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Projective and Cayley-Klein Geometries

With 69 Figures

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Preface

Projective geometry, and the Cayley-Klein geometries embedded into it, are rather ancient topics of geometry, which originated in the 19th century with the work of V. Poncelet, J. Gergonne, Ch. v. Staudt, A.-F. Möbius, A. Cayley, F. Klein, S. Lie, N. I. Lobatschewski, and many others. Although this field is one of the foundations of algebraic geometry and has many applications to differential geometry, it has been widely neglected in the teaching at German universities — and not only there. In the more recent mathematical literature these classical aspects of geometry are also scarcely taken into account. In the present book, the synthetic projective geometry and some of its recent applications, e.g. of the finite geometries, are mentioned only in passing, i.e., they form the content of some remarks. Instead, we intend to present a systematic introduction of projective geometry as based on the notion of vector space, which is the central topic of the first chapter. In the second chapter the most important classical geometries are systematically developed following the principles founded by A. Cayley and F. Klein, which rely on distinguishing an absolute and then studying the resulting invariants of geometric objects. These methods, determined by linear algebra and the theory of transformation groups, are just what is needed in algebraic as well as differential geometry. Furthermore, they may rightly be considered as an integrating factor for the development of analysis, where we mainly have in mind the harmonic or geometric analysis as based on the theory of Lie groups. Even though, wherever it does not require any extra effort, we allow for vector spaces over an arbitrary skew field, we nevertheless mainly deal with geometries over the real, the complex, or the quaternionic numbers; we also discuss the consequences of extensions as well as restrictions of the scalar domain to the geometries in question. Apart from the real projective geometry we also treat some of the complex and quaternionic geometries in detail, which are rarely presented on an elementary level. The elementary conformal or Möbius geometry is extensively discussed, and even some aspects of the symplectic projective geometry are studied. The concluding Section 2.9 contains a brief introduction into the theory of Lie transformation groups with an outline of the classification of transitive Lie group actions on the n-dimensional spheres and projective spaces. The *octonions* and the *octonion geometries*, which correspond to the exceptional groups rather than to the classical series of Lie groups remain outside of the scope of this book; for this topic we refer to H. Freudenthal [39], M. M. Postnikov [89], and H. Salzmann, D. Betten, Th. Grundhöfer, H. Hähl, R. Löwen, M. Stroppel [97].

With this systematic presentation we hope to provide a useful tool for all who intend to acquire the necessary background material all by themselves or in special seminars. The systematic organization, the detailed proofs, interspersed exercises, and an extensive index starting with a list of symbols, are meant to facilitate choosing the topic of respective interest as well as its study. The appendix contains the definitions of several notions from algebra and geometry used in the text, which are not treated in greater detail there. The book is intended to be a manual and work book, and was by no means written as the text on which a lecture should be based.

A flaw of the book which the reader will soon notice is the missing or, at best, loose connection of the material with its history. The presentation by no means follows the course of the historic development; that would have expanded the size of the book considerably and also claimed too much time and working capacity from the authors as well as the readers. Instead, we decided to take the systematics of the subject from the current point of view as the guiding principle, and to start from the structures also used in other parts of mathematics, e.g. groups or vector spaces, which in fact developed in connection with geometry. We consider labelling propositions, principles, etc., by the names of mathematicians rather as the usual way of denotation than stating any priority or authorship of the mathematician in question for the result. Interpreting these propositions as historically correct claims of authorship would, implicitly, assert to have studied the complete literature, even the correspondence or the the oral traditions up to this moment, in order to exclude the existence of any predecessor, a venture the authors do not want to pride themselves on.

We do not intend to include a complete bibliography of this field. Today it is not difficult to gather arbitrarily extensive lists of references from the available data banks. We only refer to the titles used to work on this subject, where chance and personal choice play an essential part. Frequently, we relied on the well-known basic textbooks by E. Artin [4] and R. Baer [5]. Of course, we have to thank our teachers for the suggestions and orientations we received since the time of our studies and over the long years of our cooperation, which, perhaps sometimes unknowingly, influenced the presentation. Particularly to be mentioned here are the lectures, seminars, and writings by W. Blaschke, E. B. Dynkin, L. A. Kaloujnine, A. G. Kurosch, P. K. Raschewski, and H. Reichardt.

We also want to quote some textbooks and monographs dealing with our subject from a different point of view or with differing emphasis. First, there is the report of results by J. Dieudonné [36] containing more general and more detailed results than this book as well as a comprehensive bibliography. The very interesting and brief textbook by A. Beutelspacher and U. Rosenbaum [12] contains the analytical as well as the *synthetic foundations of projective geometry,* and, in addition, recent applications to *coding theory* and *cryptography.* The *finite geometry* is also discussed there, a subject to which the complete two-volume treatise [10], [11] by A. Beutelspacher is devoted, see also L. M. Batten, A. Beutelspacher [6]. The book [7] by W. Benz presents interesting propositions characterizing geometric transformations under weak assumptions, e.g. the Theorems of Beckmann and Quarles for Euclidean isometries, and the theorem of A. D. Alexandrow concerning Lorentz transformations. In that book the geometries of Lie and Laguerre are also treated, which are not included here. Interesting concrete material discussed from the point of view of Klein's Erlanger Programm can be found in the book [8] by W. Benz. There, apart from the classical geometries, the reader may also find descriptions of the Einstein cylinder universe and the de-Sitter space. The rich summary [69] by Linus Kramer describes the recent development of projective geometry; the structure of the projective group, the relation between incidence properties and algebraic properties of the scalar domain, generalizations of the fundamental theorem, Tits buildings, Moufang planes, polar spaces, and quadratic forms are discussed there, and non-commutative scalar domains receive particular emphasis.

As prerequisites for the understanding of this book the reader is assumed to be familiar with the fundamental notions of linear algebra as well as the affine and Euclidean geometries based on it as they are usually presented in the first year of any course in mathematics or physics. This also includes the knowledge of the affine classification of quadrics and its Euclidean refinement by means of the principal axes transformation. We will frequently use these notions and results without special reference. Moreover, some experience of the metric Euclidean, non-Euclidean, and spherical elementary geometries are useful for understanding this book; as a brief and clearly written, very rich presentation we mention the recently published textbook [1] by I. Agricola and Th. Friedrich. Obviously, for the authors and the readers, who are familiar with or have at their disposal both volumes of the series "Algebra und Geometrie" [82, 83], it is advisable as well as convenient to make use of the relations to the topics discussed there. For this reason we frequently refer to these sources, e.g., Proposition 1 in § 4.2 of [82] is quoted as Proposition I.4.2.1, and, correspondingly, § 6.2 from [83] as § II.6.2. Instead of these books the reader may of course draw on other textbooks which as a rule contain similar results. Familiarity with differentiable manifolds or Lie groups, however, is not required to understand this text. The authors tried to include an extensive index. Nevertheless, the entries usually do not occur twice; e.g., "orthogonal group" is only listed as "group, orthogonal". References to the book itself have the usual format; i.e., Proposition 5 stands for the corresponding proposition in the same section, Proposition 6.3 refers to Proposition 3 in Section 6 of the same chapter, and Corollary 1.3.4 cites Corollary 4 in Section 3 of

Chapter 1. The same holds for formulas: (2.5.19) is formula (19) in Section 5 of Chapter 2.

The nearly 70 illustrations accompanying the text mostly were produced using the software *Mathematica* by St. Wolfram [114]. This program provides numerous possibilities for numerical as well as symbolic calculations and contains a varied graphic apparatus for the visualization of plane and spatial geometric objects, which is naturally tied to Euclidean geometry. On the homepage

http://www-irm.mathematik.hu-berlin.de/~sulanke

some notebooks written using this software can be found. They were developed writing this book and extend the possibilities of Mathematica. They include notebooks for pseudo-Euclidean geometry, by means of which relativity theory, Möbius geometry, Lie's sphere geometry, and, via their Killing forms, semi-simple Lie algebra may be studied. In great detail the three-dimensional Euclidean and the Möbius geometry of k-spheres, $k = 0, 1, 2$, is treated. Some of the formulas obtained by means of symbolic calculations using these notebooks, expressing Möbius invariants in terms of Euclidean invariants similarly to the Coxeter distance of hyperspheres, are included here. On the homepage one also finds a very fast orthogonalization algorithm going back to Erhard Schmidt as well as a procedure to orthogonalize sequences of vectors in pseudo-Euclidean spaces. This opens up possibilities which, because of the scope and complexity of the calculations involved, are impossible to reach by the traditional methods, i.e. “by hand”. The so-called “artificial intelligence” reaches its limits as soon as it leaves the finite algorithmic ground and turns towards the abstract conceptual thinking abounding in modern mathematics. Already implementing the naive set theory in Mathematica meets serious difficulties, as the example of a system of Mathematica notebooks and packages designed for this purpose shows, which can also be found on the homepage.

