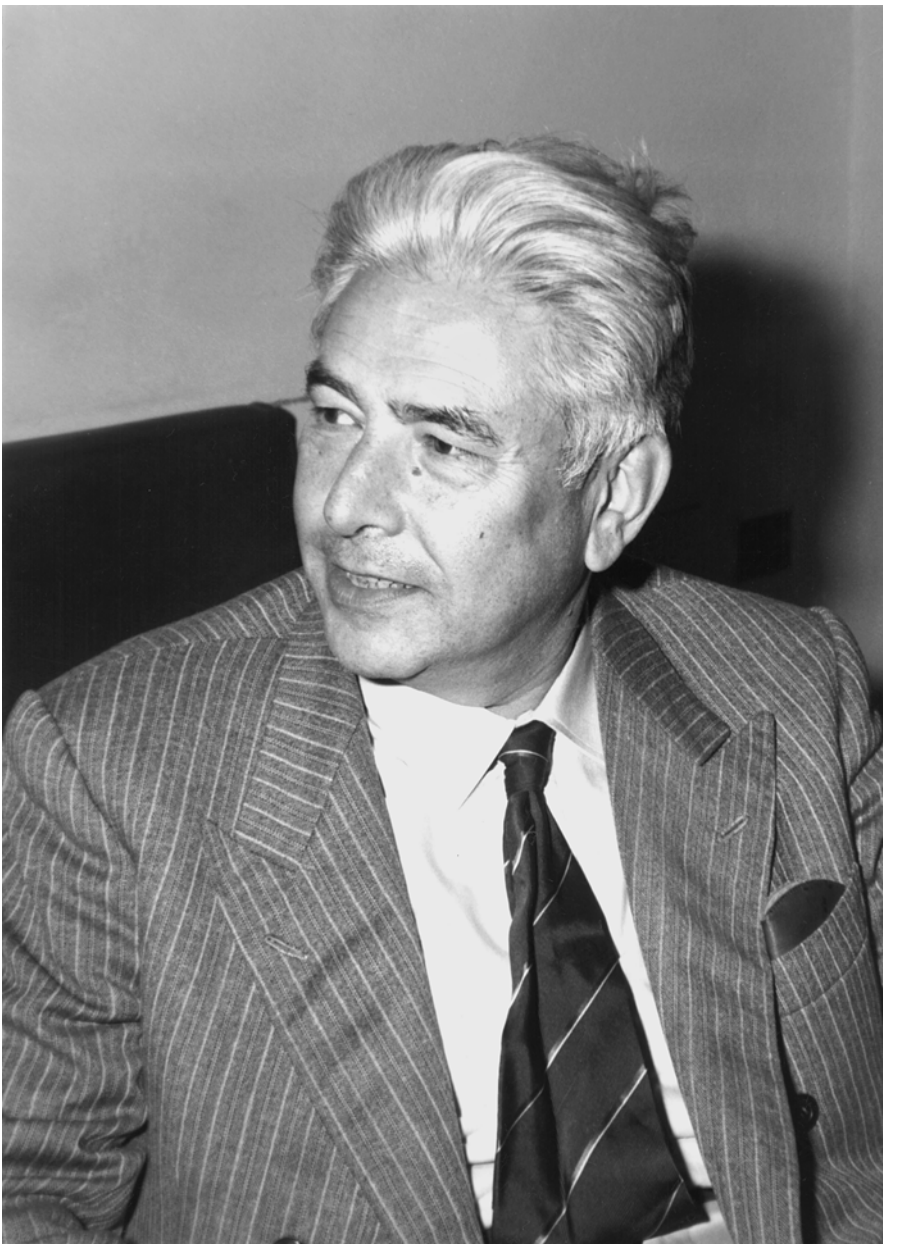
Ennio De Giorgi
Selected Papers



Ennio De Giorgi

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Ennio De Giorgi

Selected Papers

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Preface

The project of publishing some selected papers by Ennio De Giorgi was undertaken by the Scuola Normale Superiore and the Unione Matematica Italiana in 2000. The main motivations for this project have been the desire to make some of his classical papers, originally published in Italian, available to a large public and to display the broad range of his achievements and his entire intellectual path, as a problem solver and as a proponent of deep and ambitious mathematical theories.

We selected 43 papers, out of 152, and for 17 of them we decided to keep the original Italian version as well, in order to give a feeling of De Giorgi's original style.

In the development of this long project we have been helped by several collaborators. In particular we wish to thank Diego Pallara and Emanuele Paolini, for their many fundamental contributions, and Sisto Baldo, Giovanni Bellettini, Andrea Braides, Piero D'Ancona, Massimo Gobbino, Giacomo Lenzi, Roberto Monti, Maurizio Paolini, Paolo Tilli and Vincenzo Maria Tortorelli, who helped us in the typing and the translations of the papers.

