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Wave Progagation, Observation and Control in $1-d$ Flexible Multi-Structures

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Preface

This book is devoted to analyze the vibrations of simplified $1 - d$ models of multi-body structures consisting of a finite number of flexible strings distributed along planar graphs.

We first discuss issues on existence and uniqueness of solutions that can be solved by standard methods (energy arguments, semigroup theory, separation of variables, transposition,...). Then we analyze how solutions propagate along the graph as the time evolves, addressing the problem of the observation of waves. Roughly, the question of observability can be formulated as follows: Can we obtain complete information on the vibrations by making measurements in one single extreme of the network? This formulation is relevant both in the context of control and inverse problems.

Using the Fourier development of solutions and techniques of Nonharmonic Fourier Analysis, we give spectral conditions that guarantee the observability property to hold in any time larger than twice the total length of the network in a suitable Hilbert space that can be characterized in terms of Fourier series by means of properly chosen weights. When the network graph is a tree, we characterize these weights in terms of the eigenvalues of the corresponding elliptic problem. The resulting weighted observability inequality allows identifying the observable energy in Sobolev terms in some particular cases. That is the case, for instance, when the network is star-shaped and the ratios of the lengths of its strings are algebraic irrational numbers.

The observation time we obtain, twice the total length of the network, is optimal. We justify the optimality in the case of a star-shaped network consisting of three strings. We construct a solution, which is the composition of waves with small support, that vanishes at the observation point in a time-interval of length smaller than twice the total length of the network.

These observability results allow us also to solve the problem of controllability, namely, that of driving solutions to rest by means of a control acting on one of the external nodes of the network, using the classical equivalence property between observability and controllability. We describe systematically the control theoretical consequences of the observability properties we have obtained, in terms of the approximate, spectral and exact controllability of networks. More precisely, we deduce sufficient conditions on the network so that a certain subspace (a dense one in the energy space) of initial data may be driven to zero in a time equal to twice the total length of the network. This subspace may be identified to be a Sobolev space under appropriate restrictions on the shape of the network and the lengths of the strings entering in it. More generally, this space may be identified by means of the Fourier series development of solutions on the basis of the eigenfunctions of the Dirichlet laplacian on the network.

The techniques developed to handle this problem and the results we obtain, allow us solving also other similar questions. In particular, the simultaneous observability problem for strings or membranes from an interior region and the control of a network from all its nodes using a small number of different control functions are studied.

Besides, we consider other models on planar networks like Schrödinger, heat or beam-type equations. Existence and uniqueness of solutions is proved in a standard way. We then address the problem of observation and control from an extreme of the network. In order to solve these problems we use various techniques based on the Fourier representation of solutions allowing to derive properties of solutions of those equations as a consequence of those on the wave equation on the same network.

Designed as an introductory course on control and observation of networks, the book contains also some advanced topics which may be of interest for researchers in this area. The last chapter of the book also includes a list of open problems and topics for future research.

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Contents

1 Introduction

In last years a considerable effort has been devoted to the mathematical study of mechanical systems constituted by coupled flexible or elastic elements as strings, beams, membranes or plates. These systems are known as multi-link or multi-body structures. Their practical relevance is huge. However, the mathematical models describing their evolution are generally quite complex. They can be viewed as systems of Partial Differential Equations (PDE) on networks or graphs.

There is an extensive literature on this topic but a lot remains to be done in order to have a complete theory. Indeed, the interaction between the different components of a multi-link structure may generate new, unexpected pehenomena. Consequently, one can not develop a full theory by simply superposing the existintg results for PDE on domains of the euclidean space. This is particularly true for what concerns control problems. The interested reader is refered to the books [91] and [5] for an introduction the theory of Partial Differential Equations on networks which is an active subject since the early 80's ([82], [83], [97]). In [63] and [68] wide information may be found on modelling and control issues. We also refer to [66] for a systematic analysis of the application of domain decomposition techniques for networks.

But, in view of the intrinsic difficulty of these models it is hard to guess what a general theory should be. It is therefore convenient to first study simplified versions of those models to later address more complex and realistic situations.

