Vorticity and Vortex Dynamics
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With 291 Figures
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Preface

The importance of vorticity and vortex dynamics has now been well recognized at both fundamental and applied levels of fluid dynamics, as already anticipated by Truesdell half century ago when he wrote the first monograph on the subject, The Kinematics of Vorticity (1954); and as also evidenced by the appearance of several books on this field in 1990s. The present book is characterized by the following features:

1. A basic physical guide throughout the book. The material is directed by a basic observation on the splitting and coupling of two fundamental processes in fluid motion, i.e., shearing (unique to fluid) and compressing/expanding. The vorticity plays a key role in the former, and a vortex is nothing but a fluid body with high concentration of vorticity compared to its surrounding fluid. Thus, the vorticity and vortex dynamics is accordingly defined as the theory of shearing process and its coupling with compressing/expanding process.

2. A description of the vortex evolution following its entire life. This begins from the generation of vorticity to the formation of thin vortex layers and their rolling-up into vortices, from the vortex-core structure, vortex motion and interaction, to the burst of vortex layer and vortex into small-scale coherent structures which leads to the transition to turbulence, and finally to the dissipation of the smallest structures into heat.

3. Wide range of topics. In addition to fundamental theories relevant to the above subjects, their most important applications are also presented. This includes vortical structures in transitional and turbulent flows, vortical aerodynamics, and vorticity and vortices in geophysical flows. The last topic was suggested to be added by Late Sir James Lighthill, who read carefully an early draft of the planned table of contents of the book in 1994 and expressed that he likes “all the material” that we proposed there.

These basic features of the present book are a continuation and development of the spirit and logical structure of a Chinese monograph by the same authors, Introduction to Vorticity and Vortex Dynamics, Higher
Education Press, Beijing, 1993, but the material has been completely rewritten and updated. The book may fit various needs of fluid dynamics scientists, educators, engineers, as well as applied mathematicians. Its selected chapters can also be used as textbook for graduate students and senior undergraduates. The reader should have knowledge of undergraduate fluid mechanics and/or aerodynamics courses.

Many friends and colleagues have made significant contributions to improve the quality of the book, to whom we are extremely grateful. Professor Xuesong Wu read carefully the most part of Chaps. 2 through 6 of the manuscript and provided valuable comments. Professor George F. Carnevale’s detailed comments have led to a considerable improvement of the presentation of entire Chap. 12. Professors Boye Ahlhorn, Chien Cheng Chang, Sergei I. Chernyshenko, George Haller, Michael S. Howe, Yu-Ning Huang, Tsutomu Kambe, Shigeo Kida, Shi-Kuo Liu, Shi-Jun Luo, Bernd R. Noack, Rick Salmon, Yi-Peng Shi, De-Jun Sun, Shi-Xiao Wang, Susan Wu, An-Kui Xiong, and Li-Xian Zhuang reviewed sections relevant to their works and made very helpful suggestions for the revision. We have been greatly benefited from the inspiring discussions with these friends and colleagues, which sometimes evolved to very warm interactions and even led to several new results reflected in the book. However, needless to say, any mistakes and errors belong to our own.

Our own research results contained in the book were the product of our enjoyable long-term cooperations and in-depth discussions with Professors Jain-Ming Wu, Bing-Gang Tong, James C. Wu, Israel Wygnanski, Chui-Jie Wu, Xie-Yuan Yin, and Xi-Yun Lu, to whom we truly appreciate. We also thank Misses Linda Engels and Feng-Rong Zhu for their excellent work in preparing many figures, and Misters Yan-Tao Yang and Ri-Kui Zhang for their great help in the final preparation and proof reading of the manuscript.

Finally, we thank the University of Tennessee Space Institute, Peking University, and Tianjin University, without their hospitality and support the completion of the book would have to be greatly delayed. The highly professional work of the editors of Springer Verlag is also acknowledged.

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Introduction

Vortices are a special existence form of fluid motion with origin in the rotation of fluid elements. The most intuitive pictures of these organized structures range from spiral galaxies in universe to red spots of the Jupiter, from hurricanes to tornadoes, from airplane trailing vortices to swirling flows in turbines and various industrial facilities, and from vortex rings in the mushroom cloud of a nuclear explosion or at the exit of a pipe to coherent structures in turbulence. The physical quantity characterizing the rotation of fluid elements is the \[ \mathbf{\omega} = \nabla \times \mathbf{u} \] \( \mathbf{u} \) being the fluid velocity; thus, qualitatively one may say that a vortex is a connected fluid region with high concentration of vorticity compared with its surrounding.\(^1\)

