patorski93]. This book also gives a survey of various applications of the moiré effect, and an extensive bibliography on the subject. A collection of key scientific papers, both new and old, on the moiré effect and its applications can be found in [Indebetouw92].

## 1.3 The scope of the present book

The theory of the moiré phenomenon is an interdisciplinary domain whose range of applications is extremely vast. Its various theoretical and practical aspects concern the fields of physics, optics, mechanics, mathematics, image reproduction, colour printing, the human visual system, and numerous other fields. It would be in order, therefore, to clearly delimit here the scope of our present work.

Our main aim in this book is to present the moiré theory in the form of a unified and coherent text, starting from the basics of the theory, but also going in depth into recent research results. Among other topics we will discuss questions such as the minimization of moirés between regular screens, the moiré profile forms, its singular states, its periodic or almost-periodic properties, the phase of the superposed layers and of each of their eventual moirés, the relations between macro- and microstructures in the superposition, polychromatic moirés, moirés between repetitive, non-periodic layers, etc. These questions will be treated in the most general way, for any number of superposed layers having any desired forms (line-gratings, dot-screens with any dot shape, etc.).

It is clear, however, that it was impossible to include all the interesting material related to the moiré phenomenon in the present book. In the following list we enumerate some of the main points which have remained beyond the scope of this volume.

- First of all, we limit ourselves here to the analysis of moiré effects in the superposition of periodic or repetitive layers (like straight or curved line-gratings, dot-screens, etc.). Moiré effects between aperiodic or random layers are treated in *Vol. II* of the present work. Other types of moiré phenomena, such as moirés between almost-periodic or fractal structures (like Penrose tilings [Steinhardt90] or Cantor structures [Zunino03]), temporal moirés [Yule67 p. 330], etc., are not directly addressed in the present book, although they can be considered as natural extensions of the theory presented here.
- We do not consider here effects such as light scattering, light diffraction through the gratings, or any other physical questions concerning the nature of light (coherent/

incoherent) and its influence on the moiré [Patorski93 Chapter 4]. In particular, we will always assume that the line spacing in each grating is coarse enough for diffraction effects to be ignored [Durelli70 pp. 16, 35–42; Ebbeni70 p. 338; Theocaris75 p. 280].

- We do not consider here, either, the discrete nature of gratings and screen elements which are produced on digital devices such as laser printers, high-resolution filmsetters, etc., and the influence of this discrete nature on the moiré (this question is discussed in [Réveillès91 pp. 176–183]). The jagged aspect of discrete lines or dots is considered here as a real-world constraint, and we try to avoid it (or at least to reduce its influence) by producing our samples on appropriate devices with high enough resolutions (normally, at least 600 dots per inch).

- We suppose here that the different layers are superposed *in contact* (see [Post94 p. 90]), for example by overprinting, and we ignore the possible effects of the distance between the layers on the resulting moiré patterns, such as parallax-related phenomena [Huck03; Huck04] or the Talbot effect [Latimer93; Post94 pp. 76–78; Kafri90 pp. 102–103]; see also Sec. 1.8.6 in [Durelli70].

- We do not treat explicitly kinematic aspects of the moiré patterns, such as the speed of movement in the superposed layers and the speed of the resulting evolution in the moiré patterns (see, for example, Problems 7-6 and 7-7 at the end of Chapter 7). But although the kinematic aspects of the moiré theory are not explicitly developed here, they can be obtained in a rather straightforward manner by introducing the notion of time, and by considering shifts, rotations or any other layer transformations as functions of this new parameter.

- We also intentionally content ourselves here with a simplified model of the human visual system (see Sec. 2.2), and we avoid going any further into the complex questions related to the modelization of the human visual system and its performance in an inhomogeneous environment (like the perception of a moiré pattern on the irregular background of a screen superposition). More details about human vision and its modelization can be found, for example, in [Wandell95] and [Daly92].

- Finally, we usually prefer a pictorial, intuitive approach supported by mathematics over a rigorous mathematical treatment. In many cases we give informal demonstrations rather than formal proofs, or defer detailed derivations to an appendix.

