Computational Contact Mechanics
Peter Wriggers

Computational Contact Mechanics
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Preface

Contact mechanics has its application in many engineering problems. No one can walk without frictional contact, and no car would move for the same reason. Hence contact mechanics has, from an engineering point of view, a long history, beginning in ancient Egypt with the movement of large stone blocks, over first experimental contributions from leading scientists like Leonardo da Vinci and Coulomb, to today's computational methods. In the past contact conditions were often modelled in engineering analysis by more simple boundary conditions since analytical solutions were not present for real world applications. In such cases, one investigated contact as a local problem using the stress and strain fields stemming from the analysis which was performed for the entire structure. With the rapidly increasing power of modern computers, more and more numerical simulations in engineering can include contact constraints directly, which make the problems nonlinear.

This book is an account of the modern theory of nonlinear continuum mechanics and its application to contact problems, as well as of modern simulation techniques for contact problems using the finite element method. The latter includes a variety of discretization techniques for small and large deformation contact. Algorithms play another prominent role when robust and efficient techniques have to be designed for contact simulations. Finally, adaptive methods based on error controlled finite element analysis and mesh adaptation techniques are of great interest for the reliable numerical solution of contact problems. Nevertheless, all numerical models need a strong backup provided by modern continuum mechanics and its constitutive theory, which is applied in this book to the development of interface laws for normal and frictional contact.

The present text can be viewed as a textbook which is basically self-contained. It is written for students at graduate level and engineers who have to simulate contact problems in practical applications and wish to understand the theoretical and algorithmic background of modern finite element systems. The organization of the book is straightforward. After an introductory chapter which discusses relevant contact formulations in a simple matter, there
follows a chapter which provides the continuum mechanics background. The special geometrical relations needed to set up the contact constraints and constitutive equations valid at the contact interface are then discussed in detail without going into a numerical treatment. The topic of computational contact is then described in depth in the next chapters, providing different formulations, algorithms and discretization techniques which have been established so far. Here solid and beam contact is considered, as well as contact of unstable systems and thermomechanical contact. The algorithmic side includes, besides a broad range of solution methods, adaptive discretization techniques for contact analysis. However, it can be concluded for the present that there exists nothing which can be called the robust method for all different types of contact simulations. This actually also holds for other simulations, including nonlinearities. However, especially due to the fact that such a method does not exist, it is necessary to discuss those methods which are on the market in the light of good or bad behaviour.

It is finally a pleasure to thank many people who have assisted me in writing the book, and who were always available in the last twenty years for deep discussions on computational contact mechanics, including the related formulations of continuum mechanics and implementation issues. This scientific collaboration often resulted in joint work in which new papers or reports were written. In particular, I should like to mention my PhD students Anna Haraldsson, Henning Braess, Katrin Fischer, Michael Imhof, Joze Korelc, Lovre Krstulovic-Opara, Tilmann Raible, Albrecht Rieger, Oliver Scherf and Holger Tschöpe. But I have also to include colleagues who worked and still work with me on issues of computational contact mechanics: the late Mike Crisfield, Christian Miehe, Bahram Nour-Omid, the late Panos Panagiotopoulos, Karl Schweizerhof, the late Juan Simo, Bob Taylor, Giorgio Zavarise and Tarek Zohdi.

Furthermore, I would like to express my appreciation to Bob Taylor, Giorgio Zavarise and Tarek Zohdi, who read early parts of the manuscript and helped with their constructive comments and criticisms to improve the text. I would also like to thank Elke Behrend and Christian Steenbock who together with Tilman Raible, drew most of the figures in the text. Last but not least, I would like to thank Springer publishing company, for publishing this new edition of the book and for the good collaboration during the last years.
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Boundary value problems involving contact are of great importance in industry related to mechanical and civil engineering, but also in environmental and medical applications. Virtually all movements on this planet involve contact and friction, like simple walking or running, driving of cars, riding bicycles or steaming of trains. If friction were not present (see movement on ice), all these motions would not be possible. Also, the area in which a foot, a tyre or a wheel interacts with the soil, the road or the rail is not known a priori, leading to a nonlinear boundary value problem for these simple everyday tasks.