Once formed, various vortices occupy only very small portion in a flow but play a key role in organizing the flow, as “the sinews and muscles of the fluid motion” (Küchemann 1965) and “the sinews of turbulence” (Moffatt et al. 1994). Vortices are also “the voice of fluid motion” (Müller and Obermeier 1988) because at low Mach numbers they are the only source of aeroacoustic sound and noise. These identifications imply the crucial importance of the vorticity and vortices in the entire fluid mechanics. The generation, motion, evolution, instability, and decay of vorticity and vortices, as well as the interactions between vortices and solid bodies, between several vortices, and between vortices and other forms of fluid motion, are all the subject of vorticity and vortex dynamics.\(^2\)

The aim of this book is to present systematically the physical theory of vorticity and vortex dynamics. In this introductory chapter we first locate the position of vorticity and vortex dynamics in fluid mechanics, then briefly review its development. These physical and historical discussions naturally lead to an identification of the scope of vorticity and vortex dynamics, and

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\(^1\) This definition is a generalization of that given by Saffman and Baker (1979) for inviscid flow.

\(^2\) In Chinese, the words “vorticity” and “vortex” can be combined into one character sounds like “vor,” so one has created a single word “vordynamics”.
thereby determine what a book like this one should cover. An outline of every chapter concludes this chapter.

1.1 Fundamental Processes in Fluid Dynamics and Their Coupling

A very basic fact in fluid mechanics is the coexistence and interaction of two fundamental dynamic processes: the compressing/expanding process ("compressing process" for short) and the shearing process, of which a rational definition will be given later. In broader physical context these are called longitudinal and transverse processes, respectively (e.g., Morse and Feshbach 1953). They behave very differently, represented by different physical quantities governed by different equations, with different dimensionless parameters (the Mach number for compressing and the Reynolds number for shearing). These two fundamental processes and their interactions or couplings stand at the center of the entire fluid mechanics.

If we further compare a fluid with a solid, we see at once that their compressing properties have some aspects in common, e.g., both can support longitudinal waves including shock waves, but cannot be indefinitely compressed. What really makes a fluid essentially differ from a solid is their response to a shear stress. While a solid can remain in equilibrium with finite deformation under such a stress, a fluid at rest cannot stand any shear stress. For an ideal fluid with strictly zero shear viscosity, a shearing simply causes one fluid layer to "slide" over another without any resistance, and across the "slip surface" the velocity has a tangent discontinuity. But all fluids have more or less a nonzero shear viscosity, and a shear stress always puts fluid elements into spinning motion, forming rotational or vortical flow. A solid never has those beautiful vortices which are sometimes useful but sometimes harmful, nor turbulence. It is this basic feature of yielding to shear stress that makes the fluid motion extremely rich, colorful, and complicated.

Having realized this basic difference between fluid and solid, one cannot but highly admire a very insightful assertion of late Prof. Shi-Jia Lu (1911–1986), the only female student of Ludwig Prandtl, made around 1980 (private communication):

*The essence of fluid is vortices. A fluid cannot stand rubbing; once you rub it there appear vortices.*

For example, if a viscous flow has a stationary solid boundary, a strong "rubbing" must occur there since the fluid ceases to move on the boundary. A boundary layer is thereby formed, whose separation from the solid boundary is the source of various free shear layers that roll into concentrated vortices which evolve, interact, become unstable and break to turbulence, and finally dissipate into heat.

Of the two fundamental processes and their coupling in fluid, two key physical mechanisms deserve most attention. First, in the interior of a flow,
the so-called *Lamb vector* $\omega \times u$ not only leads to the richest phenomena of shearing process via its curl, such as vortex stretching and tilting as well as turbulent coherent structures formed thereby, but also serves as the crossroad of the two processes. Through the Lamb vector, shearing process can be a byproduct of strong compressing process, for example vorticity produced by a curved shock wave; or vice versa, for example sound or noise produced by vortices. Second, on flow boundaries the two processes are also coupled, but due to the viscosity and the adherence condition. In particular, a tangent pressure gradient (a compressing process) on a solid surface always produces new vorticity, which alters the existing vorticity distributed in the boundary layer and has significant effect on its later development.

The presentation of the entire material in this book will be guided by the earlier concept of two fundamental processes and their coupling.

1.2 Historical Development

Although vortices have been noticed by the mankind ever since very ancient time, rational theories were first developed for the relatively simpler compressing process, from fluid statics to the Bernoulli theorem and to ideal fluid dynamics based on the Euler equation. The theory of rotational flow of ideal fluid was founded by the three vorticity theorems of Helmholtz (1858, English translation 1867), who named such flows as “vortex motions.” His work opened a brand new field, which was enriched by, among others, Kelvin’s (1869) circulation theorem. But the inviscid fluid model on which these theorems are based cannot explain the generation of the vortices and their interaction with solid bodies. Most theoretical studies were still confined to potential flow, leaving the famous *D’Alembert’s paradox* that a uniformly translating body through the fluid would experience no drag. The situation at that time was as Sir Hinshelwood has observed, “... fluid dynamicists were divided into hydraulic engineers who observed what could not be explained, and mathematicians who explained things that could not be observed”. (Lighthill 1956). The theoretical achievements by then has been summarized in the classic monograph of Lamb (1932, first edition: 1879), in which the inviscid, incompressible, and irrotational flow occupies the central position and vortex motion is only a small part. Thus, “Sydney Goldstein has observed that one can read all of Lamb without realizing that water is wet!” (Birkhoff 1960).