It should be noted that although we occasionally use questions related to image reproduction to illustrate our discussion, this book has not been written with any specific moiré application in mind. In fact, our principal aim is to present the theoretical aspects of the moiré phenomenon in a general, application-independent way. Consequently a full discussion on the various applications of the moiré remains beyond the scope of the book; this material can be found in other books such as [Patorski93] or [Post94]. However, we felt that presenting the moiré theory without giving at least some flavour of its numerous applications would not serve the interest of the reader. Therefore, as a reasonable compromise, we have included among the problems at the end of each chapter some of the main applications of the theory being covered, along with additional references for the benefit of the interested readers. This should give the reader a general idea about the vast range of applications that the moiré effect has found in various different fields.

## 1.4 Overview of the following chapters

Chapter 2 lays the foundations for the entire book. This chapter presents the Fourier spectral approach that is the basis for our investigation of the moiré effect, and shows, step by step and in a systematic way, how this approach explains the moiré phenomenon between superposed layers: Starting with the simplest case, the superposition of cosinusoidal gratings, it gradually proceeds through the cases of binary gratings and square grids to the superposition of dot-screens. It also presents the notational system that we use for the identification, classification and labeling of the moiré effects, and introduces several fundamental notions such as the order of a moiré, singular moiré states, etc.

Chapter 3 presents the problem of moiré minimization, namely: the question of finding stable moiré-free combinations of superposed screens. In this chapter we focus on the moiré phenomenon from a different point of view: we introduce the moiré parameter space, and show how changes in the parameters of the superposed layers vary the moiré patterns in the superposition. This leads us to an algorithm for moiré minimization which provides stable moiré-free screen combinations that can be used, for example, for colour printing. Other methods for fighting unwanted moirés are also briefly reviewed (see, in particular, the problem section at the end of the chapter).

In Chapter 4 we show how, by considering not only the impulse locations in the spectrum but also their amplitudes, the Fourier-based approach provides a full quantitative analysis of the moiré intensity profiles, in addition to the qualitative geometric analysis of the moiré patterns which is already offered by the earlier classical approaches. We analyze the profile forms and intensity levels of moirés of any order which are obtained in the superposition of any periodic layers (gratings, dot-screens, etc.), and we show how they can be derived analytically from the original superposed structures, either in the spectral domain or directly in the image domain. We show how this analysis method can fully explain the surprising profile forms of the moiré patterns that are generated in the superposition of screens with any desired dot shapes, and how it can be used to synthesize moiré effects with any desired intensity profiles.

In Chapter 5 we set up a new algebraic formulation that will help us better understand the structure of the spectrum-support of the superposition, based on concepts from the theory of geometry of numbers and on linear algebra. In this discussion we completely ignore the impulse amplitudes, and we only consider their indices, their geometric locations, and the relations between them. This algebraic abstraction provides a new, important insight into the properties of the spectrum of the layer superposition and its moiré effects. Yet, this chapter can be skipped upon first reading and revisited later, when a deeper understanding is required.

In Chapter 6 we reintroduce the impulses on top of the spectrum-support, and we investigate the properties of the impulse amplitudes that are associated with the algebraic structures discussed in Chapter 5. Through the Fourier theory we see how both the structure and the amplitude properties of the spectral domain are related to properties of the layer superposition and their moirés in the image domain. In particular, we show the fundamental relationship between the Fourier expression of the layer superposition and the algebraic structure of the spectrum support.

In Chapter 7 we introduce the notion of phase, and we investigate what happens to the layer superposition (and particularly to the moiré effects) when the superposed layers are shifted on top of each other while keeping their angles and frequencies unchanged.

In Chapter 8 we focus our attention to the microstructure which occurs in the superposition and its relationship with the macro-moirés. We will see, in particular, that any moiré effect in the superposition is generated, microscopically speaking, by a repetitive alternation between zones of different microstructure in the superposition; when observed from a distance, this microstructure alternation is perceived as a repetitive gray level alternation in the superposition, i.e., as a macroscopic, visible moiré pattern.

