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The Mathematics of Arbitrage

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To Rita and Christine with love

Preface

In 1973 F. Black and M. Scholes published their pathbreaking paper [BS 73] on option pricing. The key idea — attributed to R. Merton in a footnote of the Black-Scholes paper — is the use of trading in continuous time and the notion of arbitrage. The simple and economically very convincing *"principle of no-arbitrage"* allows one to derive, in certain mathematical models of financial markets (such as the Samuelson model, [S 65], nowadays also referred to as the "Black-Scholes" model, based on geometric Brownian motion), unique prices for options and other contingent claims.

This remarkable achievement by F. Black, M. Scholes and R. Merton had a profound effect on financial markets and it shifted the paradigm of dealing with financial risks towards the use of quite sophisticated mathematical models.

It was in the late seventies that the central role of no-arbitrage arguments was crystallised in three seminal papers by M. Harrison, D. Kreps and S. Pliska ([HK 79], [HP 81], [K 81]) They considered a general framework, which allows a systematic study of different models of financial markets. The Black-Scholes model is just one, obviously very important, example embedded into the framework of a general theory. A basic insight of these papers was the intimate relation between no-arbitrage arguments on one hand, and martingale theory on the other hand. This relation is the theme of the *"Fundamental Theorem of Asset Pricing"* (this name was given by Ph. Dybvig and S. Ross [DR 87]), which is not just a single theorem but rather a general principle to relate no-arbitrage with martingale theory. Loosely speaking, it states that a mathematical model of a financial market is free of arbitrage if and only if it is a martingale under an equivalent probability measure; once this basic relation is established, one can quickly deduce precise information on the pricing and hedging of contingent claims such as options. In fact, the relation to martingale theory and stochastic integration opens the gates to the application of a powerful mathematical theory.

The mathematical challenge is to turn this general principle into precise theorems. This was first established by M. Harrison and S. Pliska in [HP 81] for the case of finite probability spaces. The typical example of a model based on a finite probability space is the "binomial" model, also known as the "Cox-Ross-Rubinstein" model in finance.

Clearly, the assumption of finite Ω is very restrictive and does not even apply to the very first examples of the theory, such as the Black-Scholes model or the much older model considered by L. Bachelier [B 00] in 1900, namely just Brownian motion. Hence the question of establishing theorems applying to more general situations than just finite probability spaces Ω remained open.

Starting with the work of D. Kreps [K 81], a long line of research of increasingly general — and mathematically rigorous — versions of the "Fundamental Theorem of Asset Pricing" was achieved in the past two decades. It turned out that this task was mathematically quite challenging and to the benefit of both theories which it links. As far as the financial aspect is concerned, it helped to develop a deeper understanding of the notions of arbitrage, trading strategies, etc., which turned out to be crucial for several applications, such as for the development of a dynamic duality theory of portfolio optimisation (compare, e.g., the survey paper [S 01a]). Furthermore, it also was fruitful for the purely mathematical aspects of stochastic integration theory, leading in the nineties to a renaissance of this theory, which had originally flourished in the sixties and seventies.

It would go beyond the framework of this preface to give an account of the many contributors to this development. We refer, e.g., to the papers [DS 94] and [DS 98], which are reprinted in Chapters 9 and 14.

In these two papers the present authors obtained a version of the "Fundamental Theorem of Asset Pricing", pertaining to general $\mathbb{R}^d$-valued semi-martingales. The arguments are quite technical. Many colleagues have asked us to provide a more accessible approach to these results as well as to several other of our related papers on Mathematical Finance, which are scattered through various journals. The idea for such a book already started in 1993 and 1994 when we visited the Department of Mathematics of Tokyo University and gave a series of lectures there.

Following the example of M. Yor [Y 01] and the advice of C. Byrne of Springer-Verlag, we finally decided to reprint updated versions of seven of our papers on Mathematical Finance, accompanied by a guided tour through the theory. This guided tour provides the background and the motivation for these research papers, hopefully making them more accessible to a broader audience.

The present book therefore is organised as follows. Part I contains the "guided tour" which is divided into eight chapters. In the introductory chapter we present, as we did before in a note in the Notices of the American Mathematical Society [DS 04], the theme of the Fundamental Theorem of Asset Pricing in a nutshell. This chapter is very informal and should serve mainly to build up some economic intuition.

In Chapter 2 we then start to present things in a mathematically rigourous way. In order to keep the technicalities as simple as possible we first restrict ourselves to the case of finite probability spaces Ω. This implies that all the function spaces $L^p(\Omega, \mathcal{F}, \mathbf{P})$ are finite-dimensional, thus reducing the functional analytic delicacies to simple linear algebra. In this chapter, which presents the theory of pricing and hedging of contingent claims in the framework of finite probability spaces, we follow closely the Saint Flour lectures given by the second author [S 03].

In Chapter 3 we still consider only finite probability spaces and develop the basic duality theory for the optimisation of dynamic portfolios. We deal with the cases of complete as well as incomplete markets and illustrate these results by applying them to the cases of the binomial as well as the trinomial model.

In Chapter 4 we give an overview of the two basic continuous-time models, the "Bachelier" and the "Black-Scholes" models. These topics are of course standard and may be found in many textbooks on Mathematical Finance. Nevertheless we hope that some of the material, e.g., the comparison of Bachelier versus Black-Scholes, based on the data used by L. Bachelier in 1900, will be of interest to the initiated reader as well.

Thus Chapters 1–4 give expositions of basic topics of Mathematical Finance and are kept at an elementary technical level. From Chapter 5 on, the level of technical sophistication has to increase rather steeply in order to build a bridge to the original research papers. We systematically study the setting of general probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$. We start by presenting, in Chapter 5, D. Kreps' version of the Fundamental Theorem of Asset Pricing involving the notion of "No Free Lunch". In Chapter 6 we apply this theory to prove the Fundamental Theorem of Asset Pricing for the case of finite, discrete time (but using a probability space that is not necessarily finite). This is the theme of the Dalang-Morton-Willinger theorem [DMW 90]. For dimension $d \geq 2$, its proof is surprisingly tricky and is sometimes called the "100 meter sprint" of Mathematical Finance, as many authors have elaborated on different proofs of this result. We deal with this topic quite extensively, considering several different proofs of this theorem. In particular, we present a proof based on the notion of "measurably parameterised subsequences" of a sequence $(f_n)_{n=1}^{\infty}$ of functions. This technique, due to Y. Kabanov and C. Stricker [KS 01], seems at present to provide the easiest approach to a proof of the Dalang-Morton-Willinger theorem.

