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The Strength of Nonstandard Analysis

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Foreword

Willst du ins Unendliche schreiten?
Geh nur im Endlichen nach alle Seiten!
Willst du dich am Ganzen erquicken,
So must du das Ganze im Kleinsten erblicken.

J. W. Goethe (*Gott, Gemüt und Welt, 1815*)

Forty-five years ago, an article appeared in the Proceedings of the Royal Academy of Sciences of the Netherlands Series A, 64, 432–440 and Indagationes Math. 23 (4), 1961, with the mysterious title "Non-standard Analysis" authored by the eminent mathematician and logician Abraham Robinson (1908–1974).

The title of the paper turned out to be a contraction of the two terms "Non-standard Model" used in model theory and "Analysis". It presents a treatment of classical analysis based on a theory of infinitesimals in the context of a non-standard model of the real number system $\mathbb{R}$.

In the Introduction of the article, Robinson states:

> "It is our main purpose to show that the models provide a natural approach to the age old problem of producing a calculus involving infinitesimal (infinitely small) and infinitely large quantities. As is well-known the use of infinitesimals strongly advocated by Leibniz and unhesitatingly accepted by Euler fell into disrepute after the advent of Cauchy's methods which put Mathematical Analysis on a firm foundation".

To bring out more clearly the importance of Robinson's creation of a rigorous theory of infinitesimals and their reciprocals, the infinitely large quantities, that has changed the landscape of analysis, I will briefly share with the reader a few highlights of the historical facts that are involved.

The invention of the "Infinitesimal Calculus" in the second half of the seventeenth century by Newton and Leibniz can be looked upon as the first fundamental new discovery in mathematics of revolutionary nature since the death of Archimedes in 212 BC. The fundamental discovery that the operations of differentiation (flux) and integration (sums of infinitesimal increments) are inverse operations using the intuitive idea that infinitesimals of higher order compared to those of lower order may be neglected became an object of severe criticism. In the "Analyst", section 35, Bishop G. Berkeley states:

> "And what are these fluxions? The velocities of evanescent increments. And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitesimally small, nor yet nothing. May we call them ghosts of departed quantities?"

The unrest and criticism concerning the lack of a rigorous foundation of the infinitesimal calculus led the Academy of Sciences of Berlin, at its public meeting on June 3, 1774, and well on the insistence of the Head of the Mathematics Section, J. L. Lagrange, to call upon the mathematical community to solve this important problem. To this end, it announced a prize contest dealing with the problem of "Infinity" in the broadest sense possible in mathematics.The announcement read:

> "The utility derived from Mathematics, the esteem it is held in and the honorable name of 'exact science' par excellence, that it justly deserves, are all due to the clarity of its principles, the rigor of its proofs and the precision of its theorems. In order to ensure the continuation of these valuable attributes in this important part of our knowledge the prize of a 50 ducat gold medal is for:
>
> A clear and precise theory of what is known as 'Infinity' in Mathematics. It is well-known that higher mathematics regularly makes use of the infinitely large and infinitely small. The geometers of antiquity and even the ancient analysts, however, took great pains to avoid anything approaching the infinity, whereas today's eminent modern analysts admit to the statement 'infinite magnitude' is a contradiction in terms. For this reason the Academy desires an explanation why it is that so many correct theorems have been deduced from a contradictory assumption, together with a formulation of a truly clear mathematical principle that may replace that of infinity without, however, rendering investigations by its use overly difficult and overly lengthy. It is requested that the subject be treated in all possible generality and with all possible rigor, clarity and simplicity."

Twenty-three answers were received before the deadline of January 1, 1786. The prize was awarded to the Swiss mathematician Simon L'Huilier for his essay with motto:

"The infinite is the abyss in which our thoughts are engulfed."

The members of the "Prize Committee" made the following noteworthy points: None of the submitted essays dealt with the question raised "why so many correct theorems have been derived from a contradictory assumption?" Furthermore, the request for clarity, simplicity and, above all, rigor was not met by the contenders, and almost all of them did not address the request for a newly formulated principle of infinity that would reach beyond the infinitesimal calculus to be meaningful also for algebra and geometry.

For a detailed account of the prize contest we refer the reader to the interesting biography of Lazare Nicolas M. Carnot (1753–1823), the father of the thermodynamicist Sadi Carnot, entitled "Lazare Carnot Savant" by Ch. C. Gillespie (Princeton Univ. Press, 1971), which contains a thorough discussion of Carnot's entry "Dissertation sur la théorie de l'infini mathématique", received by the Academy after the deadline. The above text of the query was adapted from the biography.

In retrospect, the outcome of the contest is not surprising. Nevertheless around that time the understanding of infinitesimals had reached a more sophisticated level as the books of J. L. Lagrange and L. N. Carnot published in Paris in 1797 show.