In Chapter 9 we extend our Fourier-based approach to polychromatic moirés in the superposition of any coloured periodic layers. This will be done by considering the full colour spectrum of each point in any of the superposed layers; we will be dealing, therefore, with both colour spectra and Fourier spectra simultaneously. This extension of our theory will allow a full qualitative and quantitative analysis of moirés in colour, and it will enable us to synthesize moirés of any desired colours.

In Chapter 10 we further extend the scope of our Fourier-based approach, this time, to the superposition of repetitive, non-periodic layers such as curvilinear gratings or curved screens. We will see that although the Fourier spectrum in such cases are no longer purely impulsive, the fundamental principles of the theory remain valid in these cases, too. In particular, we will obtain the *fundamental moiré theorem*, which is a generalization of the results that were obtained in Chapter 4. We will see also how this approach can be used to synthesize moiré effects having any desired geometric layout and any intensity profile.

Finally, in Chapter 11 we briefly review some of the most widely used classical methods of moiré analysis, which are not directly based on the Fourier approach. We show that these alternative methods are, in fact, encompassed by the spectral approach, so that the results they can provide are only partial to the full information which can be obtained by the spectral approach. Nevertheless, these methods remain very useful in many real-world applications in which the use of the full scale spectral approach may prove to be impractical.

The main body of the book is accompanied by several appendices:

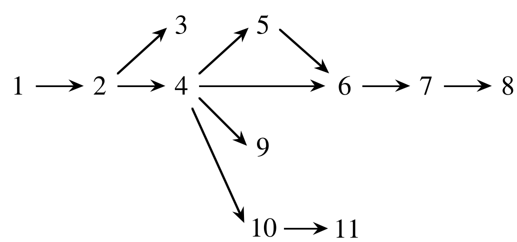
In Appendix A we review the main properties of 1D and 2D periodic functions both in the image and in the spectral domains. We also introduce the notion of step-vectors (in contrast to period-vectors), and the vector notations for 2D Fourier series that are extensively used in our work (notably in Chapters 6–10).

In Appendix B we review the main properties of 1D and 2D almost-periodic functions, both in the image and in the spectral domains. This appendix serves us as an introduction to the mathematical theory of almost periodic functions, which is not usually covered by standard textbooks on the Fourier theory.

In Appendix C we group together various issues, including the derivations of several results that we preferred, for different reasons, not to include in the main text of our work.

And finally, in Appendix D we provide a glossary of the most important terms that have been used in the present book.

The organization of Chapters 1–11 is as follows:




Although reading the chapters in their sequential order is recommended, any reader may choose to concentrate on one (or more) of the branches in this organization chart, according to his own needs and preferences.

## 1.5 About the exercises and the moiré demonstration samples

At the end of each chapter we provide a section containing a number of problems and exercises. Many of these problems are not merely routine exercises, but really intriguing and sometimes even challenging problems. Their aim is not only to aid the assimilation of the material covered by the chapter, but also to develop new insights beyond it. As already mentioned, these problems also include examples of real-world applications of the theory discussed in the chapter, along with references to existing publications on these applications (books, scientific papers, patents, etc.). We therefore highly encourage readers to dedicate some time for reviewing these exercises.

Since moiré effects are best appreciated by a hands-on experience, some of the key figures of this book have been also provided in the form of *PostScript*® programs [Adobe90], which can be printed on transparencies using any standard desktop laser printer. These PostScript programs and the instructions for using them can be found in the Internet site of this book, at the address:

`http://lspwww.epfl.ch/books/moire/`

By printing these demonstration samples the reader will obtain a kit of transparencies offering a vivid illustration of the moiré effects and their dynamic behaviour in the superposition. This demonstration set will allow the interested reader to make his own experiments by varying different parameters (angles, frequencies, etc.) in order to better understand their effects on the resulting moirés. This will not only be a valuable aid for the understanding of the material, but certainly also a source of amusement and fun.