Due to the nonlinear nature of contact mechanics, such problems in the past were often approximated by special assumptions within the design process. Due to the rapid improvement of modern computer technology, one can today apply the tools of computational mechanics to simulate applications which include contact mechanisms numerically. This can be done to an accuracy which is sufficient for design purposes. However, even now most of the standard finite element software is not fully capable of solving contact problems, including friction, with robust algorithms. Hence there is still a challenge for the finite element society to design efficient and robust methods for computational contact mechanics.

The range of application in contact mechanics starts with relatively simple problems like foundations, see Figure 1.1, in civil engineering, where the lift off
of the foundation from the soil due to eccentric forces acting on a building are considered. Furthermore, foundations including piles as supporting members or the driving of piles into the soil are of interest. The latter being a very complex problem which involves inelastic constitutive behaviour of the soil, large deformations and large sliding of the pile relative to the soil.

Also, classical bearing problems of steel constructions, the connecting of structural members by bolts or screws or the impact of cars against building structures are areas in which contact analysis enters the design process in civil engineering (see Figure 1.2). Most of these problems can usually be treated by the assumption of small strains, however problems like the car impact or steel connections need the consideration of inelastic constitutive equations and sometimes also finite deformations. But even if the contact problem can be formulated as a linear elastic problem then, due to the nature of contact
problems with an a priori unknown contact area, all applications are nonlinear and need special solution algorithms.

Applications of contact mechanics in mechanical engineering include the design of gears and bearings which usually can be treated using linear elasticity. Other contact problems are due to drilling but also occur in metal forming or cutting processes, like sheet metal or bulk forming (see Figure 1.3). The latter problems depict large deformations within the sheet and require the use of inelastic constitutive equations.

Furthermore, crash analysis of cars is of great industrial interest since its numerical simulation can reduce the development time and costs of modern cars. An example is provided in Figure 1.4, where a EURO NCAP frontal impact of a car with initial speed of 64 km/h against a deformable barrier is shown. Car crash simulations are one of the most challenging and complex contact problems. First of all, up to 10 million finite elements are needed for a sufficiently refined model. Secondly the simulation models, as can be observed in Fig. 1.4, involve finite deformations, use inelastic constitutive equa-

Fig. 1.4. Crash of a car against a deformable barrier, from Daimler Chrysler AG.

Fig. 1.5. Self-contact of a car component during crash, from ABACQUS, Inc.
Fig. 1.6. Contact of tyres with a road surface, from Michelin.

ations, include dynamic effects and have multiple contact surfaces including self-contact.

This can also be observed when components of a spatial car structure are investigated using finite element analysis. Figure 1.5 depicts a final deformation state of such a component under crash conditions. This part undergoes finite deformation and is partly in the state of self-contact.

Due to the complex mechanical behaviour, such analysis demands a deep knowledge of the engineer in mechanics and numerical methods and, from the software point of view, extremely robust computational methods.

The rolling contact of car tyres (see Figure 1.6) is, besides it is only a part of a car, today also a very challenging problem. Here implicit time integration algorithms are often applied for the solution of the contact problem together with a fine discretization of around 1 million finite elements. This number of elements is needed for a sufficient resolution of the complex tyre structure and the treads which can be observed from Figure 1.6. Furthermore tyre contact problems can also include multi-field investigations which are related to the heating of the tyre, to aquaplaning simulations or its noise production. The latter requires the consideration of the acoustic field equations.

Other applications are related to biomechanics where human joints or the implantation of teeth are of consideration. Here again, large deformation cannot be excluded in the analysis, and complicated nonlinear material models have to be applied for a successful numerical simulation.

Due to this variety, contact problems are today combined either with large elastic or inelastic deformations, including time-dependent responses. Hence a modern formulation within computational mechanics has to account for all these effects, leaving the linear theory as a special case. For most industrial applications, numerical methods have to be applied since the contacting bod-
ities have complex geometries or undergo large deformations. Today we can distinguish several branches in computational contact mechanics which are applied to solve different classes of contact problems:

- Finite element methods, applied to problems undergoing small and large deformations, as well as in the elastic or inelastic range.
- Discrete element methods, used to compute problems in which up to $10^8$ particles are coming into contact.
- Multi body systems, based on a description of the bodies as rigid ones. These systems are generally small, and can be applied to model the dynamic behaviour of engineering structures in which contact is also allowed.