The authors gratefully acknowledge the support of their work provided by the Humboldt-Universität zu Berlin. In particular, we thank Heike Pahlisch, Hubert Gollek, Jürgen Gehne, and Hans-Joachim Spitzer for their assistance and technical help. We express our sincere gratitude to Bernd Wegner from the Technische Universität Berlin for his interest in our work as well as his endeavor to publish it. We thank the Springer-Verlag for including the English translation of the book into its publishing program and the permission to publish the German manuscript on the homepage above for private use. Last but not least we thank Andreas Nestke for his careful translation of the German manuscript and the always friendly and fruitful cooperation with us during his work. We think and hope that this book may cause some of our colleagues and many students to considerations and investigations of their own.

Berlin,
June 2006

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R. Sulanke

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1

Projective Geometry

The presentation of projective geometry in this chapter will be based on linear algebra. We will start with some remarks concerning the central projection intended to be a motivation, which will then lead to the definition of a projective geometry as the lattice of subspaces of a vector space. The first fundamental result to be proved is the Main Theorem of projective geometry relating its geometric properties closely to the algebraic ones: If the dimension is at least two, i.e. for the plane and all higher-dimensional projective spaces, then geometric isomorphy implies algebraic isomorphy. There exists a collineation between two spaces, i.e. a bijective map transforming lines into lines, if and only if, first, both their dimensions coincide, and, second, their scalar fields are isomorphic. The one-dimensional case, the projective line, requires special considerations, since the collineation condition cannot work there; we will introduce the cross ratio of four collinear points and prove Staudt's Main Theorem: Invariance of the harmonic position alone implies invariance of the cross ratio. The duality of vector spaces has a projective counterpart, the duality of projective spaces. This duality enables to study correlations, i.e. bijective maps from point spaces to spaces of hyperplanes preserving incidence. Symmetric auto-correlations will be classified by reducing the problem to the classification of skew-symmetric and Hermitean biforms. The results of both classifications will be interpreted geometrically: as linear line complexes in the skew-symmetric case, and as hyperquadrics in the Hermitean case. Over the scalar domains of the real numbers, the complex numbers, and the quaternions we will obtain a complete classification. Finally, we will also investigate the geometric implications brought about by extending or restricting the scalar fields. In particular, we will describe the Hopf fibrations, which play a prominent part in topology.

1.1 Projective Spaces

1.1.1 Definitions and First Properties

There are two, in their course essentially different approaches to projective geometry, the *synthetic* and the *analytic* one. The latter should better be called the linear-algebraic approach, since linear algebra rather than analysis forms its basis and provides the methods. Nevertheless, the name "analytic geometry" became firmly established in the course of history. In synthetic geometry sets of geometric elements of various kinds, e.g. points, lines, planes, are given, and the existing relations between them are fixed axiomatically; these axioms correspond to familiar geometric concepts. Instead, here we prefer an approach based on linear algebra, which is more abstract but at the same time much more general and, in principle, even simpler. There is no need to build a special theory whose axioms have to be changed frequently, but everything relies on the notion of vector space. This structure is fundamental for analysis as well as many applications of mathematics, hence its knowledge is compulsory for every mathematician. Moreover, any undergraduate course would include this topic. Synthetic projective geometry is already systematically treated in the comprehensive two-volume treatise [106] by O. Veblen and J. W. Young that appeared in 1910 and 1918. The two-dimensional case, i.e. plane projective geometry, plays a prominent part there, cf. G. Pickert [87]. Later in this section we will present a brief discussion of its simple axiomatics. In this context, interesting connections between this synthetic axiomatics and algebraic properties of the scalar domain will arise, that can be constructed on the basis of these axioms. Conversely, these relations will become evident by looking for the analytic projective geometries satisfying certain synthetic axioms. A vivid introduction into the plane projective geometry can be found in H. S. M. Coxeter [34].

In the sequel we will always suppose that a vector space $\boldsymbol{V}$ over a skew field K is given[1]. The set of one-dimensional subspaces of $\boldsymbol{V}$ will be denoted by $\boldsymbol{P}(\boldsymbol{V})$; we will call it the *projective space associated with the vector space* $\boldsymbol{V}$. It is common to define the projective space as an extension of the affine point space by means of a so-called *hyperplane at infinity.* The one-dimensional subspaces of the vector space associated with the affine geometry bijectively correspond to the families of parallel lines in the affine space, which are determined by attaching the vectors of the subspace to the points in the affine point space. Expressed differently, each of these families is the orbit of a one-dimensional vector subspace in point space. Intuitively, one thinks of all the

[1] The only skew field essential for the following is the skew field of quaternions $\mathbf{H}$. To distinguish between a left and a right vector space is only necessary in the case of a non-commutative field K. For now it will suffice to take the fields of real or of complex numbers as K, hence the whole discussion will remain within the usual framework of vector spaces.

lines within such a family as passing through a common single point at infinity. The set of all these points at infinity corresponding to all the different families of parallels constitute the hyperplane at infinity to be attached. To make this concept more precise is elementary, but lengthy and laborious. It is much simpler to start from the definition above and then embed the affine geometry into the projective one by distinguishing an arbitrary hyperplane, which, consequently, is to be called the *hyperplane at infinity,* cf. Section 5 of this chapter. The projective plane will serve as an example to illustrate these observations. In Example 2 below we will see that the projective extension of point space is necessary and expedient to describe central projections:

Example 1. The Real Projective Plane. Let $\boldsymbol{V}= \mathbf{R}^3$ be the three-dimensional vector space over the field $\mathbf{R}$ of real numbers, and denote by $\boldsymbol{A}^3$ the associated affine space; we will frequently consider this to be the physical space. By $\boldsymbol{e} \in \boldsymbol{A}^3$ we denote the point with coordinates $(0, 0, 1)$, i.e. the unit point on the z-axis. Attaching the one-dimensional subspaces of $\boldsymbol{V}$ to the point $\boldsymbol{e}$ defines a bijection from $\boldsymbol{P}(\boldsymbol{V})$ onto the set of lines passing through $\boldsymbol{e}$, cf. Figure 1.1. Attaching in a similar way the two-dimensional subspaces of $\boldsymbol{V}$ to $\boldsymbol{e}$ defines a bijection between the set of these subspaces and the set of planes through $\boldsymbol{e}$.

Each line through $\boldsymbol{e}$, which is not parallel to the x, y-plane $\boldsymbol{H}$ $(z = 0)$, meets this plane in a unique point. Obviously, the lines through $\boldsymbol{e}$ parallel to $\boldsymbol{H}$ form a one-parameter family, whereas to describe the others we need to specify two parameters, e.g. the coordinates of their intersection point with the x, y-plane $\boldsymbol{H}$. All lines through $\boldsymbol{e}$ lying in a plane $\boldsymbol{B}$ through $\boldsymbol{e}$, i.e. whose vectors belong to the same two-dimensional subspace $\boldsymbol{W} \subset \boldsymbol{V}$, intersect $\boldsymbol{H}$ in one line, the line of intersection $\boldsymbol{B} \cap \boldsymbol{H}$. The only exception to this is the vector space $\boldsymbol{U}$ of $\boldsymbol{H}$ itself, which, by attaching it to $\boldsymbol{e}$, determines the plane $\boldsymbol{B}_0 = \boldsymbol{e} + \boldsymbol{U}$ parallel to $\boldsymbol{H}$. Thus it appears natural to extend the affine plane by a “line at infinity”, whose points just correspond to the lines through $\boldsymbol{e}$ parallel to $\boldsymbol{H}$. We will denote this extension by $\hat{\boldsymbol{H}}$. In this picture, the line at infinity appears as the intersection of the similarly extended plane $\hat{\boldsymbol{B}}_0$ with $\hat{\boldsymbol{H}}$. It is common to introduce the *real projective plane* this way: Start from the affine plane and attach a line at infinity to it, whose points correspond to the non-oriented directions of the plane, i.e. the equivalence classes of parallel lines in the affine plane. So all parallel lines in such a class appear to meet in a unique point of the line at infinity. Since there is a bijective correspondence

$$p : \boldsymbol{x} \in \boldsymbol{P}(\boldsymbol{V}) \mapsto p(\boldsymbol{x}) \in \hat{\boldsymbol{H}}$$

between the elements of $\boldsymbol{P}(\boldsymbol{V})$ and the points of the extended plane $\hat{\boldsymbol{H}}$, we arrive at a clear picture of $\boldsymbol{P}(\boldsymbol{V})$[2]. Note that, in the projective sense, the line at infinity in the picture above is equivalent to any of the other lines in the plane. □

[2] The topological shape of the real projective plane will be described in Examples 2.5.1 and 2.5.4 in the following chapter.