In Chapter 7 we give a quick overview of stochastic integration. Because of the general nature of the models we draw attention to general stochastic integration theory and therefore include processes with jumps. However, a systematic development of stochastic integration theory is beyond the scope of the present "guided tour". We suppose (at least from Chapter 7 onwards) that the reader is sufficiently familiar with this theory as presented in several beautiful textbooks (e.g., [P 90], [RY 91], [RW 00]). Nevertheless, we do highlight those aspects that are particularly important for the applications to Finance.

Finally, in Chapter 8, we discuss the proof of the Fundamental Theorem of Asset Pricing in its version obtained in [DS 94] and [DS 98]. These papers are reprinted in Chapters 9 and 14.

The main goal of our "guided tour" is to build up some intuitive insight into the Mathematics of Arbitrage. We have refrained from a logically well-ordered deductive approach; rather we have tried to pass from examples and special situations to the general theory. We did so at the cost of occasionally being somewhat incoherent, for instance when applying the theory with a degree of generality that has not yet been formally developed. A typical example is the discussion of the Bachelier and Black-Scholes models in Chapter 4, which is introduced before the formal development of the continuous time theory. This approach corresponds to our experience that the human mind works inductively rather than by logical deduction. We decided therefore on several occasions, e.g., in the introductory chapter, to jump right into the subject in order to build up the motivation for the subsequent theory, which will be formally developed only in later chapters.

In Part II we reproduce updated versions of the following papers. We have corrected a number of typographical errors and two mathematical inaccuracies (indicated by footnotes) pointed out to us over the past years by several colleagues. Here is the list of the papers.

Chapter 9: [DS 94] A General Version of the Fundamental Theorem of Asset Pricing
Chapter 10: [DS 98a] A Simple Counter-Example to Several Problems in the Theory of Asset Pricing
Chapter 11: [DS 95b] The No-Arbitrage Property under a Change of Numéraire
Chapter 12: [DS 95a] The Existence of Absolutely Continuous Local Martingale Measures
Chapter 13: [DS 97] The Banach Space of Workable Contingent Claims in Arbitrage Theory
Chapter 14: [DS 98] The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes
Chapter 15: [DS 99] A Compactness Principle for Bounded Sequences of Martingales with Applications

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This book owes much to many: in particular, we are deeply indebted to our many friends in the functional analysis, the probability, as well as the mathematical finance communities, from whom we have learned and benefitted over the years.

Zurich, November 2005, *Freddy Delbaen*
Vienna, November 2005 *Walter Schachermayer*

Contents

Part I

A Guided Tour to Arbitrage Theory

1

The Story in a Nutshell

1.1 Arbitrage

The notion of arbitrage is crucial to the modern theory of Finance. It is the corner-stone of the option pricing theory due to F. Black, R. Merton and M. Scholes [BS 73], [M 73] (published in 1973, honoured by the Nobel prize in Economics 1997).

The idea of arbitrage is best explained by telling a little joke: a professor working in Mathematical Finance and a normal person go on a walk and the normal person sees a 100 € bill lying on the street. When the normal person wants to pick it up, the professor says: don't try to do that. It is absolutely impossible that there is a 100 € bill lying on the street. Indeed, if it were lying on the street, somebody else would have picked it up before you. (end of joke)

How about financial markets? There it is already much more reasonable to assume that there are no arbitrage possibilities, i.e., that there are no 100 € bills lying around and waiting to be picked up. Let us illustrate this with an easy example.

Consider the trading of \$ versus € that takes place simultaneously at two exchanges, say in New York and Frankfurt. Assume for simplicity that in New York the \$/€ rate is $1:1$. Then it is quite obvious that in Frankfurt the exchange rate (at the same moment of time) also is $1:1$. Let us have a closer look why this is the case. Suppose to the contrary that you can buy in Frankfurt a \$ for 0.999 €. Then, indeed, the so-called "arbitrageurs" (these are people with two telephones in their hands and three screens in front of them) would quickly act to buy \$ in Frankfurt and simultaneously sell the same amount of \$ in New York, keeping the margin in their (or their bank's) pocket. Note that there is no normalising factor in front of the exchanged amount and the arbitrageur would try to do this on a scale as large as possible.

It is rather obvious that in the situation described above the market cannot be in equilibrium. A moment's reflection reveals that the market forces triggered by the arbitrageurs will make the \$ rise in Frankfurt and fall in

New York. The arbitrage possibility will disappear when the two prices become equal. Of course, "equality" here is to be understood as an approximate identity where — even for arbitrageurs with very low transaction costs — the above scheme is not profitable any more.

This brings us to a first — informal and intuitive — definition of arbitrage: an arbitrage opportunity is the possibility to make a profit in a financial market *without risk* and *without net investment of capital*. The *principle of no-arbitrage* states that a mathematical model of a financial market should not allow for arbitrage possibilities.

1.2 An Easy Model of a Financial Market

To apply this principle to less trivial cases than the Euro/Dollar example above, we consider a still extremely simple mathematical model of a financial market: there are two assets, called the bond and the stock. The bond is riskless, hence by definition we know what it is worth tomorrow. For (mainly notational) simplicity we neglect interest rates and assume that the price of a bond equals 1 € today as well as tomorrow, i.e.,

$$B_0 = B_1 = 1$$

The more interesting feature of the model is the stock which is risky: we know its value today, say (w.l.o.g.)

$$S_0 = 1,$$

but we don't know its value tomorrow. We model this uncertainty stochastically by defining S_1 to be a random variable depending on the random element $\omega \in \Omega$. To keep things as simple as possible, we let Ω consist of two elements only, g for "good" and b for "bad", with probability $\mathbf{P}[g] = \mathbf{P}[b] = \frac{1}{2}$. We define $S_1(\omega)$ by

$$S_1(\omega) = \begin{cases} 2 & \text{for } \omega = g \\ \frac{1}{2} & \text{for } \omega = b. \end{cases}$$

Now we introduce a third financial instrument in our model, an *option on the stock* with strike price K: the buyer of the option has the right — but not the obligation — to buy one stock at time $t = 1$ at a predefined price K. To fix ideas let $K = 1$. A moment's reflexion reveals that the price C_1 of the option at time $t = 1$ (where C stands for "call") equals

$$C_1 = (S_1 - K)_+,$$

i.e., in our simple example

$$C_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases}$$

Hence we know the value of the option at time $t = 1$, *contingent on the value of the stock*. But what is the price of the option today?