Thermal coupling might need to be considered within contact analysis, cooling of electronic devices, heat removal within nuclear power plant vessels or thermal insulation of astronautic vehicles, where the mechanical response and the thermal conduction interacts in the contact area. When electronic devices are considered coupling with electro-magnetic field equations can be of interest. Even stability behaviour has to be linked to contact, like wrinkling arising in metal forming problems or the shearband formation in soils (see Figure 1.7). The latter problem is also related to the simulation of avalanches. Here a contact formulation together with the correct modelling of the process in continuum mechanics can be used to compute the final position of a part of the avalanche which has sheared off.

All together, Computational Contact Mechanics (CCM) has to cover topics from tribology, including friction, lubrication, adhesion and wear. One has to establish weak forms for finite deformation mechanics, coupling to other fields like thermal or electromagnetic fields, and to derive associated algorithms to solve the nonlinear boundary value problems, which include inequality constraints. Hence, CCM is an interdisciplinary area which needs input from tribologists, mathematicians, computer scientists and people from mechanics, together with people working in other fields like heat conduction or electromagnetism.

In this book we will restrict ourselves mainly to finite element techniques for the treatment of contact problems, despite many other numerical schemes and analytical approaches which could be discussed as well. However, there are
common formulations and algorithms, and also overlapping of the methods. These will be discussed in related chapters. Generally, an overview related to modern techniques applied in discrete element methods can be found in, for example Attig and Esser (1999) and for multi-body-systems with special relation to contact in Pfeiffer and Glocke (1996) and Fremond (2002).

Before we provide a short summary of the topics covered in this book, a short historical overview on contact mechanics and computational contact mechanics is given.

**Historical remarks.** Due to this technical importance, a great number of researchers have investigated contact problems. In ancient Egypt people needed to move large stone blocks to build the pyramids, and thus had to overcome the frictional force associated with it. This is depicted in Figure 1.8, where we can see that even in ancient Egypt people knew about the process of lubrication.

There is a man standing on the sledge who pours a fluid onto the ground immediately in front of the sledge. Since friction occurs in many applications which are of technical importance, famous researchers in the past have investigated frictional contact problems, amongst them DA VINCI, who in the 15th century measured friction force and had already considered the influence of the contact area on the friction force using blocks with different contact area but the same weight (see Dowson (1979) and Figure 1.9). He found that the friction force is proportional to the weight of the blocks, and is independent of the apparent contact area. Associated results are often attributed to Amontons (1699) neglecting the contribution of DA VINCI. When putting these findings in a formula one obtains the classical equation for friction (known as COULOMB’S friction law), which every student in engineering learns during the first semesters of study:

\[ F_T = \mu N \]  

(1.1)

**Fig. 1.8.** Stone block moved by Egyptian worker.
where $F_T$ is the friction force, $N$ is the normal force and $\mu$ the coefficient of friction.

A first analysis from the mathematical point of view was carried out by Euler, who assumed triangular section asperities for the representation of surface roughness (Euler (1748b) and Euler (1748a)). His model is depicted in Figure 1.10. He had already concluded from the solution of the equations of motion for a mass on a slope that the kinetic coefficient of friction has to be smaller than the static coefficient of friction.

Actually, it was Euler who introduced the symbol $\mu$ for the friction coefficient, which is the common symbol nowadays. A comprehensive experimental study of frictional phenomena was later performed by Coulomb (1785); see Figure 1.11. He considered the following facts relating to friction: normal pressure, extent of surface area, materials and their surface coatings, ambient conditions (humidity, temperature and vacuum), and time dependency of friction force. These observations resulted in a formula for the frictional resistance to sliding of a body on a plane

$$F_T = A + \frac{N}{\mu^*},$$

(1.2)

where $F_T$ is the friction force, $N$ is the normal force and $\mu^*$ the inverse of the friction coefficient. $A$ represents cohesion, an effect which was already described in Desaguliers (1725). The second term was attributed to a ploughing
action within the interface. This result, today written as $F_T = A + \mu N$, is still acceptable, and is the basis for many developments of contact interface laws (see e.g. Tabor (1981)). Again COULOMB found that $\mu$ is nearly independent of the normal force, the sliding velocity, the contact area (see also results from DA VINCI) and from the surface roughness. However, $\mu$ depends strongly upon the material pairing in the contact interface. His further, remarkable results concerning the influence of the time of repose upon static friction are discussed in Dowson (1979).