This monograph is mainly devoted to analyze the vibrations of a simplified $1-d$ model of a multi-body structure consisting of a finite number of flexible strings distributed along a planar graph. Deformations are assumed to be perpendicular to the reference plane. Though this is an extremely simple and particular model, as we shall see, the whole mathematical picture is quite complex and requires the combination and development of different techniques. We expect the analysis we perform will contribute to clarify what the relevant aspects of the problem are, and to provide some tools for the study of more complex models.

The main goal of this book is to present in a self-contained way the state of the art of the problem of propagation, observation and control of waves on these planar $1-d$ networks. As we shall see, this requires important developments related with non-harmonic Fourier series, Diophantine approximation, graph theory and wave propagation techniques.

Though the model under consideration is, to some extent, the simplest one in the context of multi-body or multi-link continuous structures, a fine analysis of the nature of the possible vibrations of these planar networks of flexible strings is far from trivial.

The main tool for analyzing the propagation of waves along the graph will be the classical d'Alembert formula, which allows solving the $1-d$ wave equation both in the space and time directions. In the model under consideration the wave equation holds along each of the strings of the network. The d'Alembert formula allows then representing the solutions on each string explicitly. However, the overall dynamics turns out to be rather complex. This is due to the interaction of the various strings at the junction points. How the energy of waves is transferred from one string to another turns out to be a global problem in which several ingredients arise:

– the lengths of the various strings constituting the graph;
– the topology of the graph;
– the boundary conditions imposed at the extremes of the graph.

The problem of observation or observability concerns, roughly speaking, the issue of determining whether one can determine the total energy of vibrations by partial measurements made for instance, in one or several interior or external nodes of the network. In other words, the property of observability is related with the distribution or propagation of vibrations along the various components of the multi-structure. This problem is relevant, not only because it is a way of analyzing deeply the nature of vibrations, but because it is also of immediate application in the context of inverse and control problems. Part of the book is also devoted to present systematically the consequences of our analysis in what concerns control problems. In particular, we shall analyze the properties of approximate, spectral and exact controllability of networks.

As we mentioned above, graph theory and Diophantine approximation issues enter in a crucial way on the analysis of the property of observability and the topology of the graph plays a fundamental role. For instance, when the graph contains closed circuits there may exist vibrations of the network that remain concentrated and trapped in that circuit, without being propagated to the rest of the network. In those cases, obviously, it is impossible to achieve the observation and/or control property if the observer or controller is not located on the circuit where the solution is trapped. But whether a circuit may support a localized vibration depends also strongly on the mutual lengths of the strings composing the circuit. When all the ratios of the lengths of these strings are rational numbers, such a localized vibration exists. However, if some of these ratios are irrational, then, necessarily, part of the energy of the vibration will be transferred to some other components of the network.

But, in order to determine the amount of energy that is actually transferred one needs to know further properties of that irrational ratio (whether it is algebraic or not, a Liouville number....) and then to apply the existing results on Diophantine approximation.

As we shall see, the overall picture is quite complex, but we hope that this monograph will succeed on describing the main phenomena one may encounter. We shall mainly focus on three cases with different degrees of complexity and such that the corresponding results are also of quite different nature:

The star. It concerns the case where a finite number of strings are connected on a single point by one of their extremes. In this case, using d'Alembert formula, one can give sharp results characterizing the space of observation and/or control in Fourier series by means of suitable weights depending on the lengths of the strings entering in the star-shaped network. We mainly discuss the most difficult case in which observation and/or control are localized in a single extreme of the network. The weights in the corresponding norms depend on the ratios of the lengths of the strings and, in particular, on its irrationality properties. The time needed for observation turns out to be simply twice the sum of all lengths of the strings of the networks.