The classical approach, used by actuaries for centuries, is to price contingent claims by taking expectations. In our example this gives the value $C_0 := \mathbf{E}[C_1] = \frac{1}{2}$. Although this simple approach is very successful in many actuarial applications, it is not at all satisfactory in the present context. Indeed, the rationale behind taking the expected value is the following argument based on the law of large numbers: in the long run the buyer of an option will neither gain nor lose in the average. We rephrase this fact in a more financial lingo: the performance of an investment into the option would in average equal the performance of the bond (for which we have assumed an interest rate equal to zero). However, a basic feature of finance is that an investment into a risky asset should in average yield a better performance than an investment into the bond (for the sceptical reader: at least, these two values should not necessarily coincide). In our "toy example" we have chosen the numbers such that $\mathbf{E}[S_1] = 1.25 > 1 = S_0$, so that in average the stock performs better than the bond. This indicates that the option (which clearly is a risky investment) should not necessarily have the same performance (in average) as the bond. It also shows that the old method of calculating prices via expectation is not directly applicable. It already fails for the stock and hence there is no reason why the price of the option should be given by its expectation $\mathbf{E}[C_1]$.

1.3 Pricing by No-Arbitrage

A different approach to the pricing of the option goes like this: we can buy at time $t = 0$ a *portfolio* $\mathit{\Pi}$ consisting of $\frac{2}{3}$ of stock and $-\frac{1}{3}$ of bond. The reader might be puzzled about the negative sign: investing a negative amount into a bond — "going short" in the financial lingo — means borrowing money.

Note that — although normal people like most of us may not be able to do so — the "big players" can go "long" as well as "short". In fact they can do so not only with respect to the bond (i.e. to invest or borrow money at a fixed rate of interest) but can also go "long" as well as "short" in other assets like shares. In addition, they can do so at (relatively) low transaction costs, which is reflected by completely neglecting transaction costs in our present basic modelling.

Turning back to our portfolio $\mathit{\Pi}$ one verifies that the value $\mathit{\Pi}_1$ of the portfolio at time $t = 1$ equals

$$\mathit{\Pi}_1(\omega) = \begin{cases} 1 & \text{for } \omega = g \\ 0 & \text{for } \omega = b. \end{cases}$$

The portfolio "replicates" the option, i.e.,

$$C_1 \equiv \mathit{\Pi}_1, \tag{1.1}$$

or, written more explicitly,

$$C_1(g) = \Pi_1(g), \tag{1.2}$$
$$C_1(b) = \Pi_1(b). \tag{1.3}$$

We are confident that the reader now sees why we have chosen the above weights $\frac{2}{3}$ and $-\frac{1}{3}$: the mathematical complexity of determining these weights such that (1.2) and (1.3) hold true, amounts to solving two linear equations in two variables.

The portfolio Π has a well-defined price at time $t = 0$, namely $\Pi_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0 = \frac{1}{3}$. Now comes the "pricing by no-arbitrage" argument: equality (1.1) implies that we also must have

$$C_0 = \Pi_0 \tag{1.4}$$

whence $C_0 = \frac{1}{3}$. Indeed, suppose that (1.4) does not hold true; to fix ideas, suppose we have $C_0 = \frac{1}{2}$ as we had proposed above. This would allow an arbitrage by buying ("going long in") the portfolio Π and simultaneously selling ("going short in") the option C. The difference $C_0 - \Pi_0 = \frac{1}{6}$ remains as arbitrage profit at time $t = 0$, while at time $t = 1$ the two positions cancel out *independently of whether the random element ω equals g or b.*

Of course, the above considered size of the arbitrage profit by applying the above scheme to one option was only chosen for expository reasons: it is important to note that you may multiply the size of the above portfolios with your favourite power of ten, thus multiplying also your arbitrage profit.

At this stage we see that the story with the 100 € bill at the beginning of this chapter did not fully describe the idea of an arbitrage: The correct analogue would be to find instead of a single 100 € bill a "money pump", i.e., something like a box from which you can take one 100 € bill after another. While it might have happened to some of us, to occasionally find a 100 € bill lying around, we are confident that nobody ever found such a "money pump".

Another aspect where the little story at the beginning of this chapter did not fully describe the idea of arbitrage is the question of information. We shall assume throughout this book that all agents have the same information (there are no "insiders"). The theory changes completely when different agents have different information (which would correspond to the situation in the above joke). We will not address these extensions.

These arguments should convince the reader that the *"no-arbitrage principle"* is economically very appealing: in a liquid financial market there should be no arbitrage opportunities. Hence a mathematical model of a financial market should be designed in such a way that it does not permit arbitrage.

It is remarkable that this rather obvious principle yielded a unique price for the option considered in the above model.

1.4 Variations of the Example

Although the preceding "toy example" is extremely simple and, of course, far from reality, it contains the heart of the matter: the possibility of replicating a contingent claim, e.g. an option, by trading on the existing assets and to apply the no-arbitrage principle.

It is straightforward to generalise the example by passing from the time index set $\{0, 1\}$ to an arbitrary finite discrete time set $\{0, \ldots, T\}$, and by considering T independent Bernoulli random variables. This binomial model is called the Cox-Ross-Rubinstein model in finance (see Chap. 3 below).

It is also relatively simple — at least with the technology of stochastic calculus, which is available today — to pass to the (properly normalised) limit as T tends to infinity, thus ending up with a stochastic process driven by Brownian motion (see Chap. 4 below). The so-called geometric Brownian motion, i.e., Brownian motion on an exponential scale, is the celebrated *Black-Scholes model* which was proposed in 1965 by P. Samuelson, see [S 65]. In fact, already in 1900 L. Bachelier [B 00] used Brownian motion to price options in his remarkable thesis "Théorie de la spéculation" (a member of the jury and rapporteur was H. Poincaré).

In order to apply the above no-arbitrage arguments to more complex models we still need one additional, crucial concept.

1.5 Martingale Measures

To explain this notion let us turn back to our "toy example", where we have seen that the unique arbitrage free price of our option equals $C_0 = \frac{1}{3}$. We also have seen that, by taking expectations, we obtained $\mathbf{E}[C_1] = \frac{1}{2}$ as the price of the option, which was a "wrong price" as it allowed for arbitrage opportunities. The economic rationale for this discrepancy was that the expected return of the stock was higher than that of the bond.

Now make the following mind experiment: suppose that the world were governed by a different probability than $\mathbf{P}$ which assigns different weights to g and b, such that under this new probability, let's call it $\mathbf{Q}$, the expected return of the stock equals that of the bond. An elementary calculation reveals that the probability measure defined by $\mathbf{Q}[g] = \frac{1}{3}$ and $\mathbf{Q}[b] = \frac{2}{3}$ is the unique solution satisfying $\mathbf{E}_{\mathbf{Q}}[S_1] = S_0 = 1$. Mathematically speaking, the process S is a *martingale* under $\mathbf{Q}$, and $\mathbf{Q}$ is a *martingale measure* for S.