A golden age of vorticity and vortex dynamics appeared during 1894–1910s as the birth of aerodynamics associated with the realization of human power

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3 Here lies one of the hardest unsolved mathematic problems, on the finite-time existence, uniqueness, and regularity of the solutions of the Navier-Stokes equations. To quote Doering and Gibbon 1995: “It turns out that the nonlinear terms that can’t be controlled mathematically are precisely those describing what is presumed to be the basic physical mechanism for the generation of turbulence, namely vortex stretching”.

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flight. Owing to the astonishing achievements of those prominent figures such as Lanchester, Joukowski, Kutta, and Prandtl, one realized that a wing can fly with sustaining lift and relatively much smaller drag due solely to the vortex system it produces.

More specifically, in today’s terminology, the Kutta–Joukowski theorem (1902–1906) proves that the lift on an airfoil is proportional to its flight speed and surrounding velocity circulation, which is determined by the Kutta condition that the flow must be regular at the sharp trailing edge of the airfoil. The physical root of such a vortex system lies in the viscous shearing process in the thin boundary layer adjacent to the wing surface, as revealed by Prandtl (1904). The wing circulation is nothing but the net vorticity contained in the asymmetric boundary layers at upper and lower surfaces of the wing, and the Kutta condition imposed for inviscid flow is simply a synthetic consequence of these boundary layers at the trailing edge.

The wing vortex system has yet another side. The boundary layers that provide the lift also generate a friction drag. Moreover, as the direct consequence of the theorems of Helmholtz and Kelvin, these layers have to leave the wing trailing edge to become free vortex layers that roll into strong trailing vortices in the wake (already conceived by Lanchester in 1894), which cause an induced drag.

All these great discoveries made in such a short period formed the classic low-speed aerodynamics theory. Therefore, at a low Mach number all aspects of the wing-flow problem (actually any flow problems) may essentially amount to vorticity and vortex dynamics. The rapid development of aeronautical techniques in the first half of the twentieth century represented the greatest practice in the human history of utilization and control of vortices, as summarized in the six- and two-volume monographs edited by Durand (1934–1935) and Goldstein (1938), respectively.

Then, the seek for high flight speed turned aerodynamicists’ attention back to compressing process. High-speed aerodynamics is essentially a combination of compressing dynamics and boundary-layer theory (cf. Liepmann and Roshko 1957). But soon after that another golden age of vorticity and vortex dynamics appeared owing to the important finding of vortical structures of various scales in transitional and turbulent flows. In fact, the key role of vortex dynamics in turbulence had long been speculated since 1920–1930s, a concept that attracted leading scientists like Taylor and Thomson, and reflected vividly in the famous verse by Richardson (1922):

Big whirls have little whorls,
Which feed on their velocity.
And little whorls have lesser whorls,
And so on to viscosity.

4 For a detailed historical account of the times from Helmholtz to this exciting period with full references, see Giacomelli and Pistolesi (1934).
This concept was confirmed and made more precise by the discovery of turbulent coherent structures, which immediately motivated extensive studies of vortex dynamics in turbulence. The intimate link between aerodynamic vortices and turbulence has since been widely appreciated (e.g., Lilley 1983). In fact, this second golden age also received impetus from the continuous development of aerodynamics, such as the utilization of stable separated vortices from the leading edges of a slender wing at large angles of attack, the prevention of the hazardous effect of trailing vortices on a following aircraft, and the concern about vortex instability and breakdown. Meanwhile, the importance and applications of vorticity and vortex dynamics in ocean engineering, wind engineering, chemical engineering, and various fluid machineries became well recognized. On the other hand, the formation and evolution of large-scale vortices in atmosphere and ocean had long been a crucial part of geophysical fluid dynamics.