From our present state of the art, it seems that the natural place to look for a "general principle of infinity" is set theory. Not however for an "intrinsic" definition of infinity. Indeed, as Gian-Carlo Rota expressed not too long ago:

"God created infinity and man, unable to understand it, had to invent finite sets."

At this point let me digress a little for further clarification about the infinity we are dealing with. During the early development of Cantor's creation of set theory, it was E. Zermelo who realized that the attempts to prove the existence of "infinite" sets, short of assuming there is an "infinite" set or a non-finite set as in Proposition 66 of Dedekind's famous "Was sind und was sollen die Zahlen?" were fallacious. For this reason, Zermelo in his important paper "Sur les ensembles finis et le principe de l'induction complete", Acta Math. 32 (1909), 185–193 (submitted in 1907), introduced an axiom of "infinity" by postulating the existence of a set, say A, non-empty, and that for each of its elements x, the singleton $\{x\}$ is an element of it.

Returning to the request of the Academy: To discover a property that all infinite sets would have in common with the finite sets that would facilitate their use in all branches of mathematics.What comes to mind is Zermelo's well-ordering principle. Needless to say that this principle and the manifold results and consequences in all branches of mathematics have had an enormous impact on the development of mathematics since its introduction. One may ask what has this to do with the topic at hand? It so happens that the existence of non-standard models depends essentially on it as well and consequently non-standard analysis too.

The construction of the real number system (linear continuum) by Cantor and Dedekind in 1872 and the Weierstrass ε-δ technique gradually replaced the use of infinitesimals. Hilbert's characterization in 1899 of the real number system as a (Dedekind) complete field led to the discovery, in 1907, by H. Hahn, of non-archimedian totally ordered field extensions of the reals. This development brought about a renewed interest in the theory of infinitesimals. The resulting "calculus", certainly of interest by itself, lacked a process of defining extensions of the elementary and special functions, etc., of the objects of classical analysis. It is interesting that Cantor strongly rejected the existence of non-archimedian totally ordered fields. He expressed the view that no actual infinities could exist other than his transfinite cardinal numbers and that, other than 0, infinitesimals did not exist. He also offered a "proof" in which he actually assumed order completeness.

It took one hundred and seventy-five years from the time of the deadline of the Berlin Academy contest to the publication of Robinson's paper "Non-standard Analysis". As Robinson told us, his discovery did not come about as a result of his efforts to solve Leibniz' problem; far from it. Working on a paper on formal languages where the length of the sentences could be countable, it occurred to him to look up again the important paper by T. Skolem "Über die Nichtcharakterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen, Fund. Math. 23 (1934), 150–161*.

Briefly, Skolem showed in his paper the existence of models of Peano arithmetic having "infinitely large numbers". Nevertheless in his models the principle of induction holds only for subsets determined by admissible formulas from the chosen formal language used to describe Peano's axiom system. The non-empty set of the infinitely large numbers has no smallest element and so cannot be determined by a formula of the formal language and is called an external set; those that can were baptized as internal sets of the model.

Robinson, rereading Skolem's paper, wondered what systems of numbers would emerge if he would apply Skolem's method to the axiom system of the real numbers. In doing so, Robinson immediately realized that the real number

*See also: T. Skolem "Peano's Axioms and Models of Arithmetic", in Symposium on the Mathematical Interpretation of Formal Systems, North-Holland, Amsterdam 1955, 1–14.

system was a non-archimedian totally ordered field extension of the reals whose structure satisfies all the properties of the reals, and that, in particular, the set of infinitesimals lacking a least upper bound was an external set.

This is how it all started and the Academy would certainly award Robinson the gold medal.

At the end of the fifties at Caltech (California Institute of Technology) Arthur Erdelyi FRS (1908–1977) conducted a lively seminar entitled "Generalized Functions". It dealt with various areas of current research at that time in such fields as J. Mikusinski's rigorous foundation of the so-called Heaviside operational calculus and L. Schwartz' theory of distributions. In connection with Schwartz' distribution theory, Erdelyi urged us to read the just appeared papers by Laugwitz and Schmieden dealing with the representations of the Dirac-delta functions by sequences of point-functions converging to 0 pointwise except at 0 where they run to infinity. Robinson's paper fully clarified this phenomenon. Reduced powers of $\mathbb{R}$ instead of ultrapowers, as in Robinson's paper, were at play here. In my 1962 Notes on Non-standard Analysis the ultrapower construction was used, but at that time without using explicitly the Transfer Principle.

In 1967 the first International Symposium on Non-standard Analysis took place at Caltech with the support of the U.S. Office of Naval Research. At the time the use of non-standard models in other branches of mathematics started to blossom. This is the reason that the Proceedings of the Symposium carries the title: Applications of Model Theory to Algebra, Analysis and Probability.