Starting with the classical analytical work of Hertz (1882) the theory of elasticity was applied in contact mechanics. Hertz investigated the elastic contact of two spheres and derived the pressure distribution in the contact area as well as the approach of the spheres under compression. However very few problems involving contact can be solved analytically. For an overview one may consult the books of Johnson (1985) or Timoshenko and Goodier (1970), and the references therein.

The finite element method developed together with the growing power of modern computers. Hence the first attempts to solve structural problems using finite elements were published in the late fifties (see Turner et al. (1956) or Argyris (1960)). After this, the literature grew enormously since there were many problems of industrial importance which could not be solved analytically. It then took another ten years for the first papers in which methods for the solution of contact problems with finite element methods appeared. As first contributions we list the work by Wilson and Parsons (1970) or Chan and Tuba (1971), which contain early treatments of contact using the geometrically linear theory. However, even at an earlier stage Wilkins (1964) developed the explicit HEMP-hydrocode which could deal with large strains, and included a simple contact model. Following this, the explicit codes DYNA2D and DYNA3D, as well as the implicit codes NIKE2D and NIKE3D, were developed at the Lawrence Livermore Laboratory by J. HALLQUIST, beginning in the mid-seventies. For the first time these codes provided the possibility to solve contact problems undergoing finite deformations on a large scale in an efficient way. Nowadays all of the commercial finite element packages include
the possibility to solve contact problems.

**Point of departure and connection of chapters.** The design of robust algorithms to treat contact problems efficiently within the finite element method needs input from different sources. These will be considered in the book, which also provides the physical and tribological background within the contact interface. Hence several chapters are devoted to theoretical aspects of continuum mechanics, contact kinematics and the constitutive behaviour in the contact interface. Other chapters contain discretization techniques for solids, and of course, for the contact interface. Furthermore, solution algorithms are discussed, as well as adaptive techniques for contact. Chapters dealing with special contact formulations or topics are also included to complete the treatment of contact problems. An interaction between the chapters will be denoted in the following more detailed description of the contents of the different chapters.

In the first introductory chapter, several contact problems and simple discretizations are treated to present the basic ideas and difficulties of contact mechanics, including coupled and impact problems. This chapter requires no further background besides standard engineering knowledge.

The second chapter is of a more general nature, and discusses the underlying theoretical background for finite deformation solid mechanics, including kinematics, weak forms, linearizations and simple hyperelastic constitutive equations. This chapter is needed to understand the following chapters regarding the kinematics of large deformation contact, and the associated weak formulations. It can be skipped if the reader is familiar with these formulations.

The third chapter discusses contact kinematics from the continuum mechanics point of view. The formulations stated in this chapter are the basis for the derivations in later chapters.

The physical background of the constitutive behaviour in the contact interface is considered in the fourth chapter. This section can be read on its own with a classical background in engineering. It contains material regarding normal and frictional contact for different material pairings, as well as basic formulations for lubrication, adhesion and wear.

The boundary value problem for frictionless and frictional contact is stated in Chapter 6. This also contains different methods on how the contact constraints can be incorporated in the weak forms needed for finite element analysis. This chapter is based on the formulations presented in Chapters 3 and 4. This chapter also contains a section on the treatment of rolling contact based on an Arbitrary **Lagrangian Eulerian** (ALE) formulation for stationary and non-stationary processes.

The discretization of solids in contact is derived in Chapter 7 on the basis of the theoretical formulations included in Chapter 3. This chapter is only concerned with the continuum part of the bodies and hence can be skipped if the reader is familiar with this subject.
The discretization of the contact interfaces is described in Chapters 8 and 9 for linear and nonlinear geometry, respectively. Here interpolation functions and matrix formulations are given for two- and three-dimensional applications. Also, smooth interpolations are introduced to obtain more robust methods for arbitrary contact geometries. Furthermore, new techniques such as mortar or Nitsche interpolations are discussed in Chapter 8 which can be used for non-matching meshes. This chapter is based on the material derived in Chapters 4, 5, 6 and 8.