* * *

Finally, a word about our notations. Throughout this book we adopt the following notational conventions:

| | | |
|---|---|---|
| Sec. 3.2 | — | Section 2 of Chapter 3. |
| Sec. A.2 | — | Section 2 of Appendix A. |
| Fig. 3.2 | — | Figure 2 of Chapter 3. |
| Fig. A.2 | — | Figure 2 of Appendix A. |
| (3.2) | — | Equation or formula 2 of Chapter 3. |
| (A.2) | — | Equation or formula 2 of Appendix A. |

Similar conventions are also used for enumerating tables, examples, propositions, remarks, etc.; for instance, Example 3.2 is the second example of Chapter 3.

Whenever reference is made to the second volume of this work, we use the abbreviation "*Vol. II*". When referring to a section, a figure or an equation in *Vol. II*, we simply add the prefix "*II*" to the specified number; for example, Sec. *II*.3.2 refers to Sec. 2 of Chapter 3 in *Vol. II*, Fig. *II*.3.2 means Fig. 2 of Chapter 3 in *Vol. II*, and Eq. (*II*.A.2) refers to the second equation in Appendix A of *Vol. II*.

The mathematical symbols and notations used in the present volume are listed at the end; a glossary of the main terms is provided in Appendix D.

# Chapter 2

## Background and basic notions

### 2.1 Introduction

Several mathematical approaches can be used to explore the moiré phenomenon. The classical geometric approach [Nishijima64; Tollenaar64; Yule67] is based on a geometric study of the properties of the superposed layers, their periods and their angles. By considering relations between triangles, parallelograms, or other geometric entities generated between the superposed layers, this method leads to formulas that can predict, under certain limitations, the geometric properties of the moiré patterns. Another widely used classical approach is the indicial equations method (see Sec. 11.2); this is a pure algebraic approach, based on the equations of each family of lines in the superposition, which also yields the same basic formulas [Oster64]. A more recent approach, introduced in [Harthong81], analyzes the moiré phenomenon using the theory of non-standard analysis. This approach can also provide the intensity levels of the moiré in question, in addition to its basic geometric properties.

However, the best adapted approach for investigating phenomena in the superposition of periodic structures is the spectral approach, which is based on the Fourier theory. This approach, whose first applications to the study of moiré phenomena appeared in the 1960s and 1970s (see Sec. 1.2), is the basis of our work, and it will be largely developed in the present book. Unlike the previous methods, this approach enables us to analyze properties not only in the original layers and in their superposition but also in their spectral representations, and thus it offers a more profound insight into the problem and provides indispensable tools for exploring it. We will discuss the advantages that the spectral approach offers in the study of moiré phenomena at the end of this chapter (Sec. 2.14), after having introduced the basic notions of the theory.

The present chapter lays the foundations for the entire book. In Sec. 2.2 we present the background and the basic concepts of the spectral approach, and we determine the image types with which we will be concerned in our work. Then we proceed in the following sections by showing step by step, in a didactic way, how our approach explains the various moiré phenomena between superposed layers. We start in Secs. 2.3–2.4 with the simplest case, the superposition of cosinusoidal gratings, and then we gradually proceed to the more interesting cases involving binary gratings, grids and dot-screens. On our way we will also introduce some fundamental terms and notions of the theory, such as first-order and higher-order moirés, singular moirés, stable and unstable moiré-free superpositions, etc. The problems at the end of the chapter include some of the main applications of the moiré effect in various fields of science and technology, along with additional references for the interested readers.

## 2.2 The spectral approach; images and their spectra

The spectral approach is based on the duality between functions or images in the spatial image domain and their spectra in the spatial frequency domain, through the Fourier transform. A key property of the Fourier transform is its ability to allow one to examine a function or an image from the perspective of both the space and frequency domains. By allowing us to analyze properties not only in the original image itself but also in its spectral representation this approach combines the best from both worlds, namely: it accumulates the advantages offered by the analysis in each of the two domains.[1]

In this book we will be concerned with bidimensional (2D) structures in the continuous $x,y$ plane, that we will call *images*, and their 2D spectra in the continuous $u,v$ plane which are obtained by the 2D Fourier transform.[2] In fact, we will restrict ourselves only to some particular types of 2D images, such as line-gratings or dot-screens, which are liable to generate moiré effects when superposed. In this section we will review the basic properties of the image types with which we are concerned, and the implications of these properties both in the image and in the spectral domains.