A little anecdote about the meeting. When I opened the newspaper one morning during the week of the meeting, I discovered to my surprise that it had attracted the attention of the Managing Editor of the Pasadena Star News; his daily "Conversation Piece" read:

> "A Stanford Professor spoke in Pasadena this week on the subject 'Axiomatizations of Non-standard Analysis which are Conservative Extensions of Formal Systems for Standard Classical Analysis', a fact which I shall tuck away for reassurance on those days when I despair of communicating clearly."

I may add here that from the beginning Robinson was very interested in the formulation of an axiom system catching his non-standard methodology.

Unfortunately he did not live to see the solution of his problem by E. Nelson presented in the 1977 paper entitled "Internal Set Theory". A presentation by Nelson, "The virtue of Simplicity", can be found in this book.

A final observation. During the last sixty years we have all seen come about the solutions of a number of outstanding problems and conjectures, some centuries old, that have enriched mathematics. The century-old problem to create a rigorous theory of infinitesimals no doubt belongs in this category.

It is somewhat surprising that the appreciation of Robinson's creation was slow in coming. Is it possible that the finding of the solution in model theory, a branch of mathematical logic, had something to do with that?

The answer may perhaps have been given by Augustus de Morgan (1806–1871), who is well-known from De Morgan's Law, and who in collaboration with George Boole (1805–1864) reestablished formal logic as a branch of exact science in the nineteenth century, when he wrote:

> "We know that mathematicians care no more for logic than logicians for mathematics. The two eyes of exact sciences are mathematics and logic: the mathematical sect puts out the logical eye, the logical sect puts out the mathematical eye; each believing that it can see better with one eye than with two."

We owe Abraham Robinson a great deal for having taught us the use of both eyes.

This book shows clearly that we have learned our lesson well.

All the contributors are to be commended for the way they have made an effort to make their contributions that are based on the talks at the meeting "Nonstandard Mathematics 2004" as self-contained as can be expected. For further facilitating the readers, the editors have divided the papers in categories according to the subject. The whole presents a very rich assortment of the nonstandard approach to diverse areas of mathematical analysis.

I wish it many readers.

Wilhelmus A. J. Luxemburg
Pasadena, California
September 2006

Acknowledgements

The wide range of applicability of Mathematical Logic to classical Mathematics, beyond Analysis — as Robinson's terminology *Non-standard Analysis* might imply — is already apparent in the very first symposium on the area, held in Pasadena in 1967, as mentioned in the foreword.

Important indicators of the maturity of this field are the high level of foundational and pure or applied mathematics presented in this book, congresses which take place approximately every two years, as well as the experiences of teaching, which proliferated at graduate, undergraduate and secondary school level all over the world.

This book, made of peer reviewed contributions, grew out of the meeting Non Standard Mathematics 2004, which took place in July 2004 at the Department of Mathematics of the University of Aveiro (Portugal). The articles are organized into five groups, (1) Foundations, (2) Number theory, (3) Statistics, probability and measures, (4) Dynamical systems and equations and (5) Infinitesimals and education. Its cohesion is enhanced by many cross-overs.

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Our special thanks go to Wilhelmus Luxemburg of Caltech, USA, who, being one of the first to recognize the importance of Robinson's discovery of

Nonstandard Analysis, honoured us by writing the foreword. Last but not least, Salvador Sánchez-Pedreño of the University of Murcia, Spain, joined his expert knowledge of Nonstandard Analysis to a high dominion of the LaTeX editing possibilities in order to improve the consistency of the text and to obtain an outstanding typographic result.

Imme van den Berg
Vítor Neves
Editors

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Part I

Foundations

1

The strength of nonstandard analysis

H. Jerome Keisler*

Abstract

A weak theory nonstandard analysis, with types at all finite levels over both the integers and hyperintegers, is developed as a possible framework for reverse mathematics. In this weak theory, we investigate the strength of standard part principles and saturation principles which are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers.

1.1 Introduction

In this paper we revisit the work in [5] and [6], where the strength of nonstandard analysis is studied. In those papers it was shown that weak fragments of set theory become stronger when one adds saturation principles commonly used in nonstandard analysis.

The purpose of this paper is to develop a framework for reverse mathematics in nonstandard analysis. We will introduce a base theory, "weak nonstandard analysis" (*WNA*), which is proof theoretically weak but has types at all finite levels over both the integers and the hyperintegers. In *WNA* we study the strength of two principles that are prominent in nonstandard analysis, the standard part principle in Section 1.6, and the saturation principle in Section 1.9. These principles are often used in practice along with first order reasoning about the hyperintegers to obtain second order conclusions about the integers, and for this reason they can lead to the discovery of new results.