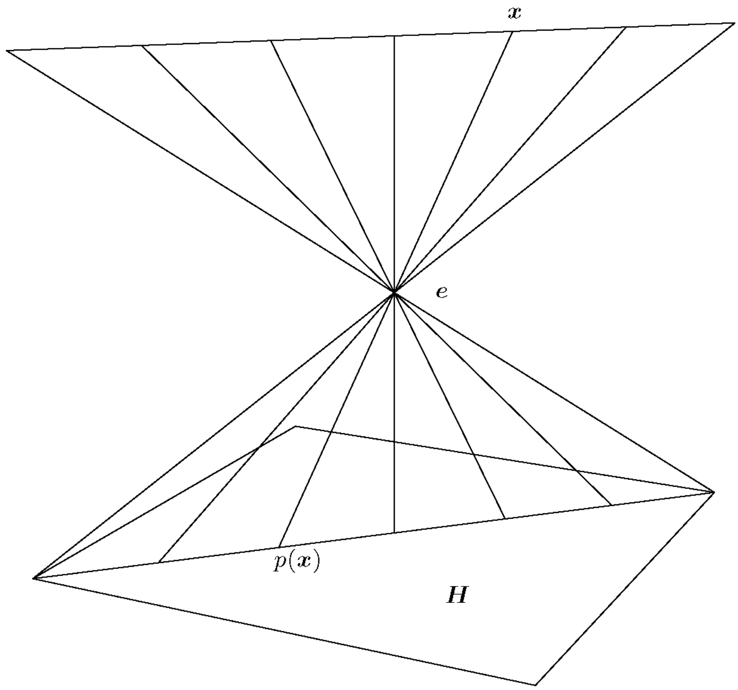

Fig. 1.1. Line bundle and the projective plane.

The bijection p described in the previous example induces a similar bijection between the two-dimensional subspaces $\boldsymbol{W} \subset \boldsymbol{V}$ of the vector space $\boldsymbol{V}$ and the set of lines $\boldsymbol{h} \subset \hat{\boldsymbol{H}}$ in the projective plane $\hat{\boldsymbol{H}}$:

$$\boldsymbol{W} \subset \boldsymbol{V} \mapsto \boldsymbol{B} = \boldsymbol{e} + \boldsymbol{W} \mapsto \boldsymbol{h} = \boldsymbol{B} \cap \hat{\boldsymbol{H}}.$$

Here, the subspace $\boldsymbol{U}$ of $\boldsymbol{H}$ corresponds to the plane parallel to $\boldsymbol{H}$ through $\boldsymbol{e}$ and the line at infinity of $\hat{\boldsymbol{H}}$. These considerations can easily be generalized to the n-dimensional case; they suggest to interpret the *lattice of subspaces of the vector space* $\boldsymbol{V}$ (cf. § I.4.2) in the following projective-geometric way extending the affine point of view that, basically, amounts to renaming the structures provided by the lattice of subspaces:

Definition 1. Let $\boldsymbol{V}$ be a right vector space over the skew field K. By $\mathfrak{P}(\boldsymbol{V})$ we will denote the lattice of subspaces of $\boldsymbol{V}$ and call it the *projective geometry of* $\boldsymbol{V}$: its elements are called *projective subspaces*. The *projective dimension* Dim of an element $\boldsymbol{W} \in \mathfrak{P}(\boldsymbol{V})$ is equal to the dimension of the vector space diminished by one:

$$\operatorname{Dim} \boldsymbol{W} := \dim \boldsymbol{W} - 1.$$

Since all right (and left, respectively) vector spaces $\boldsymbol{V}$ of equal finite dimension $n+1$ over the same skew field K are isomorphic, we will usually just write

$$\mathfrak{P}_K^n := \mathfrak{P}(\boldsymbol{V}^{n+1})$$

This is the *n-dimensional projective geometry over K*. The zero-dimensional elements in $\mathfrak{P}(\boldsymbol{V})$ are the *points* of the *projective space*

$$\boldsymbol{P}(\boldsymbol{V}) := \{\boldsymbol{a} \in \mathfrak{P}(\boldsymbol{V}) |\ \mathrm{Dim}\, \boldsymbol{a} = 0\}.$$

For the n-dimensional projective space we will also use the notation

$$\boldsymbol{P}_K^n := \boldsymbol{P}(\boldsymbol{V}^{n+1}) \qquad (n < \infty).$$

If the skew field K is clear from the context, we will omit the index K. In general, $\boldsymbol{W}$ is called a *projective k-plane* if $\mathrm{Dim}\, \boldsymbol{W} = k$; the set of k-planes of an n-dimensional projective geometry will be denoted by $\boldsymbol{P}_{n,k}$ and referred to as the *Graßmann manifold*.[3] As usual, the elements of $\boldsymbol{P}_{n,1}$ are called *lines*, those of $\boldsymbol{P}_{n,2}$ are the *planes*, and for $k = n-1$ we have the *hyperplanes*; the word "projective" will be omitted if there is no danger of confusion. Two elements $\boldsymbol{U}, \boldsymbol{W} \in \mathfrak{P}(\boldsymbol{V})$ are said to be *incident*, written as

$$\boldsymbol{U} \,\iota\, \boldsymbol{W}, \text{negation: } \boldsymbol{U} \,\bar{\iota}\, \boldsymbol{H},$$

if $\boldsymbol{U} \subset \boldsymbol{W}$ or $\boldsymbol{W} \subset \boldsymbol{U}$ holds. Keeping the notation we transfer the *inclusion relation*[4] $\subset$ of subspaces of $\boldsymbol{V}$ to $\mathfrak{P}(\boldsymbol{V})$. The vector space $\boldsymbol{V}$ itself is the largest element with respect to the order $\subset$, and the trivial subspace is the smallest, $\boldsymbol{o} := \{\mathfrak{o}\} \in \mathfrak{P}(\boldsymbol{V})$; we decided to called it the *nopoint* in projective space. The *section* of two projective subspaces $\boldsymbol{U}, \boldsymbol{W} \in \mathfrak{P}(\boldsymbol{V})$ is understood to be their (set-theoretic) intersection:

$$\boldsymbol{U} \wedge \boldsymbol{W} := \boldsymbol{U} \cap \boldsymbol{W}.$$

Their *join* is defined to be the smallest subspace containing both, i.e. the linear span of their union (cf. § I.4.2):

$$\boldsymbol{U} \vee \boldsymbol{W} := \mathfrak{L}(\boldsymbol{U} \cup \boldsymbol{W}) = \boldsymbol{U} + \boldsymbol{W}.$$

□

It is straightforward to extend the notions *section* and *join* to arbitrary families of subspaces $(\boldsymbol{H}_\iota)_{\iota \in I}$:

---

[3] In correspondence with the dimension convention, the Graßmann manifold has a differing notation in vector algebra, $G_{n+1,k+1} (= \boldsymbol{P}_{n,k})$.

[4] We do not distinguish between $\subset$ and $\subseteq$, i.e., $A \subset B$ is the same as $A \subseteq B$, equality is not excluded.

$$\bigwedge_{\iota \in I} \boldsymbol{H}_\iota := \bigcap_{\iota \in I} \boldsymbol{H}_\iota,$$
$$\bigvee_{\iota \in I} \boldsymbol{H}_\iota := \sum_{\iota \in I} \boldsymbol{H}_\iota.$$

The familiar rules of linear algebra (cf. § I.4.2) remain valid; this is summarized by stating that projective geometry is a "complete lattice" with respect to the inclusion $\subset$.