Solution methods for contact problems are contained in Chapter 10. Here different methods of algorithmic treatment are considered for the solution of contact boundary value problems which are defined in the weak sense in Chapter 5. Furthermore, search algorithms for contact are discussed for different applications with respect to global and local search.

In Chapter 11 we treat the coupled thermo-mechanical problem of contact. This chapter is concerned with the heat transfer at the contact interface, which depends upon the mechanical response. Furthermore, the associated finite element discretization for small and finite deformations and the algorithmic treatment of the coupled problem is considered. The contents of this chapter is based on formulations derived in Chapters 3, 4, 5, 8 and 9.

The contact of beam elements is of interest in, for example, the micro-mechanical modelling of woven fabrics. Since the formulations do not fit completely into the general scope, all relevant equations – from the continuous formulation to the finite element discretization – are developed for the beam contact in Chapter 12. Knowledge of the background provided in Chapters 4, 6, 7 and 10 is necessary to understand the derivations.

Stability problems which include contact constraints are discussed in Chapter 13. These problems arise in, for example, sheet metal forming, but can also occur in civil engineering applications like the drilling of deep holes. Here the associated algorithms are stated based on the formulations given in Chapters 6 and 10.

Adaptive methods for contact problems which are necessary to control the errors inherited in the finite element method are described in Chapter 14. The objective of adaptive techniques is to obtain a mesh which is optimal in the sense that the computational costs involved are minimal under the constraint that the error in the finite element solution is below a certain limit. In general, adaptive methods rely on error indicators and error estimators, which can be computed a priori or a posteriori. In Chapter 14 an overview over different techniques is given, including different error estimators and indicators. Again, the basic formulations of the solid and the contact constraints from Chapters 3, 4, 7, 8, 9 and 10 are required.
Introduction to Contact Mechanics

To introduce the basic methodology and difficulties related to contact mechanics, some simple contact problems will be discussed in this chapter. These are one-dimensional examples undergoing static, thermal or dynamic contact.

2.1 Contact in a Mass Spring System

2.1.1 General formulation

Frictionless contact. Let us consider a contact problem consisting of a point mass \( m \) under gravitational load which is supported by a spring with stiffness \( k \). The deflection of the point mass \( m \) is restricted by a rigid plane, see Figure 2.1. The energy for this system can be written as

\[
k m u h = \begin{cases} \Pi & \text{for } h > h_{\text{max}} \\ h_{\text{min}} & \text{for } h_{\text{min}} \leq h \leq h_{\text{max}} \\ 0 & \text{for } h < h_{\text{min}} \end{cases}
\]

Fig. 2.1. (a) Point mass supported by spring. (b) Energy of the mass spring system.
If we do not place any restriction on the displacement \( u \), then we can compute the extremum of (2.1) by variation, leading to

\[
\delta \Pi(u) = ku\delta u - mg\delta u = 0 .
\] (2.2)

Since the second variation of \( \Pi \) yields \( \delta^2 \Pi = k \), the extremum of (2.1) is a minimum at \( u = \frac{mg}{k} \). This is depicted in Figure 2.1b, in which the energy of the mass spring system is plotted.

The restriction of the motion of the mass by a rigid support can be described by

\[
c(u) = h - u \geq 0,
\] (2.3)

which excludes penetration as an inequality constraint. For \( c(u) > 0 \) one has a gap between point mass and rigid support. For \( c(u) = 0 \) the gap is closed.

Note that the variation \( \delta u \) is restricted at the contact surface; from (2.3) one obtains \( \delta u \leq 0 \), which means that the virtual displacement has to fulfill the constraint and can only point in the upward direction. The use of this variation in the variational form (2.2) yields an inequality

\[
k u\delta u - mg\delta u \geq 0
\] (2.4)

in which the greater sign follows from the fact that the force \( mg \) is greater than the spring force \( kh \) in the case of contact, and that the variation is \( \delta u < 0 \) at the rigid support. Equation (2.4) is called a variational inequality.