The tree. It is well known that when all but one external node of the network are observed in a tree-like configuration, the whole energy of solutions may be observed (see [68]). This can be easily seen by an energy argument. Indeed, using sidewise energy estimates for the solutions of the wave equation, one can show that the observation inequality holds in the sharp energy space in a time which is twice the length of the longest path joining the points of the network with some of the observed ends. In this case, the observation time is much smaller than twice the total length of the network, which is needed for the observation from a single end in the case of stars.

Here we analyze the opposite case in which the observation is made at one single extreme of the tree-like network. The observation time turns out to be again, as in the case of one star, twice the sum of the lengths of the strings forming the network. At this point, it is important to note that the case of a tree is a generalization of the previous case of a star. Thus, for the observability property to hold one has also to generalize the condition on the irrationality of the ratios of the lengths of the strings arising in the case of the stars. To do that it is important to observe that the fact of two strings having mutually irrational lengths can also be interpreted in spectral terms. Indeed, it means that the spectra of the two strings have empty intersection. The latter condition turns out to be the appropriate one to be extended to general trees. In this way, the tree turns out to be observable from one end if and only if the spectra of all pairs of subtrees of the tree that match on a nodal point are disjoint. Obviously, this property is also related to the values of the lengths of the strings composing the tree, but does not have an easy interpretation as in the case of the star. Nevertheless, as we shall see, generically, trees satisfy this property.

General networks. The propagation techniques we have employed in the analysis of stars and trees are hard to apply in the case of a network supported by a general graph. Indeed, in the general case we lack of a natural ordering on the graph to analyze the propagation of waves. Actually, as we mentioned above, the presence of closed circuits may trap the waves. Thus, we proceed in a different way by applying a consequence of the celebrated Beurling-Malliavin's Theorem on the completeness of families of real exponentials obtained by Haraux and Jaffard in [50] when analyzing the control of plates. Using the min-max principle, one can show that the spectral density of a general graph is the same as that of a single string whose length is the sum of the lengths of all the strings entering in the network. Then, when the time is greater than twice the total length, as a consequence of Beurling-Malliavin's Theorem, we deduce that there exist some Fourier weights so that the observation property holds in the corresponding weighted norm if and only if all the eigenfunctions of the network are observable. So far we do not know of any necessary and sufficient condition guaranteeing that all the eigenfunctions are observable in the general case. However, this condition, in the particular case of stars and trees discussed above turns out to be sharp: the lengths of the strings are mutually irrational in the case of stars or the spectra of all pairs of subtrees with a common end-point are mutually disjoint in the more general case of trees.

In view of this last result on general networks, the material in this monograph could have been presented in a completely different order. Indeed, we could have started from the most general results on the case of general networks using Beurling-Malliavin's Theorem to later discuss the particular cases of stars and trees using d'Alembert formula and Diophantine approximation, in which general results can be more easily interpreted. However, we have preferred to do all the way around. This corresponds actually to the order and chronology in which the progress was done in the field, starting from the work [75] on the case of a star composed of three strings and continuing with the series of Notes [34, 35, 36, 37].

We became interested on this subject along several discussions with Günter Leugering on this subject and his book in collaboration with Lagnese and Schmidt [68], together with the previously quoted references on PDE on networks, were a great help to start. As we said before, the model we consider in this monograph is the simplest one in the context of vibrations of networks. The interested reader is referred to [68] where many other models can be found with a description of the state of the art in what concerns the well-posedness of the initial boundary problems and the observation and/or control problems for networks of strings, beams, membranes and plates.

Before getting into the analysis of the star we discuss a simpler issue that, nevertheless, allows presenting some of the main difficulties of the theory. It concerns the simultaneous control of two strings connected at one end-point (which is in fact completely equivalent to the problem of controlling one single string from one interior point). In this case we already see the necessity that both strings have mutually irrational lengths. Moreover, we also see that the time needed to control the strings is twice the sum of the lengths of both strings for the observability property to hold. This seems to contradict a first intuition that would suggest that the time needed to control both strings simultaneously should be twice the maximum of the lengths of the strings, i.e., $2\max(\ell_1, \ell_2)$, instead of $2(\ell_1+\ell_2)$. But, in fact, the time $2(\ell_1+\ell_2)$ turns out to be sharp under the assumption that the ratio ℓ_1/ℓ_2 is irrational. In other words, even when ℓ_1/ℓ_2 is irrational, the time needed to control simultaneously the two strings together by means of the same control is $2(\ell_1+\ell_2)$, which is strictly greater than the time needed to control each string independently with two different controls that would be $2\max(\ell_1, \ell_2)$.