The standard part principle (*STP*) says that a function on the integers exists if and only if it is coded by a hyperinteger. Our main results show that in *WNA*, *STP* implies the axiom of choice for quantifier-free formulas (Theorem 17), *STP*+saturation for quantifier-free formulas implies choice for arithmetical formulas (Theorem 23), and *STP*+saturation for formulas with

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first order quantifiers implies choice for formulas with second order quantifiers (Theorem 25). The last result might be used to identify theorems that are proved using nonstandard analysis but cannot be proved by the methods commonly used in classical mathematics.

The natural models of *WNA* will have a superstructure over the standard integers $\mathbb{N}$, a superstructure over the hyperintegers $^*\mathbb{N}$, and an inclusion map $j : \mathbb{N} \to {}^*\mathbb{N}$. With the two superstructures, it makes sense to ask whether a higher order statement over the hyperintegers implies a higher order statement over the integers. As is commonly done in the standard literature on weak theories in higher types, we use functional superstructures with types of functions rather than sets. The base theory *WNA* is neutral between the internal set theory approach and the superstructure approach to nonstandard analysis, and the standard part and saturation principles considered here arise in both approaches. For background in model theory, see [2, Section 4.4].

The theory *WNA* is related to the weak nonstandard theory $NPRA^\omega$ of Avigad [1], and the base theory RCA_0^ω for higher order reverse mathematics proposed by Kohlenbach [7]. The paper [1] shows that the theory $NPRA^\omega$ is weak in the sense that it is conservative over primitive recursive arithmetic (*PRA*) for Π_2 sentences, but is still sufficient for the development of much of analysis. The theory *WNA* is also conservative over *PRA* for Π_2 sentences, but has more expressive power. In Sections 1.11 and 1.12 we will introduce a stronger, second order Standard Part Principle, and give some relationships between this principle and the theories $NPRA^\omega$ and RCA_0^ω.

1.2 The theory PRA^ω

Our starting point is the theory *PRA* of primitive recursive arithmetic, introduced by Skolem. It is a first order theory which has function symbols for each primitive recursive function (in finitely many variables), and the equality relation =. The axioms are the rules defining each primitive recursive function, and induction for quantifier-free formulas. This theory is much weaker than Peano arithmetic, which has induction for all first order formulas.

An extension of *PRA* with all finite types was introduced by Gödel [4], and several variations of this extension have been studied in the literature. Here we use the finite type theory PRA^ω as defined in Avigad [1].

There is a rich literature on constructive theories in intuitionistic logic that are very similar to PRA^ω, such as the finite type theory HA^ω over Heyting arithmetic (see, for example, [9]). However, in this paper we work exclusively in classical logic.

We first introduce a formal object N and define a collection of formal objects called **types over** N.

(1) The **base type** over N is N.

(2) If σ, τ are types over N, then $\sigma \to \tau$ is a type over N.

We now build the formal language $L(PRA^\omega)$. $L(PRA^\omega)$ is a many-sorted first order language with countably many variables of each type σ over N, and the equality symbol $=$ at the base type N only. It has the usual rules of many-sorted logic, including the rule $\exists f \forall u\, f(u) = t(u, \ldots)$ where u, f are variables of type $\sigma, \sigma \to N$ and $t(u, \ldots)$ is a term of type N in which f does not occur.

We first describe the symbols and then the corresponding axioms.

$L(PRA^\omega)$ has the following function symbols:

- A function symbol for each primitive recursive function.
- The primitive recursion operator which builds a term $R(m, f, n)$ of type N from terms of type $N, N \to N$, and N.
- The definition by cases operator which builds a term $c(n, u, v)$ of type σ from terms of type N, σ, and σ.
- The λ operator which builds a term $\lambda v.t$ of type $\sigma \to \tau$ from a variable v of type σ and a term t of type τ.
- The application operator which builds a term $t(s)$ of type τ from terms s of type σ and t of type $\sigma \to \tau$.

Given terms r, t and a variable v of the appropriate types, $r(t/v)$ denotes the result of substituting t for v in r. Given two terms s, t of type σ, $s \equiv t$ will denote the infinite scheme of formulas $r(s/v) = r(t/v)$ where v is a variable of type σ and $r(v)$ is an arbitrary term of type N. $\equiv$ is a substitute for the missing equality relations at higher types.

The axioms for PRA^ω are as follows.

- Each axiom of PRA.
- The induction scheme for quantifier-free formulas of $L(PRA^\omega)$.
- Primitive recursion: $R(m, f, 0) = m$, $R(m, f, s(n)) = f(n, R(m, f, n))$.
- Cases: $c(0, u, v) \equiv u$, $c(s(m), u, v) \equiv v$.
- Lambda abstraction: $(\lambda u.t)(s) \equiv t(s/u)$.

The order relations $<, \leq$ on type N can be defined in the usual way by quantifier-free formulas.

In [1] additional types $\sigma \times \tau$, and term-building operations for pairing and projections with corresponding axioms were also included in the language, but as explained in [1], these symbols are redundant and are often omitted in the literature.