Due to the restriction of the solution space by the constraint condition (2.3) the solution of (2.1) is not at the minimum point associated with \( \Pi_{\text{min}} \), but at the point associated with \( \Pi'_{\text{min}} \), which denotes the minimal energy within the admissible solution space, see Figure 2.1b.

Often, instead of the variation \( \delta u \), one uses the difference between a test function \( v \) and the solution \( u \): \( \delta u = v - u \). The test function has to fulfill the condition \( v - h \leq 0 \) at the contact point, as also does the solution \( u \). With the test function \( v \), (2.2) can be written as

\[
k u(v - u) - mg(v - u) = 0 .
\] (2.5)

Since \( mg > ku \) at the contact point, we have with \( v - h \leq 0 \)

\[
k u(v - h) \geq mg(v - h) .
\] (2.6)

In both cases, inequality (2.3) which constrains the displacement \( u \) leads to variational inequalities which characterize the solution of \( u \). These variational inequalities cannot be directly applied to solve the contact problem. For this one has to construct special methods. Some frequently used methods are discussed in the following sections.
Once the point mass contacts the rigid surface, a reaction force $f_R$ appears. In classical contact mechanics, we assume that the reaction force between rigid surface and point mass is negative, hence the contact pressure can only be compression. Such assumption excludes adhesion forces in the contact interface and leads to the restriction
\[ R_N \leq 0. \quad (2.7) \]
This means that either we have a compression state ($R_N < 0$) or an inactive reaction force ($R_N = 0$).

Summarizing, one has to distinguish two cases within a contact problem where the motion is constrained by (2.3):

1. The spring stiffness is sufficiently large enough that the point mass does not touch the rigid surface. In this case, the following conditions are valid:
\[ c(u) > 0 \quad \text{and} \quad R_N = 0. \quad (2.8) \]

2. The data of the system are such that the point mass comes into contact with the rigid support. In that case conditions
\[ c(u) = 0 \quad \text{and} \quad R_N < 0 \quad (2.9) \]
hold.

Both cases can be combined in the statement
\[ c(u) \geq 0, \quad R_N \leq 0 \quad \text{and} \quad R_N c(u) = 0 \quad (2.10) \]
which are known as Hertz–Signorini–Moreau conditions in contact mechanics. Such conditions coincide with Kuhn–Tucker complementary conditions in the theory of optimization.

The result of the above considerations can be depicted by plotting the reaction force versus the gap (2.3), see Figure 2.2. Since the load displacement curve has a corner it is not differentiable in the standard way. Due to that one
has to apply mathematical methods for non-smooth problems, when contact problems have to be analyzed.

**Contact with friction.** Using now the same system, we can compute also the frictional behaviour of the mass spring system. For this we assume that the mass is in contact with the rigid support, hence $R_N < 0$. Now additionally a force tangential to the supporting plane is applied, see Figure 2.3. The equilibrium equations in vertical and tangential direction follow for the state of contact with the notation provided in Figure 2.3 as

\[
\begin{align*}
R_N + mg - kh &= 0 \\
R_T - F_T &= 0
\end{align*}
\]

Friction between the mass and the rigid support is described by a constitutive equation which has to be formulated in such a way that it describes the physical phenomena of the friction process. The simplest model, widely used in engineering, is Coulomb’s law. Within this constitutive equation one differentiates between a stick and sliding state. Stick means that there is no relative tangential movement between the mass and the rigid support. During sliding there will be a relative displacement $u_T$ between the mass and the rigid support. These assumptions lead to the following set of equations which describe the frictional behaviour.

1. Coulomb’s law provides an inequality involving the normal (vertical) and tangential reaction forces

\[
f(R_N, R_T) = |R_T| + \mu R_N \leq 0.
\]

In this inequality the constitutive parameter $\mu$ is called friction coefficient. It actually can depend upon several other quantities, which will be

![Fig. 2.3. Mass spring system under tangential loading.](image-url)
discussed later. Note that the absolute value of the tangential reaction is taken, since the tangential force $F_T$ can be positive or negative. Inequality (2.13) can now be used to distinguish between stick and slip.

2. Stick occurs when

$$|R_T| < -\mu R_N.$$  \hspace{1cm} (2.14)

In that case we have no relative tangential displacement between the mass and the rigid support: $u_T = 0$. Furthermore the tangential force $R_T$ is a reaction force which can be determined from (2.12).