The nopoint $\boldsymbol{o}$ has only a formal, hardly any geometric meaning. Two projective subspaces are called *skew* if their intersection is the nopoint. By Definition 1 we have $\operatorname{Dim} \boldsymbol{o} = -1$. This definition together with the dimension theorem of linear algebra (Proposition I.4.6.3) immediately implies the analogous result for projective subspaces:

Proposition 1. *Consider $\boldsymbol{U}, \boldsymbol{W} \in \mathfrak{P}(\boldsymbol{V})$. Then the following dimension formula holds:*

$$\operatorname{Dim}(\boldsymbol{U} \wedge \boldsymbol{W}) + \operatorname{Dim}(\boldsymbol{U} \vee \boldsymbol{W}) = \operatorname{Dim} \boldsymbol{U} + \operatorname{Dim} \boldsymbol{W}.$$

□

Exercise 1. Use the dimension formula (Proposition 1) to discuss all possible position relations of projective subspaces $\boldsymbol{F}^k$, $\boldsymbol{H}^m \subset \boldsymbol{P}^n$ for $0 \leq k \leq m < n \leq 4$. Find an example for each of the resulting cases. The two subspaces are skew if and only if their join is not contained in any $(k+m)$-dimensional subspace. Compare the results with the corresponding ones in affine space (s. § I.4.6).

The next example clearly shows the advantage of the historically younger concept of projective geometry: From this point of view, the properties of central projection are derived by direct reference to the dimension formula. From the affine point of view, i.e. looking at the projective plane as the extended affine plane, it is necessary to distinguish several cases. We will nevertheless discuss them, as they also provide a useful bridge between affine and projective geometry.

Example 2. Central Projection. We start by considering the three-dimensional projective space $\boldsymbol{P}^3$. Let $\boldsymbol{H}$ and $\boldsymbol{B}$ be different projective planes in $\boldsymbol{P}^3$, and let $\boldsymbol{e}$ be a point not incident with either plane. The *central projection* from $\boldsymbol{B}$ onto $\boldsymbol{H}$ with *projection center* $\boldsymbol{e}$ is the map:

$$q : \boldsymbol{x} \in \boldsymbol{B} \longmapsto \hat{\boldsymbol{x}} = q(\boldsymbol{x}) := (\boldsymbol{x} \vee \boldsymbol{e}) \wedge \boldsymbol{H} \in \boldsymbol{H}.$$

In Section 1.3 we will give a more general definition of the central projection, which plays a very important part in projective geometry. Interpreting the definition projectively, it is easy to see that $q : \boldsymbol{B} \to \boldsymbol{H}$ is a bijection mapping lines to lines; this is a direct consequence of the dimension formula.

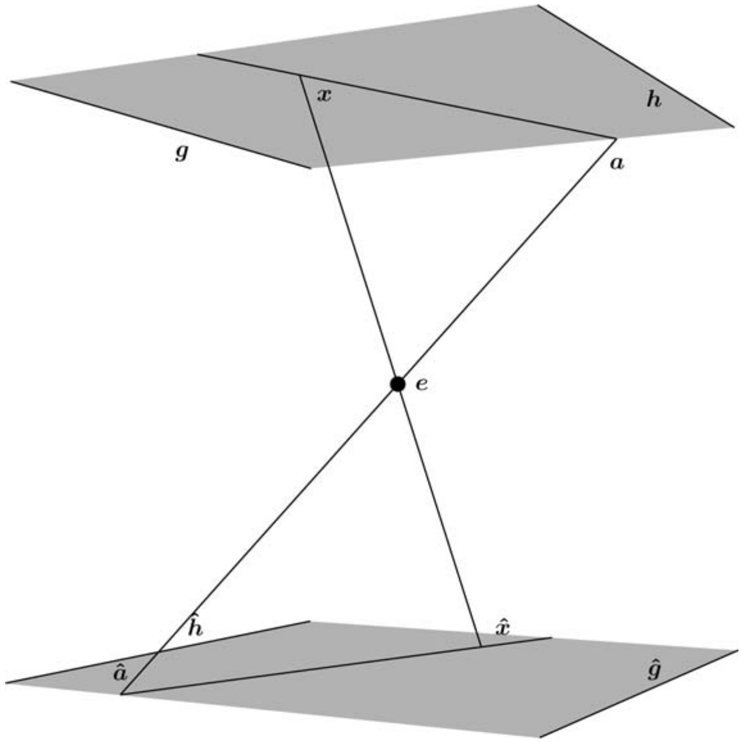

Fig. 1.2. Central projection of parallel planes.

From the affine point of view the situation turns out to be more varied. First we have to distinguish two cases.

Case 1: The planes $\boldsymbol{B}$ and $\boldsymbol{H}$ are parallel. In Figure 1.2, let the upper plane $\boldsymbol{B} = \boldsymbol{g} \vee \boldsymbol{h}$ be the pre-image, and take the lower one as the image $\boldsymbol{H} = \hat{\boldsymbol{g}} \vee \hat{\boldsymbol{h}}$. For the images of the lines $\boldsymbol{g}, \boldsymbol{h}$ and points $\boldsymbol{a}, \boldsymbol{x}$ of $\boldsymbol{B}$ above we then have:

$$q(\boldsymbol{g}) = \hat{\boldsymbol{g}},\ q(\boldsymbol{h}) = \hat{\boldsymbol{h}},\ q(\boldsymbol{a} \vee \boldsymbol{x}) = q(\boldsymbol{a}) \vee q(\boldsymbol{x}) = \hat{\boldsymbol{a}} \vee \hat{\boldsymbol{x}}.$$

In this case, q is actually an affine map between the affine, not just the extended planes; it thus maps parallel lines to parallel lines. The projectively extended planes $\boldsymbol{B}, \boldsymbol{H}$ intersect in their common line at infinity, $\boldsymbol{h}_\infty = \boldsymbol{B} \wedge \boldsymbol{H}$, containing the fixed points of the central projection q.

Case 2: The pre-image plane $\boldsymbol{B} = \boldsymbol{c} \vee \boldsymbol{d} \vee \boldsymbol{a}$ and the image plane $\boldsymbol{H} = \boldsymbol{c} \vee \boldsymbol{d} \vee \hat{\boldsymbol{a}}$ are not parallel; they intersect in a line $\boldsymbol{c} \vee \boldsymbol{d} = \boldsymbol{B} \wedge \boldsymbol{H}$ formed by the fixed points of q, cf. Fig. 1.3. Let $\boldsymbol{e} \vee \boldsymbol{u} \vee \boldsymbol{v}$ be the plane parallel to $\boldsymbol{H}$ through the projection center $\boldsymbol{e}$; it intersects $\boldsymbol{B}$ in the line $\boldsymbol{u} \vee \boldsymbol{v}$. For $\boldsymbol{w} \in \boldsymbol{u} \vee \boldsymbol{v}$ the line $\boldsymbol{w} \vee \boldsymbol{e}$ is parallel to $\boldsymbol{H}$; hence it meets the line at infinity $\boldsymbol{h}_\infty \subset \boldsymbol{H}$ of the image plane in the sense of extension as described in the previous example. Moreover,

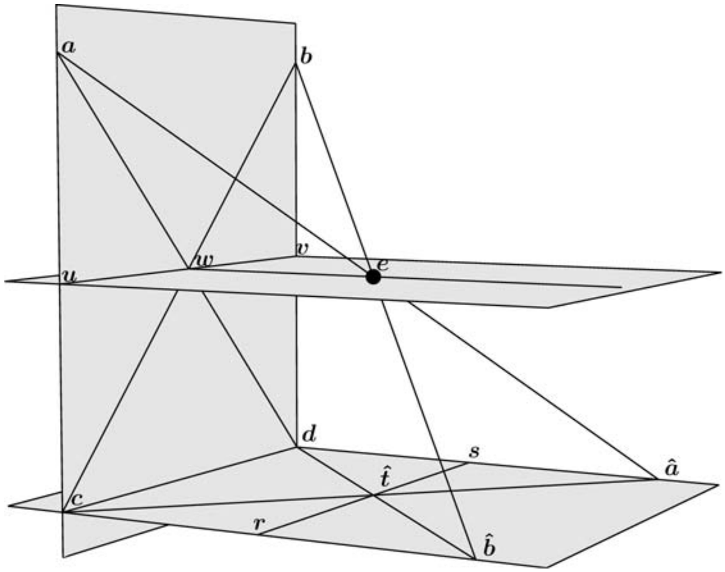

Fig. 1.3. Central projection of non-parallel planes.