On the other hand, in [1] the symbols for primitive recursive functions are not included in the language. These symbols are redundant because they can be defined from the primitive recursive operator R, but they are included here for convenience.

We state a conservative extension result from [1], which shows that PRA^ω is very weak.

Proposition 1 *PRA^ω is a conservative extension of PRA, that is, PRA^ω and PRA have the same consequences in $L(PRA)$.*

The natural model of PRA^ω is the full functional superstructure $V(\mathbb{N})$, which is defined as follows. $\mathbb{N}$ is the set of natural numbers. Define $V_N(\mathbb{N}) = \mathbb{N}$, and inductively define $V_{\sigma\to\tau}(\mathbb{N})$ to be the set of all mappings from $V_\sigma(\mathbb{N})$ into $V_\tau(\mathbb{N})$. Finally, $V(\mathbb{N}) = \bigcup_\sigma V_\sigma(\mathbb{N})$. The superstructure $V(\mathbb{N})$ becomes a model of PRA^ω when each of the symbols of $L(PRA^\omega)$ is interpreted in the obvious way indicated by the axioms. In fact, $V(\mathbb{N})$ is a model of much stronger theories than PRA^ω, since it satisfies full induction and higher order choice and comprehension principles.

1.3 The theory $NPRA^\omega$

In [1], Avigad introduced a weak nonstandard counterpart of PRA^ω, called $NPRA^\omega$. $NPRA^\omega$ adds to PRA^ω a new predicate symbol $S(\cdot)$ for the standard integers (and S-relativized quantifiers $\forall^S, \exists^S$), and a constant H for an infinite integer, axioms saying that $S(\cdot)$ is an initial segment not containing H and is closed under each primitive recursive function, and a transfer axiom scheme for universal formulas. In the following sections we will use a weakening of $NPRA^\omega$ as a part of our base theory.

In order to make $NPRA^\omega$ fit better with the present paper, we will build the formal language $L(NPRA^\omega)$ with types over a new formal object $^*N$ instead of over N. The base type over $^*N$ is $^*N$, and if σ, τ are types over $^*N$ then $\sigma \to \tau$ is a type over $^*N$.

For each type σ over N, let $^*\sigma$ be the type over $^*N$ built in the same way. For each function symbol u in $L(PRA^\omega)$ from types $\vec{\sigma}$ to type τ, $L(NPRA^\omega)$ has a corresponding function symbol $^*u$ from types $^*\vec{\sigma}$ to type $^*\tau$. $L(NPRA^\omega)$ also has the equality relation $=$ for the base type $^*N$, and the extra constant symbol H and the standardness predicate symbol S of type $^*N$.

We will use the following conventions throughout this paper. When we write a formula $A(\vec{v})$, it is understood that $\vec{v}$ is a tuple of variables that contains all the free variables of A. If we want to allow additional free variables we write $A(\vec{v}, \ldots)$. We will always let:

- $m, n, \ldots$ be variables of type N,
- $x, y, \ldots$ be variables of type ${}^*N$,
- $f, g, \ldots$ be variables of type $N \to N$.

To describe the axioms of $NPRA^\omega$ we introduce the star of a formula of $L(PRA^\omega)$. Given a formula A of $L(PRA^\omega)$, a **star of** A is a formula ${}^*A$ of $L(NPRA^\omega)$ which is obtained from A by replacing each variable of type σ in A by a variable of type ${}^*\sigma$ in a one to one fashion, and replacing each function symbol in A by its star. The order relations on ${}^*N$ will be written $<$, $\leq$ without stars.

The axioms of $NPRA^\omega$ are as follows:

- The star of each axiom of PRA^ω.
- S is an initial segment: $\neg S(H) \wedge \forall x \forall y\, [S(x) \wedge y \leq x \to S(y)]$.
- S is closed under primitive recursion.
- Transfer: $\forall^S \vec{x}\ {}^*A(\vec{x}) \to \forall \vec{x}\ {}^*A(\vec{x})$, $A(\vec{m})$ quantifier-free in $L(PRA^\omega)$.

It is shown in [1] that if $A(m, n)$ is quantifier-free in $L(PRA)$ and $NPRA^\omega$ proves $\forall^S x \exists y\, {}^*A(x, y)$, then PRA proves $\forall m \exists n\, A(m, n)$. It follows that $NPRA^\omega$ is conservative over PRA for Π_2 sentences.

The natural models of $NPRA^\omega$ are the internal structures ${}^*V(\mathbb{N})$, which are proper elementary extensions of $V(\mathbb{N})$ in the many-sorted sense, with additional symbols S for $\mathbb{N}$ and H for an element of ${}^*\mathbb{N} \setminus \mathbb{N}$.