The second golden age of vorticity and vortex dynamics has been anticipated in the writings of Truesdell (1954), Lighthill (1963), and Batchelor (1967), among others. Truesdell (1954) made the first systematic exposition of vorticity kinematics. In the introduction to his book, Batchelor (1967) claimed that “I regard flow of an incompressible viscous fluid as being at the center of fluid dynamics by virtue of its fundamental nature and its practical importance. ... most of the basic dynamic ideas are revealed clearly in a study of rotational flow of a fluid with internal friction; and for applications in geophysics, chemical engineering, hydraulics, mechanical and aeronautical engineering, this is still the key branch of fluid dynamics”. It is this emphasis on viscous shearing process, in our view, that has made Batchelor’s book a representative of the second generation of textbooks of fluid mechanics after Lamb (1932). In particular, the article of Lighthill (1963) sets an example of using vorticity to interpret a boundary layer and its separation, indicating that “although momentum considerations suffice to explain the local behavior in a boundary layer, vorticity considerations are needed to place the boundary layer correctly in the flow as a whole. It will also be shown (surprisingly, perhaps) that they illuminate the detailed development of the boundary layer ... just as clear as do momentum considerations...”. Therefore, Lighthill has placed the entire boundary layer theory (including flow separation) correctly in the realm of vorticity dynamics as a whole.

So far the second golden age is still in rapid progress. The achievements during the second half of the twentieth century have been reflected not only by innumerable research papers but also by quite a few comprehensive monographs and graduate textbooks appeared within a very short period of 1990s, e.g., Saffman (1992), Wu et al. (1993), Tong et al. (1994), Green (1995), and Lugt (1996), along with books and collected articles on special topics of this field, e.g., Tong et al. (1993), Voropayev and Afanasyev (1994), and Hunt and Vassilicos (2000). Yet not included in but relevant to this list are books on steady and unsteady flow separation, on the stability of shear flow and vortices, etc. In addition to these, very far-reaching new directions has also
emerged, such as applications to external and internal biofluidodynamics and biomimetics, and vortex control that in broad sense stands at the center of the entire field of flow control (cf. Gad-el-Hak 2000). The current fruitful progress of vortex dynamics and control in so many branches will have a very bright future.

1.3 The Contents of the Book

Based on the preceding physical and historical discussions, especially following Lu’s assertion, we consider the vorticity and vortex dynamics a branch of fluid dynamics that treats the theory of shearing process and its interaction with compressing process. This identification enables one to study as a whole the full aspects and entire life of a vortex, from its kinematics to kinetics, and from the generation of vorticity to the dissipation of vortices. But this identification also posed to ourselves a task almost impossible, since it implies that the range of the topics that should be included is too wide to be put into a single volume. Thus, certain selection has to be made based on the authors’ personal background and experience. Even so, the content of the book is still one of the widest of all relevant books.

A few words about the terminology is in order here. By the qualitative definition of a vortex given at the beginning of this section, a vortex can be identified when a vorticity concentration of arbitrary shape occurs in one or two spatial dimensions, having a layer-like or axial structure, respectively. The latter is the strongest form permissible by the solenoidal nature of vorticity, and as said before is often formed from the rolling up of the former as a further concentration of vorticity. But, conventionally layer-like structures have their special names such as boundary layer (attached vortex layer) and free shear layer or mixing layer (free vortex layer). Only axial structures are simply called vortices, which can be subdivided into disk-like vortices with diameter much larger than axial scale such as a hurricane, and columnar vortices with diameter much smaller than axial length such as a tornado Lugt (1983). While we shall follow this convention, it should be borne in mind that the layer-like and axial structures are often closely related as different temporal evolution stages and/or spatial portions of a single vortical structure.

Having said these, we now outline the organization of the book, which is divided into four parts.

Part I concerns vorticity dynamics and consists of five chapters. Chapter 2 is an overall introduction of two fundamental dynamic processes in fluid motion. After highlighting the basis of fluid kinematics and dynamics, this chapter introduces the mathematic tools for decomposing a vector field into a longitudinal part and a transverse part. This decomposition is then applied to the momentum equation, leading to an identification of each process and their coupling.
Chapter 3 gives a systematic presentation of vorticity kinematics, from spatial properties to temporal evolution, both locally and globally. The word “kinematics” is used here in the same spirit of Truesdell (1954); namely, without involving specific kinetics that identifies the cause and effect. Therefore, the results remain universal. The last section of Chap. 3 is devoted to the somewhat idealized circulation-preserving flow, in which the kinetics enters the longitudinal (compressing) process but keeps away from the transverse (shearing) process. Rich theoretical consequences follow from this situation.

Chapter 4 sets a foundation of vorticity dynamics. First, the physical mechanisms that make the shearing process no longer purely kinematic are addressed and exemplified, with emphasis on the role of viscosity. Second, the characteristic behaviors of a vorticity field at small and large Reynolds' numbers are discussed, including a section on vortex sheet dynamics as an asymptotic model when the viscosity approaches zero (but not strictly zero). Finally, formulations of viscous flow problems in terms of vorticity and velocity are discussed, which provides a theoretical basis for developing relevant numerical methods.

Chapter 5 presents theories of flow separation (more specifically and importantly, boundary-layer separation at large Reynolds' numbers). Due to separation, a boundary layer bifurcates to a free shear layer, which naturally rolls up into a concentrated vortex. Thus, typically though not always, a vortex originates from flow separation. Therefore, this chapter may serve as a transition from vorticity dynamics to vortex dynamics.