It is interesting to analyze the relation of this result with the so-called Geometric Control Condition (GCC) introduced by Bardos, Lebeau and Rauch [18] in the context of the boundary observation and/or control of the wave equation in bounded domains of $\mathbb{R}^n$. The GCC requires that all the rays of Geometric Optics enter the observation region in a finite, uniform time, which turns out to be the minimal one for observation/control. In the case of two strings observed from one common end or the equivalent problem of the string controlled at an interior point, in view of GCC, one could expect the sharp time needed for observation/control to be $2\max(\ell_1, \ell_2)$. But this is not the case, the fact that the rays pass once by the point of observation does not guarantee that the energy concentrated on that ray will be conveniently observed[1]. In fact, we need the ray to pass once more through the point of observation to be able to make a full measurement of the solution. This yields the control/observation time $2(\ell_1+\ell_2)$. But, in fact, passing twice by the observation point is not sufficient either. The irrationality of the ratio ℓ_1/ℓ_2 is needed to guarantee that, when passing through the observation point the second time, the solution is not exactly at the configuration as in the first crossing, which, of course, would make the second observation to be insufficient too. Finally, even when ℓ_1/ℓ_2 is irrational, we cannot get a uniform bound of the energy of the solution but rather a weaker measurement in a weaker norm. The nature of this norm, which is represented in Fourier series by means of some weights depending on ℓ_1/ℓ_2, depends very strongly on the irrationality class to which the number ℓ_1/ℓ_2 belongs. In fact, in the most favourable case, i.e., when ℓ_1/ℓ_2 is an algebraic number of degree two, one looses one derivative of the solution which, in Sobolev terms means that, for instance, an H^1 observation in time yields only control of the L^2-norm of

[1] The wave equation is a second order problem and therefore, even in $1-d$, for a pointwise observation mechanism to be efficient we need to measure not only the position, but also the space derivative. This implies that a necessary condition for observation/control is that all waves pass twice through the observation point.

the solution. In other more pathological cases, like when ℓ_1/ℓ_2 is a Liouville number, one may loose an infinite number of derivatives in the sense that the weights entering in the Fourier representation of the observed norm may have an exponential decay at high frequencies.

We have so far described the content of the main body of the monograph: the propagation, observation and control of waves on stars, trees and general planar networks. But these are only a few of the problems arising in this context. We have complemented this material with the discussion of two important closely related problems:

– The simultaneous observation/control of two strings from a common subinterval. In this case one obtains better results than in the case when the observer/controller is located at a single point [2]. Indeed, this time the results do hold in the sharp energy space without any loss of derivatives. This fact confirms that controlling on an open subinterval is a much more robust mechanism than controlling at a single point.

– The observation/control of general networks through all the nodal points. This is a problem of relevance in applications. From a technological point of view, putting observers/controllers at all the nodal points is feasible. However, one would like to know, for instance, if the number of applied control forces may be reduced by identifying a priori the nodes on which the same force may be applied. This is necessary in order to diminish the complexity of the applied control mechanism. Thus, we would like to know how many different control forces are needed to control the whole structure and to identify the nodes on which each control should be applied. We shall see that the total number of controls needed is four and this is a consequence of our previous analysis and the celebrated Four Colors Theorem.

So far, we have only discussed the wave equation on planar networks of strings. But of course, the same issues arise for all other models like beams, Schrödinger or heat equations. The theory of observation and control of Partial Differential Equations in open domains of $\mathbb{R}^n$ is by now quite well developed (we refer to the survey articles [121] and [123] for an updated account of the developments in this field). However, very little is known in the context of PDE's on networks.