First, let us mention that we will mainly deal here with moiré effects between monochrome, black and white images; the extension of our discussion to the fully polychromatic case will be delayed until Chapter 9. In the monochrome case each image can be represented in the image domain by a *reflectance* function, which assigns to any point $(x,y)$ of the image a value between 0 and 1 representing its light reflectance: 0 for black (i.e., no reflected light), 1 for white (i.e., full light reflectance), and intermediate values for in-between shades. In the case of transparencies, the reflectance function is replaced by a *transmittance* function which is defined in a similar way: it gives 0 for black (i.e., no light transmittance through the transparency), 1 for white (or rather *transparent*, i.e., full light transmittance through the transparency), or any intermediate value between them. A superposition of such images can be obtained by means of overprinting, or by laying printed transparencies on top of each other. Since the superposition of black and any other shade always gives here black, this suggests a *multiplicative* model for the superposition of monochrome images. Thus, when $m$ monochrome images are superposed, the reflectance of the resulting image (also called the *joint reflectance*) is given by the *product* of the reflectance functions of the individual images:

$$r(x,y) = r_1(x,y)\cdot \ldots \cdot r_m(x,y) \tag{2.1}$$

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[1] It should be emphasized that since the Fourier transform is reversible, no information is gained or lost by its application. It only reveals certain image features which were present but not explicitly apparent before the image was transformed.

[2] Note that throughout this book we adopt the Fourier transform conventions that are commonly used in optics (see [Bracewell86 p. 241] or [Gaskill78 p. 128]); thus, the Fourier transform of a function $f(x,y)$ and its inverse are given by:

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i2\pi(ux+vy)}\, dx\, dy, \qquad f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{i2\pi(ux+vy)}\, dx\, dy.$$

For alternative definitions used in literature and the relationships between them see [Bracewell86 pp. 7 and 17] or [Gaskill78 pp. 181–183].

The same rule applies also to the superposition of monochrome transparencies, in which case $r_i(x,y)$ and $r(x,y)$ simply represent transmittance rather than reflectance functions. Now, according to the convolution theorem [Bracewell86 p. 244], the Fourier transform of a function product is the convolution of the Fourier transforms of the individual functions. Therefore, if we denote the Fourier transform of each function by the respective capital letter and the 2D convolution by $**$, the spectrum of the superposition is given by:

$$R(u,v) = R_1(u,v) ** \dots ** R_m(u,v) \tag{2.2}$$

**Remark 2.1:** It should be noted, however, that the multiplicative model is not the only possible superposition rule, and in other situations different superposition rules can be appropriate. For example, when images are superposed by making multiple exposures on a positive photographic film (assuming that we do not exceed the linear part of the film's response [Shamir73 p. 85]), intensities at each point are summed up, which implies an *additive* rule of superposition. In another example, when images are superposed by making multiple exposures on a negative photographic film (again, assuming a linear response) an *inverse additive* rule can be appropriate. More exotic superposition rules (involving, for example, various Boolean operations etc.) can be artificially generated by computer, even if they do not correspond to any physical reality. The interested reader may find examples which illustrate various superposition rules in references like [Bryngdahl76], [Asundi93] or Chapter 3 of [Patorski93]. Note that different superposition rules in the image domain will have different spectrum composition rules in the spectral domain, which are determined by properties of the Fourier transform. For example, in the case of the additive superposition rule, where Eq. (2.1) is replaced by:

$$r(x,y) = r_1(x,y) + \dots + r_m(x,y) \tag{2.3}$$

the spectrum of the superposition is no longer the spectrum-convolution given by Eq. (2.2), but rather the *sum* of the individual spectra:

$$R(u,v) = R_1(u,v) + \dots + R_m(u,v) \tag{2.4}$$

As we will see in Remark 2.3 at the end of Sec. 2.3 below, this case is less interesting from the point of view of moiré generation. ■