Speaking again economically, it is not unreasonable to expect that in a world governed by $\mathbf{Q}$, the recipe of taking expected values should indeed give a price for the option which is compatible with the no-arbitrage principle. After all, our original objection, that the average performance of the stock and the bond differ, now has disappeared. A direct calculation reveals that in our "toy example" these two prices for the option indeed coincide as

$$\mathbf{E}_{\mathbf{Q}}[C_1] = \tfrac{1}{3}.$$

Clearly we suspect that this numerical match is not just a coincidence. At this stage it is, of course, the reflex of every mathematician to ask: what is precisely going on behind this phenomenon? A preliminary answer is that the expectation under the new measure $\mathbf{Q}$ defines a linear function of the span of B_1 and S_1. The price of an element in this span should therefore be the corresponding linear combination of the prices at time 0. Thus, using simple linear algebra, we get $C_0 = \frac{2}{3}S_0 - \frac{1}{3}B_0$ and moreover we identify this as $\mathbf{E}_{\mathbf{Q}}[C_1]$.

1.6 The Fundamental Theorem of Asset Pricing

To make a long story very short: for a general stochastic process $(S_t)_{0\leq t\leq T}$, modelled on a filtered probability space $(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbf{P})$, the following statement *essentially* holds true. For any "contingent claim" C_T, i.e. an $\mathcal{F}_T$-measurable random variable, the formula

$$C_0 := \mathbf{E}_{\mathbf{Q}}[C_T] \tag{1.5}$$

yields precisely the arbitrage-free prices for C_T, when $\mathbf{Q}$ runs through the probability measures on $\mathcal{F}_T$, which are equivalent to $\mathbf{P}$ and under which the process S is a martingale (*"equivalent martingale measures"*). In particular, when there is precisely one equivalent martingale measure (as it is the case in the Cox-Ross-Rubinstein, the Black-Scholes and the Bachelier model), formula (1.5) gives the unique arbitrage free price C_0 for C_T. In this case we may "replicate" the contingent claim C_T as

$$C_T = C_0 + \int_0^T H_t dS_t, \tag{1.6}$$

where $(H_t)_{0\leq t\leq T}$ is a predictable process (a *"trading strategy"*) and where H_t models the holding in the stock S during the infinitesimal interval $[t, t+dt]$.

Of course, the stochastic integral appearing in (1.6) needs some care; fortunately people like K. Itô and P.A. Meyer's school of probability in Strasbourg told us very precisely how to interpret such an integral.

The mathematical challenge of the above story consists of getting rid of the word "essentially" and to turn this program into precise theorems.

The central piece of the theory relating the no-arbitrage arguments with martingale theory is the so-called Fundamental Theorem of Asset Pricing. We quote a general version of this theorem, which is proved in Chap. 14.

Theorem 1.6.1 (Fundamental Theorem of Asset Pricing). *For an* $\mathbf{R}^d$*-valued semi-martingale* $S = (S_t)_{0\leq t\leq T}$ *t.f.a.e.:*

(i) *There exists a probability measure* $\mathbf{Q}$ *equivalent to* $\mathbf{P}$ *under which* S *is a* sigma-martingale*.*
(ii) S *does not permit a* free lunch with vanishing risk*.*

This theorem was proved for the case of a probability space Ω consisting of finitely many elements by Harrison and Pliska [HP 81]. In this case one may equivalently write *no-arbitrage* instead of *no free lunch with vanishing risk* and *martingale* instead of *sigma-martingale*.

In the general case it is unavoidable to speak about more technical concepts, such as *sigma-martingales* (which is a generalisation of the notion of a local martingale) and *free lunches*. A *free lunch* (a notion introduced by D. Kreps [K 81]) is something like an arbitrage, where — roughly speaking — agents are allowed to form integrals as in (1.6), to subsequently "throw away money" (if they want do so), and finally to pass to the limit in an appropriate topology. It was the — somewhat surprising — insight of [DS 94] (reprinted in Chap. 9) that one may take the topology of *uniform convergence* (which allows for an economic interpretation to which the term "with vanishing risk" alludes) and still get a valid theorem.

The remainder of this book is devoted to the development of this theme, as well as to its remarkable scope of applications in Finance.

2

Models of Financial Markets on Finite Probability Spaces

2.1 Description of the Model

In this section we shall develop the theory of pricing and hedging of derivative securities in financial markets.

In order to reduce the technical difficulties of the theory of option pricing to a minimum, we assume throughout this chapter that the probability space Ω underlying our model will be finite, say, $\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$ equipped with a probability measure $\mathbf{P}$ such that $\mathbf{P}[\omega_n] = p_n > 0$, for $n = 1, \dots, N$. This assumption implies that all functional-analytic delicacies pertaining to different topologies on $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, $L^1(\Omega, \mathcal{F}, \mathbf{P})$, $L^0(\Omega, \mathcal{F}, \mathbf{P})$ etc. evaporate, as all these spaces are simply $\mathbb{R}^N$ (we assume w.l.o.g. that the σ-algebra $\mathcal{F}$ is the power set of Ω). Hence all the functional analysis, which we shall need in later chapters for the case of more general processes, reduces in the setting of the present chapter to simple linear algebra. For example, the use of the Hahn-Banach theorem is replaced by the use of the separating hyperplane theorem in finite dimensional spaces.

Nevertheless we shall write $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$, $L^1(\Omega, \mathcal{F}, \mathbf{P})$ etc. (knowing very well that in the present setting these spaces are all isomorphic to $\mathbb{R}^N$) to indicate, which function spaces we shall encounter in the setting of the general theory. It also helps to see if an element of $\mathbb{R}^N$ is a contingent claim or an element of the dual space, i.e. a price vector.

In addition to the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ we fix a natural number $T \geq 1$ and a filtration $(\mathcal{F}_t)_{t=0}^T$ on Ω, i.e., an increasing sequence of σ-algebras. To avoid trivialities, we shall always assume that $\mathcal{F}_T = \mathcal{F}$; on the other hand, we shall *not* assume that $\mathcal{F}_0$ is trivial, i.e. $\mathcal{F}_0 = \{\emptyset, \Omega\}$, although this will be the case in most applications. But for technical reasons it will be more convenient to allow for general σ-algebras $\mathcal{F}_0$.

We now introduce a model of a financial market in not necessarily discounted terms. The rest of Sect. 2.1 will be devoted to reducing this situation to a model in discounted terms which, as we shall see, will make life much easier.