1.4 The theory *WNA*

We now introduce our base theory *WNA*, weak nonstandard analysis. The idea is to combine the theory PRA^ω with types over N with a weakening of the theory $NPRA^\omega$ with types over ${}^*N$, and form a link between the two by identifying the standardness predicate S of $NPRA^\omega$ with the lowest type N of PRA^ω. In this setting, it will make sense to ask whether a formula with types over ${}^*N$ implies a formula with types over N.

The language $L(WNA)$ of *WNA* has both types over N and types over ${}^*N$. It has all of the symbols of $L(PRA^\omega)$, all the symbols of $L(NPRA^\omega)$ except the primitive recursion operator ${}^*R$, and has one more function symbol j which goes from type N to type ${}^*N$.

We make the axioms of *WNA* as weak as we can so as to serve as a blank screen for viewing the relative strengths of additional statements which arise in nonstandard analysis.

The axioms of *WNA* are as follows:

- The axioms of PRA^ω.
- The star of each axiom of *PRA*.
- The stars of the Cases and Lambda abstraction axioms of PRA^ω.
- S is an initial segment: $\neg S(H) \wedge \forall x \forall y\, [S(x) \wedge y \leq x \rightarrow S(y)]$.
- S is closed under primitive recursion.
- j maps S onto $\mathbb{N}$: $\forall x\, [S(x) \leftrightarrow \exists m\, x = j(m)]$.
- Lifting: $j(\alpha(\vec{m})) = {}^*\alpha(j(\vec{m}))$ for each primitive recursive function α.

The star of a quantifier-free formula of $L(PRA)$, possibly with some variables replaced by H, will be called an **internal quantifier-free formula**. The stars of the axioms of *PRA* include the star of the defining rule for each primitive recursive function, and the induction scheme for internal quantifier-free formulas (which we will call **internal induction**).

The axioms of $NPRA^\omega$ that are left out of *WNA* are the star of the Primitive Recursion scheme, the star of the quantifier-free induction scheme of PRA^ω, and Transfer. These axioms are statements about the hyperintegers which involve terms of higher type.

Note that *WNA* is noncommittal on whether the characteristic function of S exists in type ${}^*N \rightarrow {}^*N$, while the quantifier-free induction scheme of $NPRA^\omega$ precludes this possibility.

In practice, nonstandard analysis uses very strong transfer axioms, and extends the mapping j to higher types. Strong axioms of this type will not be considered here.

Theorem 2 *$WNA + NPRA^\omega$ is a conservative extension of $NPRA^\omega$, that is, $NPRA^\omega$ and $WNA + NPRA^\omega$ have the same consequences in $L(NPRA^\omega)$.*

Proof. Let M be a model of $NPRA^\omega$, and let M^S be the restriction of M to the standardness predicate S. Then M^S is a model of *PRA*. By Proposition 1, the complete theory of M^S is consistent with PRA^ω. Therefore PRA^ω has a model K whose restriction K^N to type N is elementarily equivalent to M^S. By the compactness theorem for many-sorted logic, there is a model M_1 elementarily equivalent to M and a model K_1 elementarily equivalent to K with an isomorphism $j : M_1^S \cong K_1^N$, such that $\langle K_1, M_1, j\rangle$ is a model of $WNA + NPRA^\omega$. Thus every complete extension of $NPRA^\omega$ is consistent with $WNA + NPRA^\omega$, and the theorem follows. □

Corollary 3 *WNA is a conservative extension of PRA for* Π_2 *formulas. That is, if* $A(m,n)$ *is quantifier-free in* $L(PRA)$ *and* $WNA \vdash \forall m \exists n\, A(m,n)$*, then* $PRA \vdash \forall m \exists n\, A(m,n)$.

Proof. Suppose $WNA \vdash \forall m \exists n\, A(m,n)$. By the Lifting Axiom, $WNA \vdash \forall^S x \exists^S y\ {}^*A(x,y)$. By Theorem 2, $NPRA^\omega \vdash \forall^S x \exists^S y\ {}^*A(x,y)$. Then $PRA \vdash \forall m \exists n\, A(m,n)$ by Corollary 2.3 in [1]. □

Each model of *WNA* has a $V(\mathbb{N})$ part formed by restricting to the objects with types over N, and a $V({}^*\mathbb{N})$ part formed by restricting to the objects with types over ${}^*N$. Intuitively, the $V(\mathbb{N})$ and $V({}^*\mathbb{N})$ parts of *WNA* are completely independent of each other, except for the inclusion map j at the zeroth level. The standard part principles introduced later in this paper will provide links between types $N \to N$ and $(N \to N) \to N$ in the $V(\mathbb{N})$ part and types ${}^*N$ and ${}^*N \to {}^*N$ in the $V({}^*\mathbb{N})$ part.