The next three chapters constitute Part II as fundamentals of vortex dynamics. In Chap. 6 we present typical vortex solutions, including both exact solutions of the Navier–Stokes and Euler equations (often not fully realistic) and asymptotic solutions that are closer to reality. The last section of the chapter discusses an open issue on how to quantitatively identify a vortex. According to the evolution order of a vortex in its whole life, this chapter should appear after Chap. 7; but it seems better to introduce the vortex solutions as early as possible although this arrangement makes the logical chain of the book somewhat interrupted.

The global separated flow addressed in Chap. 7 usually has vortices as sinews and muscles, which evolve from the local flow separation processes (Chap. 5). After introducing a general topological theory as a powerful qualitative tool in analyzing separated flow, we discuss steady and unsteady separated flows. The former has two basic types: separated bubble flow and free vortex-layer separated flow, each of which can be described by an asymptotic theory as the viscosity approaches zero. In contrast, unsteady separated flow is much more complicated and no general theory is available. We thus confine ourselves to the most common situation, the unsteady separated flow behind

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5 For many authors, any time evolution of a system is considered falling into the category of dynamics.
6 The methods themselves are beyond the scope of the book.
Introduction

To describe different stages of the entire life of a vortex, various approximate theories have been developed to capture the dominant dynamic mechanisms. These are discussed in Chap. 8, including vortex-core dynamics, three-dimensional vortex filaments, two-dimensional point-vortex systems, and vortex patches, etc. The chapter also discusses typical interactions of a vortex with a solid wall and a free surface.

The vorticity plays a crucial role as flow becomes unstable, and rich patterns of vortex motion appear during the transition to turbulent flow and in fully developed turbulence. The relevant complicated mechanisms are discussed in Part III as a more advanced part of vorticity and vortex dynamics. Chapter 9 presents selected hydrodynamic stability theories for vortex layers and vortices. In addition to interpreting the basic concepts and classic results of shear-flow instability in terms of vorticity dynamics, some later developments of vortical-flow stability will be addressed. The chapter also introduces recent progresses in the study of vortex breakdown, which is a highly nonlinear process and has been a long-standing difficult issue.

Chapter 10 discusses the vortical structures in transitional and turbulent flows, starting with the concept of coherent structure and a discussion on coexistence of vortices and waves in turbulence fields. The main contents focus on the physical and qualitative understanding of the formation, evolution, and decay of coherent structures using mixing layer and boundary layer as examples, which are then extended to vortical structures in other shear flows. The understanding of coherent structure dynamics is guided by the examination of two opposite physical processes, i.e., the instability, coherence production, self-organization or negative entropy generation (the first process) and the coherent-random transfer, cascade, dissipation or entropy generation (the second process). The energy flow chart along the two processes and its impact on the philosophy of turbulent flow control is briefly discussed. Based on the earlier knowledge, typical applications of vorticity equations in studying coherent structures are shown. The relation between the vortical structures and the statistical description of turbulence field are also discussed, which may lead to some expectation on the future studies.

The topics of Part IV, including Chaps. 11 and 12, are somewhat more special. As an application of vorticity and vortex dynamics to external-flow aerodynamics, Chap. 11 presents systematically two types of theories, the projection theory and derivative-moment theory, both having the ability to reveal the local shearing process and flow structures that are responsible for the total force and moment but absent in conventional force–moment formulas. The classic aerodynamics theory will be rederived with new insight. This subject is of great interest for understanding the physical sources of the force and moment, for their diagnosis, configuration design, and effective flow control.

Chapter 12 is an introduction to vorticity and vortical structures in geophysical flow, which expands the application of vorticity and vortex dynamics

a bluff body, and focus on its phenomena and some qualitative physical interpretations.
to large geophysical scales. The most important concept in the determination of large-scale atmospheric and oceanic vortical motion is the potential vorticity. The dynamics of vorticity also gains some new characters due to the Earth’s rotation and density stratification.

Throughout the book, we put the physical understanding at the first place. Whenever possible, we shall keep the generality of the theory; but it is often necessary to be confined to as simple flow models as possible, provided the models are not oversimplified to distort the subject. Particularly, incompressible flow will be our major model for studying shearing process, due to its relative simplicity, maturity, and purity as a test bed of the theory. Obviously, to enter the full coupling of shearing and compressing processes, at least a weakly compressible flow is necessary.

The reader is assumed to be familiar with general fluid dynamics or aerodynamics at least at undergraduate level but better graduate level of major in mechanics, aerospace, and mechanical engineering. To make the book self-contained, a detailed appendix is included on vectors, tensors, and their various operations used in this book.
Part I

Vorticity Dynamics
Fundamental Processes in Fluid Motion

2.1 Basic Kinematics

For later reference, in this section we summarize the basic principles of fluid kinematics, which deals with the fluid deformation and motion in its most general continuum form, without any concern of the causes of these deformation and motion. We shall be freely using tensor notations and operations, of which a detailed introduction is given in Appendix.