Readers who are not so enthusiastic about this mainly formal and elementary reduction might proceed directly to Definition 2.1.4. On the other hand, we know from sad experience that often there is a lot of myth and confusion arising in this operation of discounting; for this reason we decided to devote this section to the clarification of this issue.

Definition 2.1.1. *A model of a* financial market *is an* $\mathbb{R}^{d+1}$*-valued stochastic process* $\widehat{S} = (\widehat{S}_t)_{t=0}^T = (\widehat{S}_t^0, \widehat{S}_t^1, \dots, \widehat{S}_t^d)_{t=0}^T$, *based on and adapted to the filtered stochastic base* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$. *We shall assume that the zero coordinate* $\widehat{S}^0$ *satisfies* $\widehat{S}_t^0 > 0$ *for all* $t = 0, \dots, T$ *and* $\widehat{S}_0^0 = 1$.

The interpretation is the following. The prices of the assets $0, \dots, d$ are measured in a fixed money unit, say Euros. For $1 \leq j \leq d$ they are not necessarily non-negative (think, e.g., of forward contracts). The asset 0 plays a special role. It is supposed to be strictly positive and will be used as a numéraire. It allows us to compare money (e.g., Euros) at time 0 to money at time $t > 0$. In many elementary models, $\widehat{S}^0$ is simply a bank account which in case of constant interest rate r is then defined as $\widehat{S}_t^0 = e^{rt}$. However, it might also be more complicated, e.g. $\widehat{S}_t^0 = \exp(r_0 h + r_1 h + \dots + r_{t-1} h)$ where $h > 0$ is the length of the time interval between $t-1$ and t (here kept fixed) and where r_{t-1} is the stochastic interest rate valid between $t-1$ and t. Other models are also possible and to prepare the reader for more general situations, we only require $\widehat{S}_t^0$ to be strictly positive. Notice that we only require that $\widehat{S}_t^0$ to be $\mathcal{F}_t$-measurable and that it is not necessarily $\mathcal{F}_{t-1}$-measurable. In other words, we assume that the process $\widehat{S}^0 = (\widehat{S}_t^0)_{t=0}^T$ is adapted, but not necessarily predictable.

An economic agent is able to buy and sell financial assets. The decision taken at time t can only use information available at time t which is modelled by the σ-algebra $\mathcal{F}_t$.

Definition 2.1.2. *A trading strategy* $(\widehat{H}_t)_{t=1}^T = (\widehat{H}_t^0, \widehat{H}_t^1, \dots, \widehat{H}_t^d)_{t=1}^T$ *is an* $\mathbb{R}^{d+1}$*-valued process which is predictable, i.e.* $\widehat{H}_t$ *is* $\mathcal{F}_{t-1}$*-measurable.*

The interpretation is that between time $t-1$ and time t, the agent holds a quantity equal to $\widehat{H}_t^j$ of asset j. The decision is taken at time $t-1$ and therefore, $\widehat{H}_t$ is required to be $\mathcal{F}_{t-1}$-measurable.

Definition 2.1.3. *A strategy* $(\widehat{H}_t)_{t=1}^T$ *is called self financing if for every* $t = 1, \dots, T-1$, *we have*

$$\left(\widehat{H}_t, \widehat{S}_t\right) = \left(\widehat{H}_{t+1}, \widehat{S}_t\right) \tag{2.1}$$

or, written more explicitly,

$$\sum_{j=0}^d \widehat{H}_t^j \widehat{S}_t^j = \sum_{j=0}^d \widehat{H}_{t+1}^j \widehat{S}_t^j. \tag{2.2}$$

The initial investment required for a strategy is $\widehat{V}_0 = (\widehat{H}_1, \widehat{S}_0) = \sum_{j=0}^d \widehat{H}_1^j \widehat{S}_0^j$.

The interpretation goes as follows. By changing the portfolio from $\widehat{H}_{t-1}$ to $\widehat{H}_t$ there is no input/outflow of money. We remark that we assume that changing a portfolio does not trigger transaction costs. Also note that $\widehat{H}_t^j$ may assume negative values, which corresponds to short selling asset j during the time interval $]t_{j-1}, t_j]$.

The $\mathcal{F}_t$-measurable random variable defined in (2.1) is interpreted as the value $\widehat{V}_t$ of the portfolio at time t defined by the trading strategy $\widehat{H}$:

$$\widehat{V}_t = (\widehat{H}_t, \widehat{S}_t) = (\widehat{H}_{t+1}, \widehat{S}_t).$$

The way in which the value $(\widehat{H}_t, \widehat{S}_t)$ evolves can be described much easier when we use discounted prices using the asset $\widehat{S}^0$ as numéraire. Discounting allows us to compare money at time t to money at time 0. For instance we could say that $\widehat{S}_t^0$ units of money at time t are the "same" as 1 unit of money, e.g., Euros, at time 0. So let us see what happens if we replace prices $\widehat{S}$ by discounted prices $\left(\frac{\widehat{S}}{\widehat{S}^0}\right) = \left(\frac{\widehat{S}^0}{\widehat{S}^0}, \frac{\widehat{S}^1}{\widehat{S}^0}, \ldots, \frac{\widehat{S}^d}{\widehat{S}^0}\right)$. We will use the notation

$$S_t^j := \frac{\widehat{S}_t^j}{\widehat{S}_t^0}, \quad \text{for } j = 1, \ldots, d \text{ and } t = 0, \ldots, T. \tag{2.3}$$

There is no need to include the coordinate 0, since obviously $S_t^0 = 1$. Let us now consider $(\widehat{H}_t)_{t=1}^T = (\widehat{H}_t^0, \widehat{H}_t^1, \ldots, \widehat{H}_t^d)_{t=1}^T$ to be a self financing strategy with initial investment $\widehat{V}_0$; we then have

$$\widehat{V}_0 = \sum_{j=0}^d \widehat{H}_1^j \widehat{S}_0^j = \widehat{H}_1^0 + \sum_{j=1}^d \widehat{H}_1^j \widehat{S}_0^j = \widehat{H}_1^0 + \sum_{j=1}^d \widehat{H}_1^j S_0^j,$$

since by definition $\widehat{S}_0^0 = 1$.

We now write $(H_t)_{t=1}^T = (H_t^1, \ldots, H_t^d)_{t=1}^T$ for the $\mathbb{R}^d$-valued process obtained by discarding the 0'th coordinate of the $\mathbb{R}^{d+1}$-valued process $(\widehat{H}_t)_{t=1}^T = (\widehat{H}_t^0, \widehat{H}_t^1, \ldots, \widehat{H}_t^d)_{t=1}^T$, i.e., $H_t^j = \widehat{H}_t^j$ for $j = 1, \ldots, d$. The reason for dropping the 0'th coordinate is, as we shall discover in a moment, that the holdings $\widehat{H}_t^0$ in the numéraire asset S_t^0 will be no longer of importance when we do the book-keeping in terms of the numéraire asset, i.e., in discounted terms.