$$q(\boldsymbol{u} \vee \boldsymbol{v}) = \boldsymbol{h}_\infty.$$

All lines in the pre-image plane $\boldsymbol{B}$ incident with $\boldsymbol{w}$, e.g. the lines $\boldsymbol{a} \vee \boldsymbol{d}$ and $\boldsymbol{b} \vee \boldsymbol{c}$, are mapped to parallel lines:

$$q(\boldsymbol{a} \vee \boldsymbol{d}) = \hat{\boldsymbol{a}} \vee \boldsymbol{d},\ q(\boldsymbol{b} \vee \boldsymbol{c}) = \hat{\boldsymbol{b}} \vee \boldsymbol{c}.$$

Thus the pencil of lines through $\boldsymbol{w}$ corresponds to the pencil of parallel lines in the common, non-oriented direction of the image plane $\boldsymbol{H}$ determined by their images. On the other hand, the images of parallel lines from $\boldsymbol{B}$ intersect:

$$q(\boldsymbol{a} \vee \boldsymbol{c}) \wedge q(\boldsymbol{b} \vee \boldsymbol{d}) = (\hat{\boldsymbol{a}} \vee \boldsymbol{c}) \wedge (\hat{\boldsymbol{b}} \vee \boldsymbol{d}) = \hat{\boldsymbol{t}}.$$

Denote by $\boldsymbol{b}_\infty$ the line at infinity of the pre-image plane $\boldsymbol{B}$ and by $\boldsymbol{t}$ the point of intersection on it determined by the parallel lines $\boldsymbol{a} \vee \boldsymbol{c}$ and $\boldsymbol{b} \vee \boldsymbol{d}$. Then we have $q(\boldsymbol{t}) = \hat{\boldsymbol{t}}$. Hence the line of intersection $\boldsymbol{r} \vee \boldsymbol{s}$ of the image plane $\boldsymbol{H}$ with the plane parallel to the pre-image plane $\boldsymbol{B}$ through $\boldsymbol{e}$ is the image of the line at infinity of $\boldsymbol{B}$: $q(\boldsymbol{b}_\infty) = \boldsymbol{r} \vee \boldsymbol{s}$.

These considerations imply that q cannot be an affine map, since it does not preserve parallelism. The situation just described is known in photography as the occurrence of plunging lines in pictures of high buildings taken with inclined camera. From the point of view of geometry, central projection is the ideal model for photographing with a pinhole camera; unfortunately, the luminosity decreases with increasing precision of the reproduction. Central projections and their generalizations are a natural point of departure for projective geometry as they map lines into lines and preserve incidence. □

Exercise 2. Introduce a cartesian coordinate system adapted to the configuration shown in Fig. 1.3 and prove the statements in the previous example by computations based on the methods of affine geometry.

For formal reasons it is often advisable to adjoin the nopoint to projective space. Therefore we define the disjoint union

$$\boldsymbol{P}^n_o := \boldsymbol{P}^n \cup \{\boldsymbol{o}\}. \tag{1}$$

As a first application consider the *canonical map*,

$$\pi : \mathfrak{x} \in \boldsymbol{V}^{n+1} \mapsto \boldsymbol{x} := [\mathfrak{x}] \in \boldsymbol{P}^n_o, \tag{2}$$

where $[\mathfrak{x}]$ denotes the linear hull $[\mathfrak{x}] = \mathfrak{L}(\{\mathfrak{x}\})$ in the vector space $\boldsymbol{V}$. The map π allows to identify the elements $\boldsymbol{H} \in \mathfrak{P}(\boldsymbol{V})$ with the point sets $\pi(\boldsymbol{H}) \subset \boldsymbol{P}^n_o$; this identification preserves the order. If not explicitly stated otherwise, we will always assume that the nopoint $\boldsymbol{o}$ is included into each projective subspace of $\boldsymbol{P}^n_o$. Since the section of projective subspaces is also a projective subspace, setting

$$M \in \mathcal{P}(\boldsymbol{P}^n_o) \longmapsto \bigvee(M) := \bigwedge_{M \subset \boldsymbol{H} \in \mathfrak{P}(\boldsymbol{V})} \boldsymbol{H} \in \mathfrak{P}(\boldsymbol{V}) \tag{3}$$

defines the *projective hull* of the subset $M \subset \boldsymbol{P}^n_o$. Using the identification above we obtain

$$\bigvee(M) = \pi(\mathfrak{L}(\pi^{-1}(M))) = \bigcap_{M \subset \boldsymbol{H} \in \mathfrak{P}(\boldsymbol{V})} \boldsymbol{H} \in \mathcal{P}(\boldsymbol{P}^n_o).$$

$\bigvee(M)$ will also be called *the projective subspace spanned by* M.

Exercise 3. Prove that (3) defines a *hull operator*, i.e. that the following properties hold:

a) $M \subset \bigvee(M)$;
b) $M \subset L$ implies $\bigvee(M) \subset \bigvee(L)$;
c) $\bigvee(\bigvee(M)) = \bigvee(M)$.

Prove, in addition, that:

d) $\bigvee(\emptyset) = \boldsymbol{o} \in \bigvee(M)$ for all $M \subset \boldsymbol{P}^n_o$.
e) The equality $\bigvee(M) = M$ holds if and only if M is a projective subspace.

Exercise 4. Prove that a subset $M \subset \boldsymbol{P}^n_o$ is a projective subspace if and only if, firstly, $\boldsymbol{o} \in M$ and, secondly, with every $\boldsymbol{x}, \boldsymbol{y} \in M, \boldsymbol{x} \neq \boldsymbol{y}$, the connecting line $\boldsymbol{x} \vee \boldsymbol{y}$ also belongs to M.

Exercise 5. Let $\boldsymbol{H}^{n-1} \subset \boldsymbol{P}_o^n$ be a projective hyperplane. Prove that

$$\bigvee(\boldsymbol{P}_o^n \setminus \boldsymbol{H}^{n-1}) = \boldsymbol{P}_o^n.$$

Now we will transfer the notion of linear independence to projective geometry.

Definition 2. Let $(\boldsymbol{x}_\iota)_{\iota \in I}$ be a family (or set) of points $\boldsymbol{x}_\iota \in \boldsymbol{P}_o^n$. $(\boldsymbol{x}_\iota)_{\iota \in I}$ is said to be *in general position* if for each of its subsequences (or subsets) with $k+1$ points, $0 \le k \le n$,

$$\text{Dim}\ \boldsymbol{x}_{\iota_0} \vee \boldsymbol{x}_{\iota_1} \vee \ldots \vee \boldsymbol{x}_{\iota_k} = k.$$

A sequence (or set) $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_r)$ of points is called *projectively independent* if

$$\text{Dim}\ \boldsymbol{x}_0 \vee \boldsymbol{x}_1 \vee \ldots \vee \boldsymbol{x}_r = r.$$

□

Exercise 6. Let $\boldsymbol{x}_i = [\mathfrak{x}_i] \in \boldsymbol{P}_o^n$. Prove: a) The sequence $(\boldsymbol{x}_0, \ldots, \boldsymbol{x}_k)$, $k \le n$, is in general position if and only if the vectors $(\mathfrak{x}_0, \ldots, \mathfrak{x}_k)$ are linearly independent. – b) An arbitrary sequence $(\boldsymbol{x}_i)_{i \in I}$ is in general position if and only if every subsequence of at most $n+1$ points is projectively independent. – c) Each subsequence of a sequence in general position is itself in general position. (Similar statements to a), b), c) hold for sets.)