We warmly thank Luis Caffarelli and Louis Nirenberg for their contributions to this volume.

Pisa, March 2005

Luigi Ambrosio
Gianni Dal Maso
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Sergio Spagnolo

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Chapter 1

Biography

Ennio De Giorgi was born in Lecce on February 8, 1928. His mother, Stefania Scopinich, came from a family of navigators from Lussino, while his father, Nicola, professor of literature in the teachers' training school of Lecce, was an esteemed scholar in Arabic Language, History, and Geography. His father died an untimely death in 1930; his mother, to whom Ennio was particularly bound, lived until 1988.

In 1946, after his high school studies in Lecce, Ennio moved to Rome, where he started his university studies in Engineering. The following year he switched to Mathematics, and graduated in 1950 under the direction of Mauro Picone. Soon afterwards he obtained a fellowship at the Istituto per le Applicazioni del Calcolo, and in 1951 he became assistant to Picone at the Mathematical Institute "Guido Castelnuovo" of the University of Rome.

In 1958 he was awarded the Chair of Mathematical Analysis by the University of Messina, where he started in this post in December. In the autumn of 1959, following a proposal by Alessandro Faedo, he was hired by the Scuola Normale of Pisa, where he held the Chair of Algebraic and Infinitesimal Mathematical Analysis for almost forty years.

In September 1996 he was admitted to the hospital in Pisa. He underwent surgical treatment, and passed away on October 25.

1.1 Prizes and academic awards

In 1960 the Italian Mathematical Union awarded him the Caccioppoli Prize, founded in the same year. In 1973 the Accademia dei Lincei awarded him the Prize of the President of the Republic. In 1990 he received the prestigious Wolf Prize in Tel Aviv.

In 1983, during a solemn ceremony at Sorbonne, he received the degree *honoris causa* in Mathematics at the University of Paris. In 1992 the University of Lecce awarded him the degree *honoris causa* in Philosophy, of which he was particularly proud.

He was a member of the most important scientific institutions, in particular of the Accademia dei Lincei and of the Pontifical Academy of Sciences, where he had an active role until his last days. In 1995 he became a member of the Académie des Sciences de l'Institut de France and of the National Academy of Sciences of the United States.

1.2 Teaching and academic engagements

In all the activities he was engaged in, De Giorgi showed endeavour and availability well beyond the mere academic duties. He set only one limit on his collaboration: the firm refusal to take administrative or bureaucratic positions.

At the end of the 1950s, together with Enrico Magenes, Giovanni Prodi, Carlo Pucci and others, he founded an association of young researchers, named CONARM, that was the starting point for the future National Groups for Mathematics.

He was a member of the selection committee for the entrance examination of the Scuola Normale almost every year from 1960 and 1980, and many original problems proposed in that period show his imprint.

At the end of the 1960s he worked intensely on the Technical Committee of the Faculty of Sciences of the University of Lecce; ten years later he was a member of the Founding Committee of SISSA in Trieste.

In 1964 he entered the Scientific Council of the Unione Matematica Italiana and, later, the Director's Committee of the Istituto Nazionale di Alta Matematica. From 1979 on he offered his collaboration to CIMPA in Nice, an international centre that promotes teaching and research in Mathematics in developing countries.

In Pisa, De Giorgi used to give two courses each year, normally on Tuesday and Wednesday, from 11 a.m. to 13 p.m. The tone of these lectures was very relaxed, with frequent questions asked by participants. Sometimes the lecture was interrupted in the middle for about twenty minutes, and the whole class moved to a nearby café. Even if the details were not carefully prepared, his lectures were fascinating; those on Measure Theory given in the 1960s have become a classic. The notes of some of these courses have been written by his students, and carefully revised by him.

Beginning in the mid 1970s, De Giorgi reserved the Wednesday course to Foundations of Mathematics, while the other course continued to be devoted to the Calculus of Variations or to Geometric Measure Theory.

In 1967–68 he wanted to experience a "service course", namely Mathematics for Chemists. The notes of this course, written by Mario Miranda, are a model of essentiality and clarity; together with the (unfinished) textbook "Ghizzetti-De Giorgi" on Advanced Analysis, this is one of the very few books by Ennio.

1.3 The activities away from Pisa

At the end of the 1950s De Giorgi received pressing invitations from American universities; in particular, in 1960 Robert Oppenheimer repeatedly invited him to visit the Institute for Advanced Study in Princeton. But, maybe because of his scarce familiarity with the English language, he went to the United States only in 1964, when, accepting an invitation by Wendell Fleming, he spent four months between Brown and Stanford University.