Second, until Chapter 10 we will be basically interested in *periodic* images, such as line-gratings or dot-screens, and in their superpositions. This implies that the spectrum of the image on the $u,v$ plane is not smooth but rather consists of impulses, which represent the frequencies in the Fourier series decomposition of the periodic image [Bracewell86 p. 204].[3] A strong impulse in the spectrum indicates a pronounced periodic component in the original image at the frequency and direction of that impulse.

Each impulse in the 2D spectrum is characterized by three main properties: its *label* (which is its index in the Fourier series development); its *geometric location* (or *impulse*

[3] A short survey of the spectral Fourier representation of periodic functions is also provided in Appendix A.

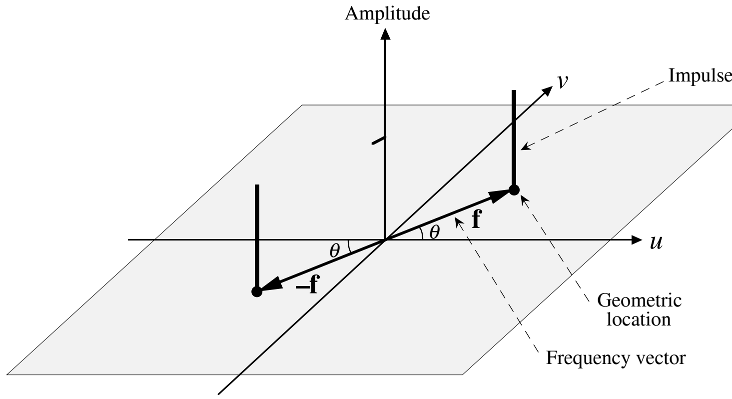


**Figure 2.1:** The *geometric location* and *amplitude* of impulses in the 2D spectrum. To each impulse is attached its *frequency vector*, which points to the geometric location of the impulse in the $u$,$v$ spectrum plane.

*location*); and its *amplitude* (see Fig. 2.1). To the geometric location of any impulse is attached a *frequency vector* **f** in the spectrum plane, which connects the spectrum origin to the geometric location of the impulse. This vector can be expressed either by its polar coordinates ($f$,$\theta$), where $\theta$ is the direction of the impulse and $f$ is its distance from the origin (i.e., its frequency in that direction), or by its Cartesian coordinates ($u$,$v$), where $u$ and $v$ are the horizontal and vertical components of the frequency. In terms of the original image, the *geometric location* of an impulse in the spectrum determines the frequency $f$ and the direction $\theta$ of the corresponding periodic component in the image, and the *amplitude* of the impulse represents the intensity of that periodic component in the image.[4] Note, however, that the impulse which is located on the spectrum origin is rather unique in its properties and requires some particular attention: It represents the zero frequency, which corresponds in the image domain to the constant component of the image, and its amplitude corresponds to the intensity of this constant component.[5] This particular impulse is traditionally called the *DC impulse* (because it represents in electrical transmission theory the direct current component, i.e., the constant term in the frequency decomposition of an electric wave), and we will maintain here this convention.

The periodic images with which we will be dealing will normally be of a symmetric nature (gratings, grids, etc.). For the sake of simplicity we also assume, unless otherwise mentioned, that the given images are not shifted, but indeed centered symmetrically about

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[4] It should be stressed that the direction $\theta$ of the impulse, i.e., the direction of the corresponding periodic component in the image, is *perpendicular* to the corrugations of the periodic component (see, for example, gratings (a) and (b) and their spectra (d) and (e) in Fig. 2.2).