Corollary 2. *In n-dimensional projective space there are $n+1$ projectively independent points; every sequence or set with $r > n+1$ points is projectively dependent. There is at least one set M containing $n+2$ points in general position in $\boldsymbol{P}_o^n$.*

Proof. First, note that the points $(\mathfrak{a}_i)_{i=0,\ldots,n}$ of $\boldsymbol{V}^{n+1}$ corresponding to any basis $\boldsymbol{a}_i = [\mathfrak{a}_i]$ are projectively independent. Then add the *unit point* $\boldsymbol{e} = [\mathfrak{a}_0 + \mathfrak{a}_1 + \ldots + \mathfrak{a}_n]$ to the set $\{\boldsymbol{a}_0, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_n\}$ of these so-called *base points*. Together, we obtain $n+2$ points in general position. □

Example 3. Projective Lines. Consider the projective line $\boldsymbol{P}^1$ over the skew field K. Let $(\mathfrak{a}_0, \mathfrak{a}_1)$ be a basis of the associated vector space $\boldsymbol{V}^2$. The *projective scale* corresponding to $(\mathfrak{a}_0, \mathfrak{a}_1)$ is defined as follows[5]:

$$\xi : \boldsymbol{x} = [\mathfrak{a}_0 x^0 + \mathfrak{a}_1 x^1] \in \boldsymbol{P}^1 \mapsto \xi(\boldsymbol{x}) := \begin{cases} x^1(x^0)^{-1} \in K & \text{if } x^0 \neq 0, \\ \infty & \text{if } x^0 = 0. \end{cases} \tag{4}$$

Obviously, this definition is independent of the chosen representatives; to both, $\mathfrak{x}$ and $\mathfrak{x}\lambda$, $\lambda \neq 0$,, the same value is assigned. Setting $\hat{K} = K \cup \{\infty\}$

[5] As is common in tensor calculus, the upper indices label the coordinates of a vector, here 0, 1; they are not to be confused with powers.

the map $\xi : \boldsymbol{P}^1 \to \hat{K}$ becomes a bijection establishing a close and natural relation between the properties of K and those of the projective geometry. The point $\boldsymbol{a}_0 = [\mathfrak{a}_0]$ is called the *zero point*, $\boldsymbol{a}_1 = [\mathfrak{a}_1]$ the *point at infinity*, and $\boldsymbol{e} = [\mathfrak{a}_0 + \mathfrak{a}_1]$ the *unit point* of the projective scale; the following relations are obvious:

$$\xi(\boldsymbol{a}_0) = 0,\ \xi(\boldsymbol{a}_1) = \infty,\ \xi(\boldsymbol{e}) = 1.$$

In the cases $K = \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$, a metric as well as a topology are defined on the scalar domain. Then there is a standard topology on $\hat{K}$, that of the so-called *Alexandroff compactification*, cf. W. Rinow [94], § 28: This is nothing but the usual topology of K to which the complements in $\hat{K}$ of all compact sets in K are added as the neighborhoods of ∞. Considering them as homeomorphisms the projective scales transfer the topologies defined on $\hat{K}$ onto the corresponding projective lines. The space obtained this way is completely different from the affine and the Euclidean one: Topologically, the real projective line $\boldsymbol{P}^1_{\mathbf{R}}$ is a circle S^1, the complex one, $\boldsymbol{P}^1_{\mathbf{C}}$, is the *Riemann sphere* S^2, and the ***quaternionic projective line*** $\boldsymbol{P}^1_{\mathbf{H}}$ is homeomorphic to the four-dimensional sphere S^4. Later we will see that the above topology does not depend on the chosen projective scale, i.e. the basis $(\mathfrak{a}_0, \mathfrak{a}_1)$, cf. Exercise 2.4. □

A first example for the *duality* in projective geometry, which will be treated in detail in Section 1.6 below, is the following proposition:

Proposition 3. *Let $\mathfrak{P}^n$ be a projective geometry over K. Then:*

a) For each couple of points $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{P}^n$, $\boldsymbol{x} \neq \boldsymbol{y}$, there is a unique line $\boldsymbol{h} \in \boldsymbol{P}_{n,1}$ incident with both, the connecting line $\boldsymbol{h} = \boldsymbol{x} \vee \boldsymbol{y}$.

b) For each couple of hyperplanes $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{P}_{n,n-1}$, $\boldsymbol{X} \neq \boldsymbol{Y}$, there is a unique $(n-2)$-plane $\boldsymbol{H} \in \boldsymbol{P}_{n,n-2}$ incident with both, the $(n-2)$-plane of intersection $\boldsymbol{H} = \boldsymbol{X} \wedge \boldsymbol{Y}$.

Proof. a) $\boldsymbol{x} \neq \boldsymbol{y}$ implies $\boldsymbol{x} \wedge \boldsymbol{y} = \boldsymbol{o}$. Since $\operatorname{Dim} \boldsymbol{o} = -1$, the dimension formula yields $\operatorname{Dim} \boldsymbol{x} \vee \boldsymbol{y} = 1$. If $\boldsymbol{h}_1$ is any line incident with $\boldsymbol{x}$ and $\boldsymbol{y}$, then we have $\boldsymbol{h} = \boldsymbol{x} \vee \boldsymbol{y} \subset \boldsymbol{h}_1$, and hence $\operatorname{Dim} \boldsymbol{h} = \operatorname{Dim} \boldsymbol{h}_1 = 1$ implies $\boldsymbol{h} = \boldsymbol{h}_1$.
b) From $\boldsymbol{X} \neq \boldsymbol{Y}$ we conclude $\boldsymbol{X} \vee \boldsymbol{Y} = \boldsymbol{P}^n$. Since $\operatorname{Dim} \boldsymbol{P}^n = n$, the dimension formula yields $\operatorname{Dim} \boldsymbol{X} \wedge \boldsymbol{Y} = n-2$. If $\boldsymbol{H}_1$ is any $(n-2)$-plane incident with $\boldsymbol{X}$ and $\boldsymbol{Y}$, then we have $\boldsymbol{H} = \boldsymbol{X} \wedge \boldsymbol{Y} \supset \boldsymbol{H}_1$, and hence $\operatorname{Dim} \boldsymbol{H} = \operatorname{Dim} \boldsymbol{H}_1 = n-2$, i.e. $\boldsymbol{H} = \boldsymbol{H}_1$. □

1.1.2 Plane Projective Geometry

Example 4. Plane Incidence Geometry. For $n = 2$ Proposition 3 and Corollary 2 comprise the following statements:

P.1. *Any two different points are incident with exactly one line.*
P.2. *Any two different lines are incident with exactly one point.*
P.3. *There are four points no three of which lie on the same line.*

These three statements may serve as axioms for a synthetic projective *plane incidence geometry* Π. Such a geometry has two basic sets: points $x \in P$ and lines $X \in G$, together with a single relation, the incidence ι. Points and lines may be incident, $x \,\iota\, X$, or they may not be incident, $x \,\bar{\iota}\, Y$. The properties of this relations are described by the axioms P.1–3. However, these axioms do not suffice to guarantee the existence of a skew field K whose plane projective geometry is isomorphic to the one determined by the axioms. The existence of such a scalar domain can be achieved by additional axioms. In this context, interesting interrelations between more general scalar domains and synthetic-geometric properties of the projective planes arise, cf. G. Pickert [87]. For more recent developments concerning this subject we refer to the survey article [69] by L. Kramer, where in particular the case of non-commutative scalar domains is stressed. A very vivid presentation of the real plane projective geometry based on a synthetic system of axioms goes back to H. S. M. Coxeter [34]. The multifaceted textbook [12] by A. Beutelspacher and U. Rosenbaum treats the synthetic as well as the analytic foundations of projective geometry. The books [106] by O. Veblen and J. W. Young contain a synthetic presentation of higher-dimensional projective geometry. Note however, that the axioms necessary to describe incidence in three-dimensional space restrict the scalar field decisively stronger than in the case of planar incidence geometry. □

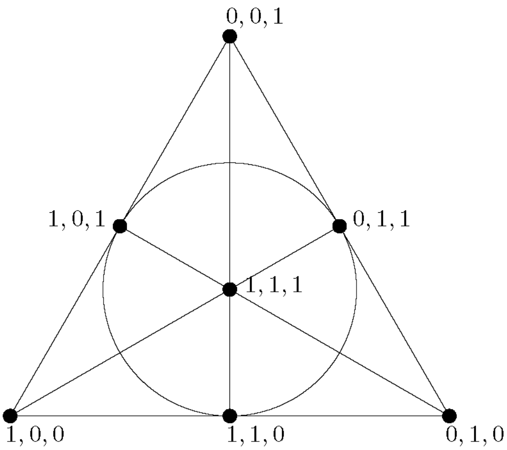

Fig. 1.4. The projective plane over $\mathbf{Z}_2$.