His visits to Paris, however, were very frequent. Starting from the 1960s, he used to go to Paris almost every year for 3–4 weeks, invited by Jean Leray or Jacques-Louis Lions. In Paris Ennio felt at home. From these stays in Paris came his preference for *Le Monde*, which he used to read almost every day, even when he was in Italy.

In 1966 the International Congress of Mathematicians took place in Moscow, and Ivan G. Petrovskij invited De Giorgi to give one of the plenary lectures; but, after having prepared the text of his lecture, Ennio did not go. That text was read to the Congress by Edoardo Vesentini, and is an account of the most recent results on the theory of multi-dimensional minimal surfaces.

Several years later, in 1983, De Giorgi accepted an invitation to give a plenary lecture at the ICM in Warsaw. It was the time of Solidarność and Jaruzelski, and the Congress, already postponed for one year, took place in a very difficult political climate. Ennio began his lecture on Γ-convergence by manifesting a great admiration towards Poland. On the same occasion he publicly expressed one of his deepest beliefs, declaring that man's thirst for knowledge was, in his opinion, the "sign of a secret desire to see some ray of the glory of God".

De Giorgi was very interested, perhaps also for family tradition, in all that concerned far-away countries, like Brazil or Japan (where he never had the opportunity to go). In 1966 he accepted with enthusiasm Giovanni Prodi's proposal to spend some months as a teacher in the small University of Asmara, managed by Italian nuns. So, up to 1973, i.e., as long as the political situation in Eritrea made it possible, he spent one month in Asmara each year. When returning to Pisa, he used to speak at great length about his "African ventures".

In Italy Ennio had friends and former students almost everywhere. He was often away from Pisa for seminars or conferences, mainly in Pavia, Perugia, Naples, and Trento, not to mention, obviously, Rome and Lecce. He was a constant participant in the conferences on the Calculus of Variations held in Elba Island and in Villa Madruzzo, in Trento, where he felt particularly at ease. On these occasions he seemed to be indefatigable, promoting endless discussions and formulating new ideas and conjectures.

Beginning in 1988, when his first health problems appeared, Ennio began to spend long periods in Lecce, mainly in the summer, visiting his sister Rosa, his brother Mario, and their children and grandchildren. This was the occasion for this man, who was used to living alone in Pisa, lodging in a room at Collegio Timpano, to experience family life. These stays also gave him the opportunity for frequent meetings with his students and collaborators living in Lecce, either

on the beaches of the Adriatic or Ionian coasts of Salento, or in the department which is now called after him.

1.4 Civil, political, and religious engagement

Ever since the first years he spent in Pisa, De Giorgi wanted to serve in several charitable activities. For many years he took care of some poor families of the town, which he used to visit together with his young students. He was very generous, and his generosity was made more acceptable by the respect he instinctively showed towards any person, regardless of his social or cultural status.

In 1969 he undertook the task of teaching in an evening school for adults, at the junior high school level. He sometimes used puzzle magazines as an auxiliary teaching tool. The students appreciated his teaching, even if they found it a little abstract.

Among the De Giorgi's commitment to social issues, the most important was, without any doubt, the defense of human rights. This commitment, which continued until the very last days of his life, started around 1973 with the campaign to defend the Ukrainian dissident Leonid Plioutsch, who had been locked up in a madhouse in Dniepropetrovsk. Thanks to the efforts of many scientists all over the world, like Lipman Bers, Laurent Schwartz, and De Giorgi himself, Plioutsch became a symbol of the struggle for freedom of opinion and he finally was released in 1976. In Italy Ennio succeeded in involving in this battle hundreds of persons with different political ideas. Later he continued his activity to defend a great number of persons persecuted for their political and religious ideas. He became an active member of Amnesty International, and was among the founders of Amnesty's group in Pisa. He often took the opportunity to illustrate and publicize the Universal Declaration of Human Rights.

Without taking active part in national politics, which he considered too far from the great universal problems, De Giorgi always showed a lively interest in the main themes that animated Italian life. In particular he took a public position several times on the problem of abortion, on freedom of teaching, and on the relationships between science and faith.

He was a deeply religious man. This is witnessed by the serenity he was able to convey to those who were close to him in the hard trials of his last days. He did not hide his religious beliefs, but his attitude towards a continuous search, his natural curiosity, and his open-mindedness towards all ideas, made it easy for him to have a constructive dialogue with others on these themes, too.

Besides St. John's *Apocalypse* and the *Book of Proverbs*, one of his favourite books was *Pensées* by Pascal.