[5] In fact, as shown in Appendix C.2, the amplitude of the DC impulse represents the average intensity level of the image (which is, in our case, a number between 0 and 1, since our images can only take values between 0 and 1).

the origin. As a result, we will normally deal with images (and image superpositions) which are *real-valued* and *symmetric*, and whose spectra are consequently also real-valued and symmetric [Bracewell86 pp. 14–15]. This means that each impulse in the spectrum (except for the DC impulse at the origin) is always accompanied by a twin impulse of an identical amplitude, which is symmetrically located at the other side of the origin as in Fig. 2.1 (their frequency vectors being **f** and –**f**). Note, however, that if the original image is not symmetric about the origin (but, of course, still real-valued), the amplitudes of the twin impulses at **f** and –**f** are complex conjugates; in this case the amplitude of each impulse in the spectrum (except for the DC impulse) may also have a non-zero imaginary component. We will return to such cases in more detail in Chapter 7, where we will discuss the superposition of non-centered or shifted images. It is important to understand, however, that even in such cases each frequency **f** of the image is still represented in the spectrum by a pair of impulses, whose geometric locations are **f** and –**f**. [6]

However, the question of whether or not an impulse pair in the spectrum represents a *visible* periodic component in the image strongly depends on properties of the human visual system. The fact that the eye cannot distinguish fine details above a certain frequency (i.e., below a certain period) suggests that the human visual system model includes a low-pass filtering stage. This is a bidimensional bell-shaped filter whose form is anisotropic (since it appears that the eye is less sensitive to small details in diagonal directions such as 45° [Ulichney88 pp. 79–84]).[7] However, for the sake of simplicity this low-pass filter can be approximated by the *visibility circle,* a circular step-function around the spectrum origin whose radius represents the *cutoff frequency* (i.e., the threshold frequency beyond which fine detail is no longer detected by the eye). Obviously, its radius depends on several factors such as the contrast of the observed details, the viewing distance, light conditions, etc. If the frequencies of the image details are beyond the border of the visibility circle in the spectrum, the eye can no longer see them; but if a strong enough impulse in the spectrum of the image superposition falls inside the visibility circle, then a moiré effect becomes visible in the superposed image. (In fact, the visibility circle has a hole in its center, since very low frequencies cannot be seen, either.)

Another possible property of our images (although it is not necessarily a requirement) comes from the fact that most printing devices are only bilevel, namely: they are only capable of printing solid ink or leaving the paper unprinted, but they cannot produce intermediate ink tones. (This is also true for most colour printing devices, where each of the printed primary colours is bilevel.) In such devices the *visual impression* of intermediate tone levels is usually obtained by means of the halftoning technique, i.e., by breaking the continuous-tone image into small dots whose size depends on the tone level (see Sec. 3.2). Therefore, in most practical cases the reflectance function of a printed

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[6] For the sake of completeness we mention here that this conjugate symmetry property in the spectrum only breaks up in the case of complex-valued images. For example, a single impulse at the point $(u,v)$ in the spectrum corresponds to the complex-valued function $p(x,y) = e^{-2\pi i(ux+vy)}$ in the image domain. We will rarely be concerned with such cases, since all physically realizable images are purely real.

[7] For a more detailed account on the human visual system and its properties the reader is referred to specialized references on this subject such as [Cornsweet70], [Wandell95] or Chapter 34 in [Boff86].