Exercise 7. Prove that in every plane projective incidence geometry Π satisfying the axioms P.1–3 there are four lines no three of which meet in a point. Hint: First prove that on the lines $A \in G$ connecting the points in a configuration as in axiom P.3 there are at least three different points.

Exercise 8. Consider a projective geometry $\mathfrak{P}_K^n$ over a field K. Prove: a) If there exists a line $\boldsymbol{h} \in \mathfrak{P}_K^n$ containing exactly $p+1$ points, p any prime, then $K = \mathbf{Z}_p$, the field of cosets mod p. – b) The projective geometry $\mathfrak{P}_{\mathbf{Z}_2}^2$ consists of exactly seven points and seven lines with the incidence relation described by Fig. 1.4. The circle together with the six segments represent the seven lines. This geometry is also known as the *Fano plane* . – c) In $\mathfrak{P}_{\mathbf{Z}_2}^n$ there are exactly $2^{n+1} - 1$ points.

Definition 3. A set of points $M \in \boldsymbol{P}_o^n$ is called *collinear*, if there is a line $\boldsymbol{h} \in \boldsymbol{P}_{n,1}$ such that $M \subset \boldsymbol{h}$. A set $\mathfrak{M} \subset \mathfrak{P}^n$ is called *concentric*, if there is a point $\boldsymbol{z} \in \boldsymbol{P}^n$ such that $\boldsymbol{z} \iota \boldsymbol{A}$ for all $\boldsymbol{A} \in \mathfrak{M}$. □

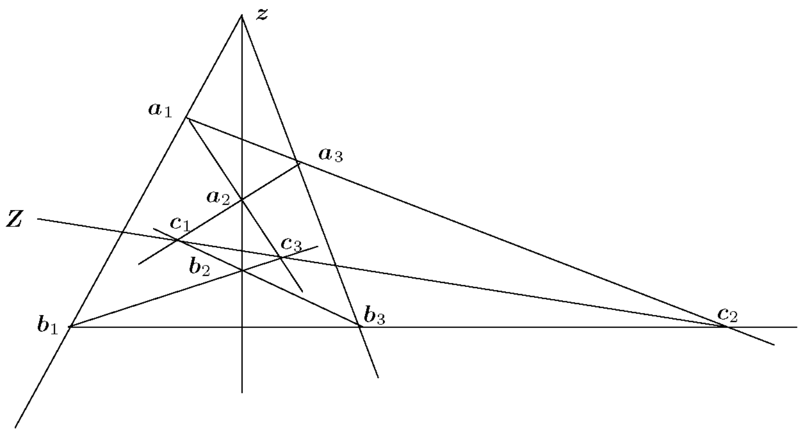

Fig. 1.5. Desargues' Theorem.

The proposition stated in the next exercise goes back to *Desargues* and is particularly interesting for the plane projective geometry, since it does not follow from the Axioms P.1–3 of plane incidence geometry; incidence planes satisfying it are called *Desarguesian planes*. It can be shown that Desargues' Property is equivalent to the associativity of the scalar domain, cf. G. Pickert [87]. In the incidence geometry of three-dimensional space Desargues' Theorem already follows from the other incidence axioms, cf. L. Heffter [48], Section A.5. Hence there are no projective spaces of dimension $n > 2$ with a non-associative scalar domain. The projective plane over the *octonions* is perhaps the most important example for a plane not having Desargues' Property, cf. H. Freudenthal [39] or H. Salzmann et al. [97]. In the geometries based on skew fields as they are discussed here the following proposition holds in all dimensions $n \geq 2$.

Exercise 9 (Desargues' Theorem). Let $\mathfrak{P}_K^n$ be a projective geometry over a skew field K, $n \geq 2$. Consider six different points $\boldsymbol{a}_i, \boldsymbol{b}_i \in \boldsymbol{P}^n$, $i = 1, 2, 3$, whose connecting lines $\boldsymbol{h}_i := \boldsymbol{a}_i \vee \boldsymbol{b}_i$, $i = 1, 2, 3$, are concentric. Prove that "corresponding sides" intersect in points, i.e.

$$\begin{aligned} \boldsymbol{c}_1 &= (\boldsymbol{a}_2 \vee \boldsymbol{a}_3) \wedge (\boldsymbol{b}_2 \vee \boldsymbol{b}_3), \\ \boldsymbol{c}_2 &= (\boldsymbol{a}_1 \vee \boldsymbol{a}_3) \wedge (\boldsymbol{b}_1 \vee \boldsymbol{b}_3), \\ \boldsymbol{c}_3 &= (\boldsymbol{a}_1 \vee \boldsymbol{a}_2) \wedge (\boldsymbol{b}_1 \vee \boldsymbol{b}_2), \end{aligned}$$

and that, moreover, these points are collinear, cf. Fig. 1.5.

For the proof of *Pappos'* Theorem, which does not follow from the incidence axioms P.1–3 as well, the commutativity of the scalar field is essential (cf. Exercise 2.10):

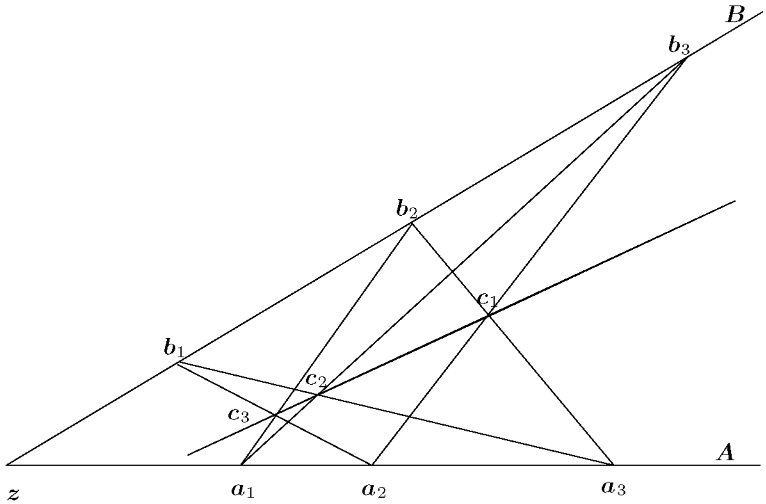

Fig. 1.6. Pappos' Theorem.

Exercise 10 (Pappos' Theorem). Let $\boldsymbol{P}_K^2$ be a projective plane over a field K. Let $\boldsymbol{A}$, $\boldsymbol{B}$ be two lines intersecting in the point $\boldsymbol{z} = \boldsymbol{A} \wedge \boldsymbol{B}$. Consider three points, different from one another and from $\boldsymbol{z}$, on each of the lines $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{a}_i \in \boldsymbol{A}$, $\boldsymbol{b}_i \in \boldsymbol{B}$, $i = 1, 2, 3,$, cf. Fig. 1.6. Prove that under these conditions the points

$$\begin{aligned} \boldsymbol{c}_1 &= (\boldsymbol{a}_2 \vee \boldsymbol{b}_3) \wedge (\boldsymbol{a}_3 \vee \boldsymbol{b}_2), \\ \boldsymbol{c}_2 &= (\boldsymbol{a}_1 \vee \boldsymbol{b}_3) \wedge (\boldsymbol{a}_3 \vee \boldsymbol{b}_1), \\ \boldsymbol{c}_3 &= (\boldsymbol{a}_1 \vee \boldsymbol{b}_2) \wedge (\boldsymbol{a}_2 \vee \boldsymbol{b}_1) \end{aligned}$$

are collinear.

The *finite geometries* some examples of which we have been mentioning here are studied systematically in the monographs [10], [11] by A. Beutelspacher. For more recent developments we in addition refer to L. M. Batten,

A. Beutelspacher [6], K. Metsch [76], and J. W. P. Hirschfeld [54]. The latter also contains an extensive bibliography. Finite geometries are applied to numerous kinds of problems and subjects, among others in the field of combinatorics.