Chapter 2

The scientific work of Ennio De Giorgi

This list completes and updates the one contained in De Giorgi's obituary, appeared in 1999 on Boll. UMI, Sez. B, **(8) 2**. We decided to include also some writings (as for instance [125]) that are not publications in a strict sense. De Giorgi, in particular in the last years of his life, used to circulated them among friends and colleagues, asking for opinions. We plan in the future to collect and to make available all these unpublished writings.

The publications appearing in this volume are marked with a star.

2.1 Complete list of De Giorgi's scientific publications

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[2] E. De Giorgi: Un criterio generale di compattezza per lo spazio delle successioni, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat.* **(8) 9,** (1950), 238–242.12-728, (V. L. Klee). See also Addenda and Errata, vol. 13, p.1139.

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[4] E. De Giorgi: Sulla sommabilità delle funzioni assolutamente integrabili, *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Nat.* **(8) 12** (1952), 507–510.14-257 (T. H. Hildebrandt)

[5] E. De Giorgi: Compiuta ricerca dell'estremo inferiore di un particolare funzionale, *Rend. Accad. Sci. Fis. Mat. Napoli* **(4) 19** (1952), 29–41.14-291, 14-1278 (J. M. Danskin).

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2.2 From the juvenile papers to Plateau's Problem and the solution of Hilbert's 19th Problem

2.2.1 The first articles

The first papers written by Ennio De Giorgi for scientific journals ([1]–[5] and [8]) were published in the years 1950–53. Each of them is the result of a strong student–teacher relationship with his advisor Mauro Picone, who was the director of the Istituto per le Applicazioni del Calcolo, where De Giorgi obtained his first job.

The papers [1] and [2] can be considered as the outcome of his active participation in a university course held by Picone in the academic year 1949–50. In both papers we read the remark "we follow the terminology adopted by Professor Picone in his course of Functional Analysis". As for the content of the papers, in the first one "there is shown how to obtain a limit point, in the space of closed subsets (under Hausdorff's metric) of a compactum, of any given sequence of closed subsets". This is the entire review by R. Arens in *Mathematical Reviews*.

The second paper considers the space Σ of sequences of real numbers and three elementary properties of distance functions on this space, which lead to an elementary characterization of compactness in Σ. The review for *Mathematical Reviews*, signed by V.L. Klee, Jr., as all those we shall refer to, is very detailed and contains some suggestions to simplify the original paper.

The paper [5] is a very careful study of the minimum problem for a one-dimensional quadratic functional. This was a problem presented by Picone in his book "Corso di Analisi Superiore–Calcolo delle Variazioni", published by the Circolo Matematico di Catania in 1922. The hasty review by J.M. Danskin seems to say that the paper contains a not short solution of a not difficult exercise.

The paper [3] considers a problem that was probably proposed by Picone in his course "Introduzione al Calcolo delle Variazioni", held in Rome in the academic year 1950–51. The problem is the minimization of a quadratic functional depending on vector–valued functions of one real variable. In the review L.M. Graves is not impressed by the results, as he writes "this paper contains a few remarks on quadratic variational problems".

The paper [4] presents a necessary and sufficient condition for the integrability of a quasi-continuous function, with respect to an elementary mass. For a general discussion on the problem the author refers to the monograph by M. Picone and T. Viola "Lezioni sulla teoria moderna dell'integrazione", Ediz. Scient. Einaudi, 1952. The review of the paper appeared just after that of the monograph; both were signed by T.H. Hildebrandt.

The paper [8] proves the holomorphy of the sum of a series of homogeneous polynomials, uniformly convergent in a real neighbourhood of the origin, under suitable boundedness conditions in the complex field. The origin of this paper is explained by the author, who writes "I prove a theorem that has been for a long time a conjecture of Professor Mauro Picone, who has often exposed it to

his students; as far as I know this theorem has never had the simple proof I am going to present here". The review is by J. Favard.

2.2.2 The development of the theory of perimeters

In the second series of papers ([6], [7], [10], [14], and [18]) we see the exceptional abilities and the great self-confidence of the young De Giorgi. These gifts were acknowledged by Renato Caccioppoli, in the memorable meeting we will describe later.

The paper [6] is the text of a communication to the Congress of the Austrian Mathematical Society held in Salzburg from 9 to 15 September 1952. It contains a theorem à la Weierstrass, on the existence of maxima and minima of a real-valued function defined on the family of all measurable sets with equi-bounded perimeter in a given bounded set of the Euclidean n-dimensional space. The author remarks that the notion of perimeter is equivalent to the $(n-1)$-dimensional measure of the oriented boundary, introduced by Caccioppoli in two papers in 1952. The perimeter is computed analytically, in an original and extremely effective way, through the convolution product of the characteristic function of the set by a Gaussian function. In the same paper the author announces the validity of the isoperimetric inequality for every Lebesgue measurable set. The paper was reviewed by L.M. Graves, who did not notice the great interest of these results for the Calculus of Variations.