One can make the following easy, but crucial observation: for *every* $\mathbb{R}^d$-valued, predictable process $(H_t)_{t=1}^T = (H_t^1, \ldots, H_t^d)_{t=1}^T$ there exists a unique *self financing* $\mathbb{R}^{d+1}$-valued predictable process $(\widehat{H}_t)_{t=1}^T = (\widehat{H}_t^0, \widehat{H}_t^1, \ldots, \widehat{H}_t^d)_{t=1}^T$ such that $(\widehat{H}_t^j)_{t=1}^T = (H_t^j)_{t=1}^T$ for $j = 1, \ldots, d$ and $\widehat{H}_1^0 = 0$. Indeed, one determines the values of $\widehat{H}_{t+1}^0$, for $t = 1, \ldots, T-1$, by inductively applying (2.2). The strict positivity of $(\widehat{S}_t^0)_{t=0}^{T-1}$ implies that there is precisely one function $\widehat{H}_{t+1}^0$ such that equality (2.2) holds true. Clearly such a function $\widehat{H}_{t+1}^0$ is

$\mathcal{F}_t$-measurable. In economic terms the above argument is rather obvious: for any given trading strategy $(H_t)_{t=1}^T = (H_t^1, \dots, H_t^d)_{t=1}^T$ in the "risky" assets $j = 1, \dots, d$, we may always add a trading strategy $(\widehat{H}_t^0)_{t=1}^T$ in the numéraire asset 0 such that the total strategy becomes self financing. Moreover, by normalising $\widehat{H}_1^0 = 0$, this trading strategy becomes unique. This can be particularly well visualised when interpreting the asset 0 as a cash account, into which at all times $t = 1, \dots, T-1$, the gains and losses occurring from the investments in the d risky assets are absorbed and from which the investments in the risky assets are financed. If we normalise this procedure by requiring $\widehat{H}_1^0 = 0$, i.e., by starting with an empty cash account, then clearly the subsequent evolution of the holdings in the cash account is uniquely determined by the holdings in the "risky" assets $1, \dots, d$. From now on we fix two processes $(\widehat{H}_t)_{t=1}^T = (\widehat{H}_t^0, \widehat{H}_t^1, \dots, \widehat{H}_t^d)_{t=1}^T$ and $(H_t)_{t=1}^T = (H_t^1, \dots, H_t^d)_{t=1}^T$ corresponding uniquely one to each other in the above described way.

Now one can make a second straightforward observation: the investment $(\widehat{H}_t^0)_{t=1}^T$ in the numéraire asset does not change the *discounted* value $(V_t)_{t=0}^T$ of the portfolio. Indeed, by definition — and rather trivially — the numéraire asset remains constant in discounted terms (i.e., expressed in units of itself).

Hence the discounted value V_t of the portfolio

$$V_t = \frac{\widehat{V}_t}{\widehat{S}_t^0}, \quad t = 0, \dots, T,$$

depends only on the $\mathbb{R}^d$-dimensional process $(H_t)_{t=1}^T = (H_t^1, \dots, H_t^d)_{t=1}^T$.

More precisely, in view of the normalisation $\widehat{S}_0^0 = 1$ and $\widehat{H}_1^0 = 0$ we have

$$\widehat{V}_0 = V_0 = \sum_{j=1}^d H_1^j S_0^j.$$

For the increment $\Delta V_{t+1} = V_{t+1} - V_t$ we find, using (2.2),

$$\begin{aligned}
\Delta V_{t+1} = V_{t+1} - V_t &= \frac{\widehat{V}_{t+1}}{\widehat{S}_{t+1}^0} - \frac{\widehat{V}_t}{\widehat{S}_t^0} \\
&= \sum_{j=0}^d \widehat{H}_{t+1}^j \frac{\widehat{S}_{t+1}^j}{\widehat{S}_{t+1}^0} - \sum_{j=0}^d \widehat{H}_{t+1}^j \frac{\widehat{S}_t^j}{\widehat{S}_t^0} \\
&= \widehat{H}_{t+1}^0 (1-1) + \sum_{j=1}^d \widehat{H}_{t+1}^j \left(S_{t+1}^j - S_t^j \right) \\
&= \left(H_{t+1}^j, \Delta S_{t+1}^j \right),
\end{aligned}$$

where $(. , .)$ now denotes the inner product in $\mathbb{R}^d$.

In particular, the final value V_T of the portfolio becomes (in discounted units)

$$V_T = V_0 + \sum_{t=1}^{T} (H_t, \Delta S_t) = V_0 + (H \cdot S)_T,$$

where $(H \cdot S)_T = \sum_{t=1}^{T} (H_t, \Delta S_t)$ is the notation for a stochastic integral familiar from the theory of stochastic integration. In our discrete time framework the "stochastic integral" is simply a finite Riemann sum.

In order to know the value V_T of the portfolio in real money, we still would have to multiply by $\widehat{S}_T^0$, i.e., we have $\widehat{V}_T = V_T \widehat{S}_T^0$. This, however, is rarely needed.

We can therefore replace Definition 2.1.2 by the following definition in discounted terms, which will turn out to be much easier to handle.

Definition 2.1.4. *Let $S = (S^1, \ldots, S^d)$ be a model of a financial market in discounted terms. A* trading strategy *is an $\mathbb{R}^d$-valued process $(H_t)_{t=1}^T = (H_t^1, H_t^2, \ldots, H_t^d)_{t=1}^T$ which is predictable, i.e., each H_t is $\mathcal{F}_{t-1}$-measurable. We denote by $\mathcal{H}$ the set of all such trading strategies.*

We then define the stochastic integral $H \cdot S$ as the $\mathbb{R}$-valued process $((H \cdot S)_t)_{t=0}^T$ given by

$$(H \cdot S)_t = \sum_{u=1}^{t} (H_u, \Delta S_u), \quad t = 0, \ldots, T, \tag{2.4}$$

where $(. , .)$ denotes the inner product in $\mathbb{R}^d$. The random variable

$$(H \cdot S)_t = \sum_{u=1}^{t} (H_u, \Delta S_u)$$

models — when following the trading strategy H — the gain or loss occurred up to time t in discounted terms.