WNA has two natural models, the "internal model" $\langle V(\mathbb{N}), {}^*V(\mathbb{N}), j\rangle$ which contains the natural model ${}^*V(\mathbb{N})$ of $NPRA^\omega$, and the "full model" $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ which contains the full superstructure $V({}^*\mathbb{N})$ over ${}^*\mathbb{N}$. In both models, j is the inclusion map from $\mathbb{N}$ into ${}^*\mathbb{N}$. The full natural model $\langle V(\mathbb{N}), V({}^*\mathbb{N}), j\rangle$ of *WNA* does not satisfy the axioms $NPRA^\omega$. In particular, the star of quantifier-free induction fails in this model, because the characteristic function of S exists as an object of type ${}^*N \to {}^*N$.

1.5 Bounded minima and overspill

In this section we prove some useful consequences of the *WNA* axioms.

Given a formula $A(x,\ldots)$ of $L(WNA)$, the **bounded minimum** operator is defined by

$$u = (\mu x < y)\, A(x,\ldots) \leftrightarrow \big[u \le y \wedge (\forall x < u)\neg A(x,\ldots) \wedge [A(u,\ldots) \vee u = y]\big],$$

where u is a new variable. By this we mean that the expression to the left of the $\leftrightarrow$ symbol is an abbreviation for the formula to the right of the $\leftrightarrow$ symbol. In particular, if z does not occur in A, $(\mu z < 1)\, A(\ldots)$ is the (inverted) characteristic function of A, which has the value 0 when A is true and the value 1 when A is false.

In *PRA*, the bounded minimum operator is defined similarly.

Lemma 4 *Let $A(m, \vec{n})$ be a quantifier-free formula of $L(PRA)$ and let $\alpha(p, \vec{n})$ be the primitive recursive function such that in PRA,*

$$\alpha(p, \vec{n}) = (\mu m < p)\, A(m, \vec{n}).$$

Then

(i) $WNA \vdash {}^*\alpha(y, \vec{z}) = (\mu x < y)\, {}^*A(x, \vec{z})$.

(ii) *In WNA, there is a quantifier-free formula $B(p, \ldots)$ such that*

$$(\forall m < p)\, A(m, \ldots) \leftrightarrow B(p, \ldots), \qquad (\forall x < y)\, {}^*A(x, \ldots) \leftrightarrow {}^*B(y, \ldots).$$

Similarly for $(\exists x < y)\, {}^*A(x, \ldots)$, and $u = (\mu x < y)\, {}^*A(x, \ldots)$.

Proof. (i) By the axioms of *WNA*, the defining rule for ${}^*\alpha$ is the star of the defining rule for α.

(ii) Apply (i) and observe that in *WNA*,

$$(\forall x < y)\, {}^*A(x, \ldots) \leftrightarrow y = (\mu x < y)\, \neg {}^*A(x, \ldots). \qquad \square$$

Let us write $\forall^\infty x\, A(x, \ldots)$ for $\forall x\, [\neg S(x) \rightarrow A(x, \ldots)]$ and $\exists^\infty x\, A(x, \ldots)$ for $\exists x\, [\neg S(x) \wedge A(x, \ldots)]$.

Lemma 5 *(Overspill) Let $A(x, \ldots)$ be an internal quantifier-free formula. In WNA,*

$$\forall^S x\, A(x, \ldots) \rightarrow \exists^\infty x\, A(x, \ldots) \quad \textit{and} \quad \forall^\infty x\, A(x, \ldots) \rightarrow \exists^S x\, A(x, \ldots).$$

Proof. Work in *WNA*. If $A(H, \ldots)$ we may take $x = H$. Assume $\forall^S x A(x, \ldots)$ and $\neg A(H, \ldots)$. By Lemma 4 (ii) we may take $u = (\mu x < H)\, \neg A(x, \ldots)$. Then $\neg S(u)$. Let $x = u - 1$. We have $x < u$, so $A(x, \ldots)$. Since S is closed under the successor function, $\neg S(x)$. $\square$

We now give a consequence of *WNA* in the language of *PRA* which is similar to Proposition 4.3 in [1] for $NPRA^\omega$. **Σ_1-collection** in $L(PRA)$ is the scheme

$$(\forall m < p) \exists n\, B(m, n, \vec{r}) \rightarrow \exists k (\forall m < p)(\exists n < k) B(m, n, \vec{r})$$

where B is a formula of $L(PRA)$ of the form $\exists q\, C$, C quantifier-free.

Proposition 6 *Σ_1-collection in $L(PRA)$ is provable in WNA.*

Proof. We work in *WNA*. By pairing existential quantifiers, we may assume that $B(m, n, \vec{r})$ is quantifier-free. Assume $(\forall m < p)\exists n\, B(m, n, \vec{r})$. Let $^*B$ be the formula obtained by starring each function symbol in B and replacing variables of type N by variables of type $^*N$.