2.1.1 Descriptions and Visualizations of Fluid Motion

As is well known, the fluid motion in space and time can be described in two ways. The first description follows every fluid particle, exactly the same as in the particle mechanics. Assume a fluid body \( V \) moves arbitrarily in the space, where a fixed Cartesian coordinate system is introduced. Let a fluid particle in \( V \) locate at \( X = (X_1, X_2, X_3) \) at an initial time \( \tau = 0 \), then \( X \) is the label of this particle at any time.\(^1\) This implies that

\[
\frac{\partial X}{\partial \tau} = 0.
\]  

Assume at a later time \( \tau \) the fluid particle moves smoothly to \( x = (x_1, x_2, x_3) \). Then all \( x \) in \( V \) can be considered as differentiable functions of \( X \) and \( \tau \):

\[
x = \phi(X, \tau),
\]  

where \( \phi(X, 0) = X \). For fixed \( X \) and varying \( \tau \), (2.2) gives the path of the particle labeled \( X \); while for fixed \( \tau \) and varying \( X \), it determines the spatial region \( V(\tau) \) of the whole fluid body at that moment. This description is called material description or Lagrangian description, and \( (X, \tau) \) are material or Lagrangian variables.

\(^1\) More generally, the label of a fluid particle can be any set of three numbers which are one-to-one mappings of the particle’s initial coordinates.
Equation (2.2) is a continuous mapping of the physical space onto itself with parameter \( \tau \). But functionally the spaces spanned by \( \mathbf{X} \) and \( \mathbf{x} \) are different. We call the former reference space or simply the \( \mathbf{X} \)-space. In this space, differenting (2.2) with respect to \( \mathbf{X} \), gives a tensor of rank 2 called the deformation gradient tensor:

\[
\mathbf{F} = \nabla_{\mathbf{X}} \mathbf{x} \quad \text{or} \quad F_{\alpha i} = x_{i,\alpha},
\]

which describes the displacement of all particles initially neighboring to the particle \( \mathbf{X} \). Hereafter we use Greek letters for the indices of the tensor components in the reference space, and Latin letters for those in the physical space. The gradient with respect to \( \mathbf{X} \) is denoted by \( \nabla_{\mathbf{X}} \), while the gradient with no suffix is with respect to \( \mathbf{x} \). \( (\cdot)_\alpha \) is a simplified notation of \( \partial(\cdot)/\partial X_\alpha \).

The deformation gradient tensor \( \mathbf{F} \) defines an infinitesimal transformation from the reference space to physical space. Indeed, assume that at \( \tau = 0 \) a fluid element occupies a cubic volume \( dV \), and at some \( \tau \) it moves to the neighborhood of \( \mathbf{x} \), occupying a volume \( dv \). Then according to the theory of multivariable functions and the algebra of mixing product of vectors, we see that

\[
dv = J dV,
\]

where the Jacobian

\[
J \equiv \frac{\partial(x_1,x_2,x_3)}{\partial(X_1,X_2,X_3)} = \det \mathbf{F}
\]

represents the expansion or compression of an infinitesimal volume element during the motion. Moreover, keeping the labels of particles, any variation of \( J \) can only be caused by that of \( \mathbf{x} \). By using (2.4), an infinitesimal change of \( J \) is given by (for an explicit proof see Appendix, A.4.1)

\[
\delta J = J \nabla \cdot \delta \mathbf{x}.
\]

Initially separated particles cannot merge to a single point at later time, even though they may be tightly squeezed together; meanwhile, a single particle initially having one label cannot be split into several different ones. Thus we can always trace back to the particle’s initial position from its position \( \mathbf{x} \) at any \( \tau > 0 \). This means the mapping (2.2) is one-to-one and has inverse

\[
\mathbf{X} = \Phi(\mathbf{x}, t).
\]

Here \( t = \tau \) is the same time variable but used along with \( \mathbf{x} \). Functions \( \Phi \) and \( \phi \) are assumed to have derivatives of sufficiently many orders. Since (2.2) is invertible, \( J \) must be regular, i.e.,

\[
0 < J < \infty.
\]
2.1 Basic Kinematics

\[ u = \frac{\partial x}{\partial \tau}, \quad a = \frac{\partial^2 x}{\partial \tau^2} = \frac{\partial u}{\partial \tau}, \]

respectively. Therefore, the study of fluid motion in this description amounts to solving a dynamic system of infinitely many degrees of freedom. Unlike solids which can only have relatively small deformation gradient, a fluid cannot stand shearing and hence \( F \) may have very complicated behavior. Thus, the Lagrangian approach is generally inconvenient. However, it finds some important applications in both theoretical and computational vorticity dynamics. Special topics of Lagrangian description useful in some later chapters are given in Appendix A.4.