Summing up: by following the good old actuarial tradition of discounting, i.e. by passing from the process $\widehat{S}$, denoted in units of money, to the process S, denoted in terms of the numéraire asset (e.g., the cash account), things become considerably simpler and more transparent. In particular the value process V of an agent starting with initial wealth $V_0 = 0$ and subsequently applying the trading strategy H, is given by the stochastic integral $V_t = (H \cdot S)_t$ defined in (2.4).

We still emphasize that the choice of the numéraire is not unique; only for notational convenience we have fixed it to be the asset indexed by 0. But it may be chosen as any traded asset, provided only that it always remains strictly positive. We shall deal with this topic in more detail in Sect. 2.5 below.

From now on we shall work in terms of the discounted $\mathbb{R}^d$-valued process, denoted by S.

2.2 No-Arbitrage and the Fundamental Theorem of Asset Pricing

Definition 2.2.1. *We call the subspace K of $L^0(\Omega, \mathcal{F}, \mathbf{P})$ defined by*

$$K = \{(H \cdot S)_T \mid H \in \mathcal{H}\},$$

the set of contingent claims attainable at price 0, *where $\mathcal{H}$ denotes the set of predictable, $\mathbb{R}^d$-valued processes $H = (H_t)_{t=1}^T$.*

We leave it to the reader to check that K is indeed a vector space.

The economic interpretation is the following: the random variables $f = (H \cdot S)_T$ are precisely those contingent claims, i.e., the pay-off functions at time T, depending on $\omega \in \Omega$, that an economic agent may replicate with zero initial investment by pursuing some predictable trading strategy H.

For $a \in \mathbb{R}$, we call the *set of contingent claims attainable at price a* the affine space $K_a = a + K$, obtained by shifting K by the constant function a, in other words, the space of all the random variables of the form $a + (H \cdot S)_T$, for some trading strategy H. Again the economic interpretation is that these are precisely the contingent claims that an economic agent may replicate with an initial investment of a by pursuing some predictable trading strategy H.

Definition 2.2.2. *We call the convex cone C in $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ defined by*

$$C = \{g \in L^\infty(\Omega, \mathcal{F}, \mathbf{P}) \mid \textit{there exists } f \in K \textit{ with } f \geq g\}.$$

the set of contingent claims super-replicable at price 0.

Economically speaking, a contingent claim $g \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ is *super-replicable at price* 0, if we can achieve it with zero net investment by pursuing some predictable trading strategy H. Thus we arrive at some contingent claim f and if necessary we "throw away money" to arrive at g. This operation of "throwing away money" or "free disposal" may seem awkward at this stage, but we shall see later that the set C plays an important role in the development of the theory. Observe that C is a convex cone containing the negative orthant $L^\infty_-(\Omega, \mathcal{F}, \mathbf{P})$. Again we may define $C_a = a + C$ as the *contingent claims super-replicable at price a*, if we shift C by the constant function a.

Definition 2.2.3. *A financial market S satisfies the* no-arbitrage condition *(NA) if*

$$K \cap L^0_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\}$$

or, equivalently,

$$C \cap L^0_+(\Omega, \mathcal{F}, \mathbf{P}) = \{0\}$$

where 0 *denotes the function identically equal to zero.*

Recall that $L^0(\Omega, \mathcal{F}, \mathbf{P})$ denotes the space of all $\mathcal{F}$-measurable real-valued functions and $L^0_+(\Omega, \mathcal{F}, \mathbf{P})$ its positive orthant.

We now have formalised the concept of an arbitrage possibility: it means the existence of a trading strategy H such that — starting from an initial investment zero — the resulting contingent claim $f = (H \cdot S)_T$ is non-negative and not identically equal to zero. Such an opportunity is of course the dream of every arbitrageur. If a financial market does not allow for arbitrage opportunities, we say it satisfies the *no-arbitrage condition (NA)*.

Proposition 2.2.4. *Assume S satisfies (NA) then*

$$C \cap (-C) = K.$$

Proof. Let $g \in C \cap (-C)$ then $g = f_1 - h_1$ with $f_1 \in K$, $h_1 \in L^\infty_+$ and $g = f_2 + h_2$ with $f_2 \in K$ and $h_2 \in L^\infty_+$. Then $f_1 - f_2 = h_1 + h_2 \in L^\infty_+$ and hence $f_1 - f_2 \in K \cap L^\infty_+ = \{0\}$. It follows that $f_1 = f_2$ and $h_1 + h_2 = 0$, hence $h_1 = h_2 = 0$. This means that $g = f_1 = f_2 \in K$. □

Definition 2.2.5. *A probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ is called an* equivalent martingale measure *for S, if $\mathbf{Q} \sim \mathbf{P}$ and S is a martingale under $\mathbf{Q}$, i.e., $\mathbf{E}_\mathbf{Q}[S_{t+1}|\mathcal{F}_t] = S_t$ for $t = 0, \dots, T-1$.*

We denote by $\mathcal{M}^e(S)$ the set of equivalent martingale measures and by $\mathcal{M}^a(S)$ the set of all (not necessarily equivalent) martingale probability measures. The letter a stands for "absolutely continuous with respect to $\mathbf{P}$" which in the present setting (finite Ω and $\mathbf{P}$ having full support) automatically holds true, but which will be of relevance for general probability spaces $(\Omega, \mathcal{F}, \mathbf{P})$ later. Note that in the present setting of a finite probability space Ω with $\mathbf{P}[\omega] > 0$ for each $\omega \in \Omega$, we have that $\mathbf{Q} \sim \mathbf{P}$ iff $\mathbf{Q}[\omega] > 0$, for each $\omega \in \Omega$. We shall often identify a measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ with its Radon-Nikodým derivative $\frac{d\mathbf{Q}}{d\mathbf{P}} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$. In the present setting of finite Ω, this simply means

$$\frac{d\mathbf{Q}}{d\mathbf{P}}(\omega) = \frac{\mathbf{Q}[\omega]}{\mathbf{P}[\omega]}.$$

In statistics this quantity is also called the likelihood ratio.

Lemma 2.2.6. *For a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ the following are equivalent:*

(i) $\mathbf{Q} \in \mathcal{M}^a(S)$,
(ii) $\mathbf{E}_\mathbf{Q}[f] = 0$, *for all* $f \in K$,
(iii) $\mathbf{E}_\mathbf{Q}[g] \le 0$, *for all* $g \in C$.