3. Slip occurs when

$$|R_T| = -\mu R_N.$$  \hspace{1cm} (2.15)

In that case we have a relative tangential displacement between the mass and the rigid support: $u_T \neq 0$ and $R_T$ follows directly from the above equation. The direction of $u_T$ will be opposite to the tangential reaction force $R_T$.

Again the inequalities formulated above can be written in a form of the KUHN–TUCKER, see (2.10). Here we formulate

$$|u_T| \geq 0, \quad f \leq 0 \quad \text{and} \quad |u_T| f = 0$$  \hspace{1cm} (2.16)

where the absolute value of the tangential displacement enters since the tangential force $F_T$ can act in positive or negative direction.

The above analysis leads to load displacement diagram for the tangential loading versus the tangential displacement in case of friction. It is shown in Figure 2.4. As in the frictionless case, see Figure 2.2, the frictional load-displacement curve depicts non-smooth behaviour. This leads to mathematical difficulties, due to the non-differentiability at the corners, when treating frictional contact problems.

### 2.1.2 Lagrange multiplier method

The solution of a contact problem in which the motion is constrained by an inequality (2.3) can be obtained using the method of LAGRANGE multipliers.
For this we assume that a constraint is active, which means condition (2.9) is fulfilled by the solution. Therefore, the Lagrange multiplier method adds to the energy of the system (2.1) a term which contains the constraint and yields
\[ \Pi(u, \lambda) = \frac{1}{2} k u^2 - m g u + \lambda c(u). \]  
(2.17)

A comparison with (2.10) shows that the Lagrange multiplier \( \lambda \) is equivalent to the reaction force \( f_R \). The variation of (2.17) leads to two equations, since \( \delta u \) and \( \delta \lambda \) can be varied independently:
\[ k u \delta u - m g \delta u - \lambda \delta u = 0, \]  
(2.18)
\[ c(u) \delta \lambda = 0. \]  
(2.19)

The first equation represents the equilibrium for the point mass including the reaction force when it touches the rigid surface (see also Figure 2.5), and the second equation states the fulfillment of the kinematical constraint equation (2.3) for contact: \( u = h \). Due to that, the variation is no longer restricted, and one can solve for Lagrange multiplier \( \lambda \) which is equivalent with the reaction force \( R_N \), see (2.7),
\[ \lambda = k h - m g = R_N. \]  
(2.20)

However condition (2.7) still has to be checked and fulfilled by the solution (2.20). If this condition is not met, and hence an adhesion force is computed, then the assumption of contact no longer holds. This means the inequality constraint is inactive and the correct solution can be computed from (2.2) as \( u = \frac{m g}{k} \); furthermore, the reaction force or Lagrange multiplier is zero.
2.1 Contact in a Mass Spring System

2.1.3 Penalty method

Another well known method which is often applied in finite element analysis of contact problems is the penalty approach. Here for an active constraint one adds a penalty term to the energy (2.1) as follows:

\[ \Pi(u) = \frac{1}{2} ku^2 - mg u + \frac{1}{2} \epsilon [c(u)]^2 \text{ with } \epsilon > 0. \]  

(2.21)

As can be seen in Figure 2.6, the penalty parameter \( \epsilon \) can be interpreted as a spring stiffness in the contact interface between point mass and rigid support. This is due to the fact that the energy of the penalty term has the same structure as the potential energy of a simple spring. The variation of (2.21) yields for the assumption of contact

\[ ku \delta u - mg \delta u - \epsilon c(u) \delta u = 0, \]

(2.22)

from which the solution

\[ u = \frac{mg + \epsilon h}{k + \epsilon} \]

(2.23)

can be derived. The value of the constraint equation is then

\[ c(u) = h - u = \frac{kh - mg}{k + \epsilon}. \]

(2.24)

Since \( mg \geq kh \) in the case of contact, equation (2.24) means that a penetration of the point mass into the rigid support occurs, which is physically equivalent to a compression of the spring, see Figure 2.6. Note that the penetration depends upon the penalty parameter. The constraint equation is only fulfilled in the limit \( \epsilon \to \infty \implies c(u) \to 0 \). Hence, in the penalty method we can distinguish two limiting cases:
1. $\epsilon \to \infty \implies u - h \to 0$, which means that one approaches the correct solution for very large penalty parameters. Intuitively, this is clear since that means the penalty spring stiffness is very large, and hence only very small penetration occurs.