1.2 Homogeneous Coordinates

Let K be an arbitrary skew field. Then $K\boldsymbol{P}^n_o$ (and $K\boldsymbol{P}^n$, respectively) denotes the projective space associated with the right vector space K^{n+1} of $(n+1)$-tuples: $K\boldsymbol{P}^n := \boldsymbol{P}(K^{n+1})$. The elements of $K\boldsymbol{P}^n_o$ are called *homogeneous $(n+1)$-tuples*; they are the orbits of the multiplicative group K^* of K for the right action

$$((x^i), \lambda) \in K^{n+1} \times K^* \longmapsto (x^i \lambda) \in K^{n+1}, \tag{1}$$

where in the case of $K\boldsymbol{P}^n$ the zero tuple is excluded. Just as the vector space K^m of m-tuples from K serves as the standard coordinate space for all the m-dimensional vector spaces over K, the space $K\boldsymbol{P}^n_o$ will now be used as the coordinate space for any n-dimensional projective space over K. Using homogeneous $(n+1)$-tuples, i.e. admitting a superfluous degree of freedom in the form of an arbitrary factor λ, allows to work with just a single coordinate system for the whole projective space. Moreover, it turns out to be possible to describe projective coordinate transformations in a way similar to the familiar representation of the transformations of vector coordinates, i.e. by means of matrix calculus.

1.2.1 Definition. Simplices

Now we consider an arbitrary $(n+1)$-dimensional right vector space $\boldsymbol{V}$ over K and the associated projective space $\boldsymbol{P}^n = \boldsymbol{P}(\boldsymbol{V})$. Let $(\mathfrak{a}_i)$ be a basis of $\boldsymbol{V}$, and let $(\mathfrak{e}_i)$ be the standard basis of K^{n+1}, $i = 0, \ldots, n$. Linearly extending the relations $\varphi(\mathfrak{a}_i) = \mathfrak{e}_i$, $i = 0, \ldots, n$, defines a linear isomorphism $\varphi : K^{n+1} \to \boldsymbol{V}$ of vector spaces, which in turn determines vector coordinates on $\boldsymbol{V}$. Denoting by π both the respective canonical maps the linearity of φ immediately implies

$$\tilde{\varphi}(\pi(\mathfrak{x})) = \pi(\varphi(\mathfrak{x})) \qquad (\mathfrak{x} \in \boldsymbol{V}), \tag{2}$$

i.e., the map $\tilde{\varphi} : \boldsymbol{P}^n_o \to K\boldsymbol{P}^n_o$ does not depend on the chosen representative $\mathfrak{x} \in \pi^{-1}(\boldsymbol{x})$. $\tilde{\varphi}$ is called the *homogeneous coordinate system determined by* $(\mathfrak{a}_i)$ on the projective space $\boldsymbol{P}^n$; correspondingly, the x^i defined by $\tilde{\varphi}(\boldsymbol{x}) = (x^i)K^*$ are called the *homogeneous coordinates* of $\boldsymbol{x}$ with respect to $\tilde{\varphi}$. They are determined only up to a common factor $\lambda \in K^*$. The points $\boldsymbol{a}_i := \pi(\mathfrak{a}_i)$ are called the *base points*, and $\boldsymbol{e} := \pi(\sum_{i=0}^n \mathfrak{a}_i)$ is the *unit point* of the homogeneous coordinate system. The j-th base point has homogeneous coordinates $(\delta^i_j)K^*$, where δ^i_j is the *Kronecker symbol*:

$$\delta^i_j = \begin{cases} 0 & \text{for } i \neq j, \\ 1 & \text{for } i = j, \end{cases} \quad i, j = 0, \ldots, n.$$

The coordinates of the unit point are all equal $(1, \ldots, 1)K^*$. Obviously, the sequence $(\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n; \boldsymbol{e})$ formed by the base points and the unit point of a homogeneous coordinate system is in general position, cf. Exercise 1.6. Hence any family of $k + 1$ base points of a coordinate system spans a *coordinate k-plane*:

$$\boldsymbol{H}_{i_0 \ldots i_k} = \boldsymbol{a}_{i_0} \vee \ldots \vee \boldsymbol{a}_{i_k}.$$

In general, two projective subspaces $\boldsymbol{A}$, $\boldsymbol{B}$ are called *complementary*, if $\boldsymbol{A} \wedge \boldsymbol{B} = \boldsymbol{o}$ and $\boldsymbol{A} \vee \boldsymbol{B} = \boldsymbol{P}^n$. For every coordinate k-plane $\boldsymbol{H}_{i_0 \ldots i_k}$ there is exactly one complementary coordinate plane, namely $\boldsymbol{H}_{j_1 \ldots j_l}$, where $\{j_1, \ldots, j_l\}$ is the complement of $\{i_0, \ldots, i_k\}$ in the set $\{0, \ldots, n\}$; its dimension is $n - k - 1$.

Exercise 1. Let $\boldsymbol{A}, \boldsymbol{B} \subset \boldsymbol{P}^n$ be complementary subspaces. Prove: a) For every point $\boldsymbol{z} \in \boldsymbol{P}^n \setminus (\boldsymbol{A} \cup \boldsymbol{B})$ there is exactly one line $\boldsymbol{H}$ such that $\boldsymbol{z} \in \boldsymbol{H}$, $\boldsymbol{H} \cap \boldsymbol{A} \neq \boldsymbol{o}$ and $\boldsymbol{H} \cap \boldsymbol{B} \neq \boldsymbol{o}$. – b) Setting

$$p : \boldsymbol{x} \in \boldsymbol{P}^n_o \longmapsto (\boldsymbol{x} \vee \boldsymbol{B}) \wedge \boldsymbol{A} \in \boldsymbol{A}$$

defines a surjective map $p : \boldsymbol{P}^n_o \to \boldsymbol{A}$; the equality $p(\boldsymbol{x}) = \boldsymbol{o}$ holds if and only if $\boldsymbol{x} \in \boldsymbol{B}$. – c) The map p satisfies $p^2 = p$.

The following example illustrates the situation:

Example 1. Simplices. A projectively independent sequence of $k+1$ points $(\boldsymbol{b}_0, \ldots, \boldsymbol{b}_k)$ is called a *k-dimensional simplex*, *k-simplex* for short. The points themselves are the *vertices*, and the subsequences form the *faces* of the simplex. Frequently we will also use the term face of the simplex to denote the whole projective subspace spanned by the face of the simplex as well. One-dimensional faces will also be called *edges*. For each vertex $\boldsymbol{b}_j$ of a k-simplex the $(k - 1)$-face not containing $\boldsymbol{b}_j$ is called its *opposite face* $\boldsymbol{B}_j$. The base points of a homogeneous coordinate system for $\boldsymbol{P}^n$ form the *coordinate simplex*. Denoting by $\boldsymbol{B}_j$ the opposite face of $\boldsymbol{a}_j$ the points in the hyperplane spanned by $\boldsymbol{B}_j$ are characterized by the equation $x^j = 0$. The *j-th projection* p_j of a point $\boldsymbol{x}$ onto the face $\boldsymbol{B}_j$ (from the point $\boldsymbol{a}_j$) is defined by

$$p_j : \boldsymbol{x} \in \boldsymbol{P}^n \longmapsto \boldsymbol{x}_j := \boldsymbol{B}_j \wedge (\boldsymbol{a}_j \vee \boldsymbol{x}) \in \boldsymbol{B}_j; \tag{3}$$

thus $p_j(\boldsymbol{a}_j) = \boldsymbol{o}$. The unit point in the face $\boldsymbol{B}_j$ is the image of the unit point under the corresponding projection, $\boldsymbol{e}_j := p_j(\boldsymbol{e})$. Fig. 1.7 illustrates the situation for the plane. Take the face $\boldsymbol{B}_0$ as the line at infinity, cf. Example 1.1. Its complement, the affine plane $\boldsymbol{A}^2 := \boldsymbol{P}^2 \setminus \boldsymbol{B}_0$, is characterized by $x^0 \neq 0$. The homogeneous coordinates there can be normalized so that $x^0 = 1$ holds. The coordinates x^1, x^2 of the triples normalized in this way then become the cartesian coordinates of the point $\boldsymbol{x} \in \boldsymbol{A}^2$. They can also be computed as the scale values of the projections of $\boldsymbol{x}$ onto the axes of the