Proof. The equivalences are rather trivial. (ii) is tantamount to the very definition of S being a martingale under $\mathbf{Q}$, i.e., to the validity of

$$\mathbf{E}_\mathbf{Q}[S_t \mid \mathcal{F}_{t-1}] = S_{t-1}, \quad \text{for } t = 1, \dots, T. \tag{2.5}$$

Indeed, (2.5) holds true iff for each $\mathcal{F}_{t-1}$-measurable set A we have $\mathbf{E_Q}[\chi_A(S_t - S_{t-1})] = 0 \in \mathbb{R}^d$, in other words $\mathbf{E_Q}[(x\chi_A, \Delta S_t)] = 0$, for each x. By linearity this relation extends to K which shows (ii).

The equivalence of (ii) and (iii) is straightforward. □

After having fixed these formalities we may formulate and prove the central result of the theory of pricing and hedging by no-arbitrage, sometimes called the "Fundamental Theorem of Asset Pricing", which in its present form (i.e., finite Ω) is due to M. Harrison and S.R. Pliska [HP 81].

Theorem 2.2.7 (Fundamental Theorem of Asset Pricing). *For a financial market S modelled on a finite stochastic base $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, \mathbf{P})$, the following are equivalent:*

(i) *S satisfies (NA),*
(ii) *$\mathcal{M}^e(S) \neq \emptyset$.*

Proof. (ii) $\Rightarrow$ (i): This is the obvious implication. If there is some $\mathbf{Q} \in \mathcal{M}^e(S)$ then by Lemma 2.2.6 we have that

$$\mathbf{E_Q}[g] \leq 0, \quad \text{for } g \in C.$$

On the other hand, if there were $g \in C \cap L_+^\infty$, $g \neq 0$, then, using the assumption that $\mathbf{Q}$ is equivalent to $\mathbf{P}$, we would have

$$\mathbf{E_Q}[g] > 0,$$

a contradiction.

(i) $\Rightarrow$ (ii) This implication is the important message of the theorem which will allow us to link the no-arbitrage arguments with martingale theory. We give a functional analytic existence proof, which will be extendable — in spirit — to more general situations.

By assumption the space K intersects L_+^∞ only at 0. We want to separate the disjoint convex sets $L_+^\infty \setminus \{0\}$ and K by a hyperplane induced by a linear functional $\mathbf{Q} \in L^1(\Omega, \mathcal{F}, \mathbf{P})$. In order to get a strict separation of K and $L_+^\infty \setminus \{0\}$ we have to be a little careful since the standard separation theorems do not directly apply.

One way to overcome this difficulty (in finite dimension) is to consider the convex hull of the unit vectors $\left(\mathbf{1}_{\{\omega_n\}}\right)_{n=1}^N$ in $L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ i.e.

$$P := \left\{ \sum_{n=1}^N \mu_n \mathbf{1}_{\{\omega_n\}} \,\middle|\, \mu_n \geq 0, \sum_{n=1}^N \mu_n = 1 \right\}.$$

This is a convex, compact subset of $L_+^\infty(\Omega, \mathcal{F}, \mathbf{P})$ and, by the *(NA)* assumption, disjoint from K. Hence we may strictly separate the convex compact set P from the convex closed set K by a linear functional $\mathbf{Q} \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})^* = L^1(\Omega, \mathcal{F}, \mathbf{P})$, i.e., find $\alpha < \beta$ such that

$$\begin{aligned} (\mathbf{Q}, f) &\leq \alpha, \quad \text{for } f \in K, \\ (\mathbf{Q}, h) &\geq \beta, \quad \text{for } h \in P. \end{aligned}$$

Since K is a linear space, we have $\alpha \geq 0$ and may replace α by 0. Hence $\beta > 0$. Defining by I the constant vector $I = (1, \ldots, 1)$, we have $(\mathbf{Q}, I) > 0$. We may normalise $\mathbf{Q}$ such that $(\mathbf{Q}, I) = 1$. As $\mathbf{Q}$ is strictly positive on each $\mathbf{1}_{\{\omega_n\}}$, we therefore have found a probability measure $\mathbf{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbf{P}$ such that condition (ii) of Lemma 2.2.6 holds true. In other words, we found an equivalent martingale measure $\mathbf{Q}$ for the process S. □

The name "Fundamental Theorem of Asset Pricing" was, as far as we are aware, first used in [DR 87]. We shall see that it plays a truly fundamental role in the theory of pricing and hedging of derivative securities (or, synonymously, contingent claims, i.e., elements of $L^0(\Omega, \mathcal{F}, \mathbf{P})$) by no-arbitrage arguments.

It seems worthwhile to discuss the intuitive interpretation of this basic result: a martingale S (say, under the original measure $\mathbf{P}$) is a mathematical model for a *perfectly fair* game. Applying any strategy $H \in \mathcal{H}$ we always have $\mathbf{E}[(H \cdot S)_T] = 0$, i.e., an investor can neither win nor lose in expectation.

On the other hand, a process S allowing for arbitrage, is a model for an utterly unfair game: choosing a good strategy $H \in \mathcal{H}$, an investor can make "something out of nothing". Applying H, the investor is sure not to lose, but has strictly positive probability to gain something.

In reality, there are many processes S which do not belong to either of these two extreme classes. Nevertheless, the above theorem tells us that there is a sharp dichotomy by allowing to *change the odds*. Either a process S is utterly unfair, in the sense that it allows for arbitrage. In this case there is no remedy to make the process fair by changing the odds: it never becomes a martingale. In fact, the possibility of making an arbitrage is not affected by changing the odds, i.e., by passing to an equivalent probability $\mathbf{Q}$. On the other hand, discarding this extreme case of processes allowing for arbitrage, we can always pass from $\mathbf{P}$ to an equivalent measure $\mathbf{Q}$ under which S is a martingale, i.e., a perfectly fair game. Note that the passage from $\mathbf{P}$ to $\mathbf{Q}$ may change the probabilities (the "odds") but not the impossible events (i.e. the null sets).

We believe that this dichotomy is a remarkable fact, also from a purely intuitive point of view.

Corollary 2.2.8. *Let S satisfy (NA) and let $f \in L^\infty(\Omega, \mathcal{F}, \mathbf{P})$ be an attainable contingent claim. In other words f is of the form*

$$f = a + (H \cdot S)_T, \tag{2.6}$$

for some $a \in \mathbb{R}$ and some trading strategy H. Then the constant a and the process $(H \cdot S)_t$ are uniquely determined by (2.6) and satisfy, for every $\mathbf{Q} \in \mathcal{M}^e(S)$,

$$a = \mathbf{E}_\mathbf{Q}[f], \quad \text{and} \quad a + (H \cdot S)_t = \mathbf{E}_\mathbf{Q}[f \mid \mathcal{F}_t], \quad \text{for} \quad 0 \leq t \leq T. \tag{2.7}$$