The connection of these new results with the Calculus of Variations was underlined by De Giorgi himself in the paper [7], submitted to the Accademia dei Lincei on 14 March 1953. In it he draws attention to the close connection between his results and the results announced by Caccioppoli. The paper was reviewed by L.C. Young, who did not see this connection, although he had expressed a very negative opinion on the papers by Caccioppoli, which he had reviewed before De Giorgi's paper.

This is not the only strange fact in the Caccioppoli–De Giorgi relationship, a recognized master and a young mathematician facing his first challenges. De Giorgi worked on his theory of perimeters, following with a remarkable self-confidence the path just seen by Caccioppoli, without having ever spoken with him. He exploited the papers by Caccioppoli, to which Picone, probably, had drawn his attention. In the academic year 1953–54, on the occasion of a lecture delivered in Rome, Caccioppoli met De Giorgi, who made him aware of his work on oriented boundaries. This meeting was recalled by Edoardo Vesentini during the funeral service for De Giorgi in Pisa, on October 27, 1996: "Before touching the mathematical aspects of De Giorgi's remark, Caccioppoli quoted a sentence by André Gide: nothing is more barbarous than a pure spirit; then, turning to Ennio, he added: it seems that you are an exception".

De Giorgi saw the confirmation of the acknowledgement of his great value in the review of the paper [10], which contains the detailed proof of all previously announced results. Young's review is not only positive, but it also contains a self-criticism of his previous negative opinion on Caccioppoli's work: "Although the author's definitions and theorems are given in precise terms, he is able to

show that his definition of perimeter coincides with one proposed by Caccioppoli for $(r-1)$-dimensional boundary measure. This makes it possible to judge more clearly the precise scope of Caccioppoli's definitions". De Giorgi considered this review as the best one ever published in *Mathematical Reviews* on his work.

In the paper [14] De Giorgi analyses the geometrical properties of the boundaries of sets with finite perimeter. He singles out a part of the boundary, called the reduced boundary, which carries the measure associated with the gradient of the characteristic function of the set, and on which the total variation of this measure coincides with the $(n-1)$-dimensional Hausdorff measure. At each point of the reduced boundary De Giorgi proves also the existence of a tangent hyperplane to the boundary of the set.

These results, which are totally nontrivial and will be the starting point for the study of the regularity of the boundaries, when these are solutions of suitable variational problems, did not obtain due regard in the review by L.C. Young.

Instead Young underlined the value of the isoperimetric property of the hypersphere, which is proven in [18]. This paper concludes, in 1958, the cycle devoted to the theory of sets with finite perimeter, started at the Conference in Salzburg in September 1952.

2.2.3 The solution of Hilbert's 19th Problem

The four papers [15], [16], [17], and [26] are devoted to the solution of Hilbert's 19th Problem. The first one is the text of a communication to the fifth Congress of the Italian Mathematical Union, held in Pavia and Turin from 6 to 9 October 1955. This text is very important from the historical point of view. The title shows that, when De Giorgi decided to participate in the Congress, he intended to present his results on sets with finite perimeter, the isoperimetric inequality, and their applications to the Calculus of Variations. The first chapter of the paper is indeed devoted to these topics. The second chapter presents the regularity result for the minima of regular functionals of the Calculus of Variations. There is certainly a connection between the two chapters, since the proof of the regularity result uses the isoperimetric property of the hypersphere. But the fact that the solution of Hilbert's 19th Problem was not mentioned in the title confirms that De Giorgi had not yet obtained this result when he registered for the Congress.

The events which led to the proof of the regularity theorem, reported by Enrico Magenes in the Commemoration of De Giorgi at the Accademia dei Lincei, happened with breath-taking speed. In August 1955, during a hike near Pordoi Pass, in the Dolomites, De Giorgi was informed by Guido Stampacchia about the existence of the 19th Problem. He must have immediately seen the possibility of applying to the solution of this problem the results of his research on the geometry of subsets of multi-dimensional Euclidean spaces. Indeed he was able, in less than two months, to present his solution of Hilbert's Problem to the UMI Congress. This story points out an aspect of De Giorgi's scientific personality: a striking intuition, combined with the prodigious ability to obtain from it a complete proof, with all minor details. The other aspect of De Giorgi's personality