2. $\epsilon \to 0$ represents the unconstrained solution, and thus is only valid for inactive constraints. In the case of contact, a solution with a very small penalty parameter $\epsilon$ leads to a high penetration, see (2.24).

The reaction force for a penalty method is computed (see (2.22)) from $R_N = \epsilon c(u)$. For this example, one arrives with (2.24) at

$$R_N = \lambda = \epsilon c(u) = \frac{\epsilon}{k + \epsilon} (k h - m g),$$

(2.25)

which in the limit $\epsilon \to \infty$ yields the correct solution obtained with the Lagrange multiplier method, see (2.20).

### 2.2 Finite Element Analysis of the Contact of Two Bars

This example shows that even for a system which is built from two simple bars with geometrically linear and elastic behaviour a nonlinear response curve occurs in the case of contact. This is due to the change of stiffness within the contact process.

The potential energy of a bar loaded by $i$ point loads is given by

$$\Pi = \frac{1}{2} \int_{(l)}^{} EA |u'(x)|^2 dx - \sum_i F_i u(x_i)$$

(2.26)

when distributed forces along the bar are neglected. $EA$ denotes the axial stiffness, $u(x)$ is the displacement of the bar and $F_i$ describes a point load at point $x_i$. The problem depicted in Figure 2.7 shows a system consisting of two bars which are separated by a gap $g$. When the force $F$, acting at $x = l$, is large enough the gap will close. We assume that a penetration of bar 1 into bar 2 is impossible. Due to Figure 2.7, this yields the constraint equation

![Fig. 2.7. System of two bars and loading.](image-url)
For $u_l - u_r < g$ no contact occurs, whereas contact takes place for $u_l - u_r = g$.

This system is discretized using three finite elements, two for the left bar and one for the right bar. Linear shape functions are chosen (see Figure 2.8) which already fulfill the boundary conditions at the left and right end of the structure, see also Figure 2.7. The explicit form of the shape functions and their derivatives is given within the elements as

$$
0 \leq x \leq l : u(x) = \frac{x}{l} u_1, \quad u'(x) = \frac{u_1}{l},
$$

$$
l < x \leq 2l : u(x) = (2 - \frac{x}{l}) u_1 + (\frac{x}{l} - 1) u_2, \quad u'(x) = -\frac{u_1}{l} + \frac{u_2}{l},
$$

$$
2l < x \leq 3l : u(x) = (3 - \frac{x}{l}) u_3, \quad u'(x) = -\frac{u_3}{l}.
$$

By inserting these interpolations into (2.26), the discretized form of the potential energy can then be derived by integration, leading for the bar system to

$$
\Pi = \frac{1}{2} EA l \left[ u_1^2 + (u_2 - u_1)^2 + u_3^2 \right] - F u_1.
$$

The variation of $\Pi$ yields

$$
\delta \Pi = \frac{EA}{l} \left[ u_1 \delta u_1 + (u_2 - u_1) (\delta u_2 - \delta u_1) + u_3 \delta u_3 \right] - F \delta u_1 = 0.
$$

The constraint condition (2.27) is now given by $u_2 - u_3 \leq g$:

i) For $u_2 - u_3 < g$ displacement $u_3 = 0$ and no contact occurs. One says that the constraint equation is not active, since the gap is open. In this case, the solution follows directly from (2.30), which has the matrix form

$$
\begin{bmatrix}
2 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix} F \\
0 \\
0
\end{bmatrix}
$$

leading for arbitrary virtual displacements $\delta u_i$ to the equation system

$$
\frac{EA}{l} \begin{bmatrix}
2 & -1 & 0 \\
-1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix} F \\
0 \\
0
\end{bmatrix}
$$

with the solution

$$
\begin{bmatrix}
u_1 \\
u_2 \\
u_3
\end{bmatrix}
= \begin{bmatrix} \frac{F l}{EA} \\
0 \\
0
\end{bmatrix},
$$

Fig. 2.8. Shape functions.