Instead of following the motion of each particle, we may concentrate on the spatial distribution of physical quantities and their temporal variation at every point \( x \). This leads to the field description or Eulerian description. By (2.2) and (2.7), any field quantity \( F(x, t) \) can be expressed as

\[ F(x, t) = F(\phi(X, \tau), \tau), \]

which is also a function of \( (X, \tau) \), and vice versa. Now there is

\[ \frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial t} + \frac{\partial x_i}{\partial \tau} \frac{\partial F}{\partial x_i}, \]

thus

\[ \frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + u \cdot \nabla \equiv \frac{D}{Dt} \quad \text{(2.9)} \]

is the operator of material derivative (i.e., the rate of change following the same particles) in the field description. Here \( \partial / \partial t \) implies the local rate of change at fixed \( x \), while \( u \cdot \nabla \) is the rate of change due to advection. Clearly, \( \partial / \partial \tau \) for fixed \( X \) and \( \partial / \partial t \) for fixed \( x \) are different operations. Thus, (2.1) and \( a = \partial u / \partial \tau \) in the Eulerian description become, respectively,

\[ \frac{DX}{Dt} = 0, \quad \text{(2.10)} \]

\[ a = \frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \cdot \nabla u. \quad \text{(2.11)} \]

Moreover, from (2.6) and (2.9) follows the classic Euler formula for the rate of change of \( J \):

\[ \frac{DJ}{Dt} = J \nabla \cdot u. \quad \text{(2.12)} \]

As a field theory, Eulerian description does not care which fluid particle occupies a position \( x \) at time \( t \). As long as two flow fields have the same velocity distribution in space and time, they will be considered kinematically identical, no matter if their individual particles experience the same motion. In other words, fluid particles are allowed to be relabeled during the motion.
in the Eulerian description. This is the case in most applications. Hence, the information in the Eulerian description is less than that in the Lagrangian description.

Some developments of fluid mechanics, however, are based on taking (2.11) as a nonlinear system and thus require tracing individual particles. This requirement happens in the study of, e.g., chaos and mixing, as well as in variational approaches of vortical flows. In these situations one has to keep the particle labels $X$ unchanged. This is simply done by adding to the Eulerian description condition (2.10), known as Lin’s constraint (cf. Serrin 1959). Then the two descriptions become fully equivalent (for more discussion see Sect. 3.6).

It is appropriate here to distinct three different types of curves in a flow field, defined based on the earlier two descriptions. First, as noted earlier, a pathline is the curve created by the motion of a particle $X$ as time goes on, described by (2.2), where $X$ is fixed and $0 < \tau < \infty$. Equation (2.2) is the solution curve of the ordinary differential equation

$$\frac{dx_i}{dt} = u_i(x,t) \tag{2.13}$$

under the initial condition $x(0) = X$.

Next, a curve tangent to the velocity $u(x,t)$ everywhere at a time $t$ is a streamline at this time. Its equation follows from eliminating $dt$ in (2.13):

$$\frac{dx_1}{u_1(x,t)} = \frac{dx_2}{u_2(x,t)} = \frac{dx_3}{u_3(x,t)}, \tag{2.14}$$

of which the solution curve $f_1(x,t) = 0, f_2(x,t) = 0$ passing a given $x$ at a given $t$ is the required streamline.

Then, consider all the fluid particles which have passed a point $x_0$ at any $t < t_0$ and continue to move ahead. The set of their spatial positions at $t_0$ constitute a curve passing $x_0$, called a streakline passing $(x_0,t_0)$. By (2.7), the labels of these particles are $\Phi(x_0,t), -\infty < t \leq t_0$, where $t$ becomes a parameter for identifying different particles. The positions of these particles at $t_0$ follow from (2.2):

$$x(x_0,t_0,t) = \phi(\Phi(x_0,t), t_0), -\infty < t \leq t_0. \tag{2.15}$$

It is easily seen that at a given $t_0$, the streamline passing $x_0$, the pathline of a particle locating at $x_0$ and the streakline passing $x_0$ have a common tangent vector at $x_0$. When the flow is steady, i.e., in (2.13) $u$ is independent of $t$, the three curves coincide. But even in this case the streamlines may not necessarily be well-ordered; in Sect. 3.2.3 we shall see a famous counter-example. For more general unsteady flows, then, the three curves are entirely different. The behavior of streamlines and pathlines vary drastically as the observer changes from a fixed frame of reference to a moving one, but the streaklines will remain the same (Taneda 1985). Figure 2.1 sketches the unsteady streamlines and streaklines due to the instability traveling waves in a flat-plate boundary
Fig. 2.1. Schematic streamlines (viewed in different frames) and streaklines in a boundary layer with traveling instability waves. $C$ is the wave speed. Reproduced from Taneda (1985)

Layer, viewed from different frames of reference. Note that the streamlines in the frame moving with the wave exhibit some vortex-like structure (so-called “cat-eyes”), but whether or not these cat-eyes can be classified as vortices should be judged by the concentration of vorticity rather than merely by the frame-dependent streamlines. A discussion on vortex definition will be made in Sect. 6.6.