By the Lifting Axiom and the axiom that S is an initial segment,

$$(\forall x < p)\exists^S y\ ^*B(x, y, j(\vec{r})).$$

Then

$$\forall^\infty w(\forall x < p)(\exists y < w)\ ^*B(x, y, j(\vec{r})).$$

By Lemma 4 and Overspill,

$$\exists^S w(\forall x < p)(\exists y < w)\ ^*B(x, y, j(\vec{r})).$$

By the Lifting Axiom again,

$$\exists k(\forall m < p)(\exists n < k)\, B(m, n, \vec{r}).$$ □

1.6 Standard parts

This section introduces a standard part notion which formalizes a construction commonly used in nonstandard analysis, and provides a link between the type $N \to N$ and the type $^*N$.

In type N let $(n)_k$ be the power of the k-th prime in n, and in type $^*N$ let $(x)_y$ be the power of the y-th prime in x. The function $(n, k) \mapsto (n)_k$ is primitive recursive, and its star is the function $(x, y) \mapsto (x)_y$.

Hereafter, when it is clear from the context, we will write t instead of $j(t)$ in formulas of $L(WNA)$.

Intuitively, we identify $j(t)$ with t, but officially, they are different because t has type N while $j(t)$ has type $^*N$. This will make formulas easier to read. When a term t of type N appears in a place of type $^*N$, it really is $j(t)$.

In the theory *WNA*, we say that x is **near-standard**, in symbols $ns(x)$, if $\forall^S z\, S((x)_z)$. Note that this is equivalent to $\forall n S((x)_n)$. We employ the usual convention for relativized quantifiers, so that $\forall^{ns} x\, B$ means $\forall x\, [ns(x) \to B]$ and $\exists^{ns} x\, B$ means $\exists x\, [ns(x) \wedge B]$. We write

$$x \approx y \text{ if } ns(x) \wedge \forall^S z\, (x)_z = (y)_z.$$

This is equivalent to $ns(x) \wedge \forall n\, (x)_n = (y)_n$. We write $f = {}^o x$, and say f is the **standard part of** x and x is a **lifting of** f, if

$$ns(x) \wedge \forall n f(n) = (x)_n.$$

Note that the operation $x \mapsto {}^o x$ goes from type ${}^*N$ to type $N \to N$. In nonstandard analysis, this often allows one to obtain results about functions of type $N \to N$ by reasoning about hyperintegers of type ${}^*N$.

Lemma 7 *In WNA, suppose that x is near-standard. Then*

(i) If $x \approx y$ then $ns(y)$ and $y \approx x$.

(ii) $(\exists y < H)\, x \approx y$.

Proof. (i) Suppose $x \approx y$. If $S(z)$ then $S((x)_z)$ and $(y)_z = (x)_z$, so $S((y)_z)$. Therefore $ns(y)$, and $y \approx x$ follows trivially.

(ii) Let β be the primitive recursive function $\beta(m,n) = \Pi_{i<m}\, p_i^{(n)_i}$. By Lifting and defining rules for β and ${}^*\beta$, $\forall x \forall u \forall z\, [z < u \to (x)_z = ({}^*\beta(u,x))_z]$. Therefore $\forall^\infty u\, \forall^S z\, (x)_z = ({}^*\beta(u,x))_z$, and hence $\forall^\infty u\, x \approx {}^*\beta(u,x)$. We have $\forall^S w\, w^w < H$, and by Overspill, there exists w with $\neg S(w) \wedge w^w < H$. Since x is near-standard, $\forall^S u\, [u \le w \wedge (\forall z < u)\, p_z^{(x)_z} < w]$. By Overspill,

$$\exists^\infty u\, [u \le w \wedge (\forall z < u)\, p_z^{(x)_z} < w].$$

Let $y = {}^*\beta(u,x)$. Then $x \approx y$. By internal induction,

$$\forall u\, [(\forall z < u)\, p_z^{(x)_z} < w \to {}^*\beta(u,x) < w^u].$$

Then $y \le w^u \le w^w < H$. □

We now state the Standard Part Principle, which says that every near-standard x has a standard part and every f has a lifting.

Standard Part Principle (*STP*):

$$\forall^{ns} x \exists f\, f = {}^o x \wedge \forall f \exists x\, f = {}^o x.$$

The following corollary is an easy consequence of Lemma 7.

Corollary 8 *In WNA, STP is equivalent to*

$$(\forall^{ns} x < H)\exists f\, f = {}^o x \wedge \forall f (\exists x < H)\, f = {}^o x.$$

The **Weak Koenig Lemma** is the statement that every infinite binary tree has an infinite branch. The work in reverse mathematics shows that many classical mathematical statements are equivalent to the Weak Koenig Lemma.

Theorem 9 *The Weak Koenig Lemma is provable in WNA + STP.*