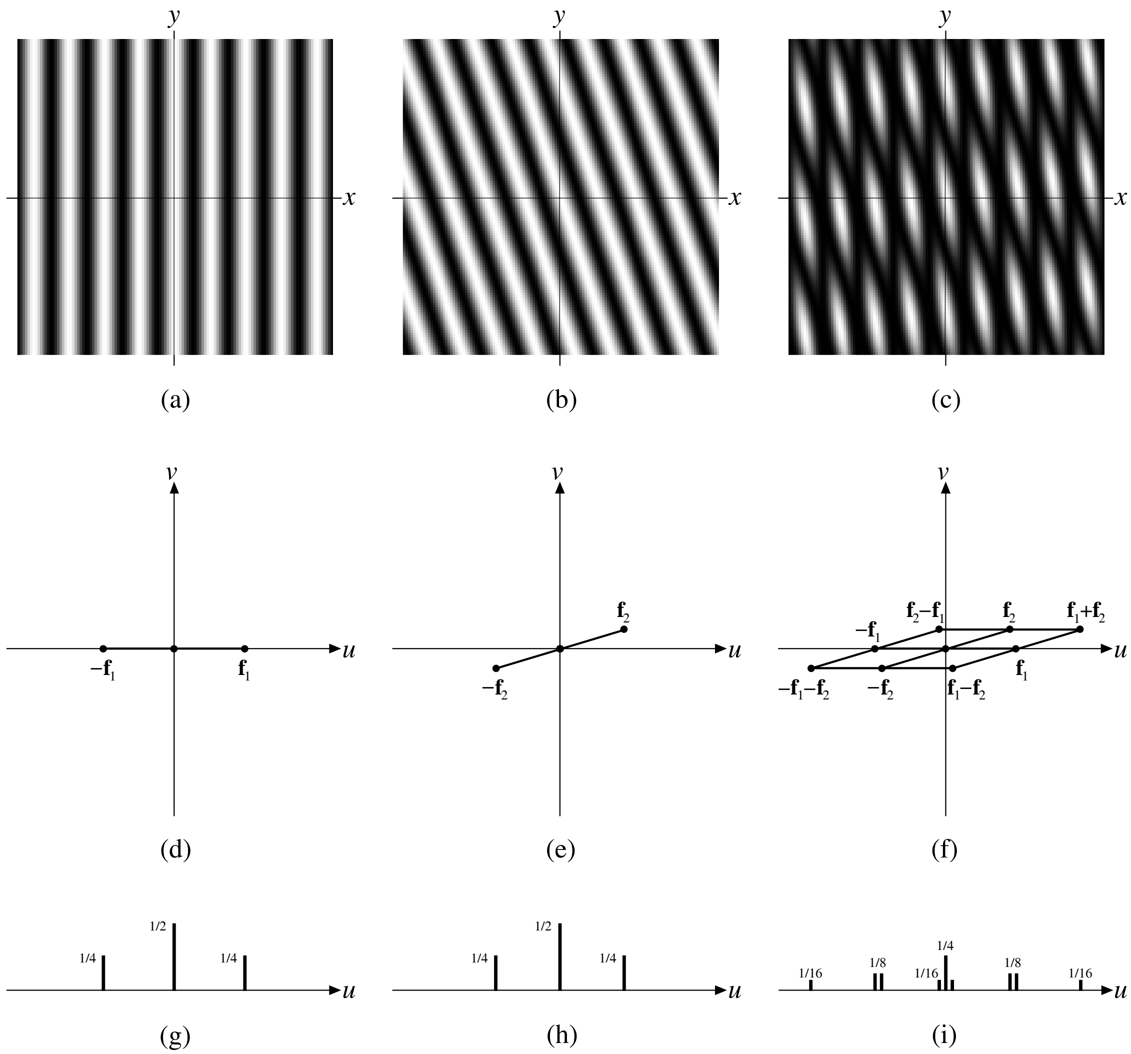


**Figure 2.2:** First row: cosinusoidal gratings (a) and (b) and their superposition (c) in the image domain. Second row: top view of the respective spectra (d), (e) and their convolution (f). Black dots in the spectra indicate the geometric locations of the impulses; the line segments connecting them have been added only in order to clarify the geometric relations. (g), (h), (i): Side view of the same spectra, showing the impulse amplitudes. Note the two new impulse pairs which have appeared in the spectrum convolution (f); the isolated contributions of these two impulse pairs to the superposition (c) are shown in (j) and (k): (j) is the periodic component contributed by the new impulse pair which is located at the difference frequencies $\mathbf{f}_1-\mathbf{f}_2$ and $\mathbf{f}_2-\mathbf{f}_1$, and (k) is the periodic component contributed by the new impulse pair at the sum frequencies, $\mathbf{f}_1+\mathbf{f}_2$ and $-\mathbf{f}_1-\mathbf{f}_2$.