Flow visualization is a powerful and intuitive means in understanding various vortical flows (see Van Dyke 1982), of which the foundation is a clear distinction of the earlier three types of curves. If one introduces a tracer particle into the fluid and photographs its motion with a long time exposure, he/she obtains a pathline. If one spreads the tracer particles and takes photo with very short time exposure, then he/she sees a set of short line segments, of which a smooth connection can represent a family of instant streamlines. If one introduces some dyed fluid continuously at a fixed point $x_0$ and takes a fast photograph at a later time $t_0$, then he/she obtains the streakline consisting of all particles passing $x_0$ at any $t \leq t_0$.

A pathline or a streakline can intersect itself, but a streamline cannot. Most visualization experiments with vortical flows give at least streaklines. But their interpretation needs great care, since vortical flows are inherently more or less unsteady. Ignoring the difference of these three types of lines in an unsteady flow may lead to serious misunderstanding. Figure 2.2 shows both streamlines and streaklines due to the unsteady vortex shedding from a
circular cylinder, where their difference is obvious. However, while streaklines can tell where the vorticity resides in a flow, it tells very little about the surrounding fluid and the entrainment process. In this regard instantaneous streamlines in an unsteady flow are still useful; and it can be shown that over a very short time interval streaklines, pathlines, and instantaneous streamlines are identical (Perry et al. 1982).

Finally, if one inserts a straight metal wire across a moving fluid (say, water) and introduces pulsating current with fixed frequency $\omega$ through it, then the wire will electrolyze the water and release hydrogen bubbles periodically, which are advected by local flow velocity. Hence the pulsating appearance of bubbles will form a velocity profile along the wire (Fig. 2.3). These pulsating lines are called *time-lines*. Referring to Fig. 2.3 and assume the metal line is located along the $y$-axis. Then at time $t$, the time line released from all points $x_0 = (0, y)$ at initial time $t_0 < t$ is given by

$$
    x(x_0, t_0, t; \omega) = \phi(\Phi(0, y, t_0), t; \omega) \quad \text{for all} \quad y, t > t_0.
$$

(2.16)

### 2.1.2 Deformation Kinematics. Vorticity and Dilatation

The central issue of fluid kinematics concerns the deformation rate of a material fluid line, surface, and volume element. We first consider the rate of change of a material line element $dx$. By using (2.2), since $X$ is fixed, there is

$$
    \frac{D}{Dt}(dx_i) = \frac{D}{Dt}(x_{i,\alpha} dX_\alpha) = u_{i,\alpha} dX_\alpha = u_{i,\alpha} X_{\alpha,j} dx_j = u_{i,j} dx_j,
$$

Fig. 2.2. Streamlines and streaklines in unsteady vortex shedding from a circular cylinder. From Taneda (1985)
Thus
\[
\frac{D}{Dt}(dx) = dx \cdot \nabla u = du. \tag{2.17}
\]
Therefore, the rate of change of a material line element results in the velocity difference \(du\) at the two ends of the element. This rate of change includes both magnitude and direction, which should be analyzed separately. But the more fundamental quantity in (2.17) is the velocity gradient \(\nabla u\), the deformation-rate tensor (not to be confused with the deformation gradient tensor \(F\) defined by (2.3)), which is independent of the choice of specific line element. The intrinsic decomposition of \(\nabla u\) into symmetric and antisymmetric parts gives
\[
\nabla u = D + \Omega, \tag{2.18}
\]
where \(D\) and \(\Omega\) are the strain-rate tensor and vorticity tensor (or spin tensor), defined by
\[
D = \frac{1}{2} [\nabla u + (\nabla u)^T], \quad \text{with} \quad D_{ii} = \vartheta, \tag{2.19a}
\]
\[
\Omega = \frac{1}{2} [\nabla u - (\nabla u)^T], \quad \text{with} \quad \epsilon_{ijk} \Omega_{jk} = \omega_i, \tag{2.19b}
\]
respectively, with the superscript \(T\) denoting transpose. Here,
\[
\vartheta = \nabla \cdot u, \quad \omega = \nabla \times u \tag{2.20}
\]
are the dilatation and vorticity, respectively, which are the central concepts of this book and serve as the kinematic representations of the compressing and shearing processes.\(^2\)

\(^2\) In a shearing process the vorticity usually coexists with the rate of strain, see the example of simple shear flow discussed later and a general triple decomposition.