The last part of this monograph is devoted to discuss those three models. Roughly speaking, we show that the results proved in the previous sections on the wave equation yield similar results for those three models. To do that we employ two different results. In the case of the heat equation on the network, we use a classical result by Russell [105] guaranteeing that, whenever the wave equation is controllable in some time, then the heat equation is controllable in an arbitrarily small time. The results of this monograph on the observation and/or control of the wave equation on the network then immediately imply similar results on the corresponding heat model. In what concerns

[2] According to the analysis in [42] the problem of pointwise control may be viewed as a singular limit of that of controlling in a subinterval shrinking to that point.

the Schödinger and beams models we use the fact that the time frequencies of the complex exponentials involved in the Fourier representation of solutions of these two models are the squares of those entering in the solutions of the wave equation. Thus, the gap between consecutive eigenfrequencies increases. This allows obtaining observability inequalities for Schrödinger and beam equations from the Fourier representation of those previously obtained for the wave equation. But, this time, as expected, due to the infinite speed of propagation, the observability inequalities hold in an arbitrarily small time.

As we have already mentioned this monograph collects the existing results on simple $1-d$ models on networks. Much remains to be done in this field. At the end of this book we include a list of open problems and possible subjects of future research. We hope this book to attract the attention to this challenging field of research.

For those who will address these topics for the first time, we refer to [84] for an introduction to some of the most elementary tools on the controllability of PDE's and to the survey articles [121] and [123], for a description of the state of the art in this field.

Finally, some comments on the notations used along this book are in order. The numbering of objects is made locally in each chapter. The sections, subsections, theorems, lemmas, formulas, etc., have a first number to indicate the chapter in which they appear. Thus, Proposition 3.4, is the fourth proposition of Chapter 3. Concerning the constants, they all have been denoted by C. Thus, C may stand for numbers that are different from line to line of the text, but that remain uniform with respect to the relevant parameters. Only when we intend to explicitly indicate the dependence of C on some parameter, or to avoid ambiguities, we use some more complete notations.

We would like to emphasize that the book is mainly self-contained and that it has been designed as an introductory course to the controllability and observability of networks for graduate students. The text may be covered in the order presented or, if a simplified approach is desired, it is possible to restrict oneself to Chapters 2, 3, 4 and 8, as the remaining chapters are more technical. However, many other variants are also possible.

2

Preliminaries

2.1 The Elastic String

Let us start with a simple example. Consider an elastic string of length one which is fixed at its ends. The deformation of the string is given by the function $\phi(t,x) : \mathbb{R} \times (0,1) \to \mathbb{R}$ which is the unique solution of the wave equation

$$\begin{array}{ll} \phi_{tt} - \phi_{xx} = 0 & \text{in } \mathbb{R} \times (0,1), \\ \phi(t,0) = \phi(t,1) = 0 & \text{in } \mathbb{R}, \\ \phi(0,x) = \phi_0(x), \quad \phi_x(0,x) = \phi_1(x) & \text{in } (0,1), \end{array} \tag{2.1}$$

where ϕ_0 and ϕ_1 are the initial deformation and velocity of the string, respectively.

The solution of system (2.1) may be expressed by the Fourier formula

$$\phi(t,x) = \sum_{n=1}^{\infty} (a_n \cos n\pi t + \frac{b_n}{n\pi} \sin n\pi t) \sin n\pi x, \tag{2.2}$$

where (a_n) and (b_n) are the sequences of Fourier coefficients in the orthogonal basis of $L^2(0,1)$:

$$\theta_n(x) = \sin n\pi x, \quad n = 1, 2, ...$$

The energy of the solution ϕ is defined as

$$E_\phi(\phi_0, \phi_1, t) = \frac{1}{2} \int_0^1 \left(|\phi_x(t,x)|^2 + |\phi_t(t,x)|^2 \right).$$

It is easy to prove that the energy of a solution is constant[1], that is $E_\phi(t) = E_\phi(0)$. The energy is a norm in the space $H_0^1(0,1) \times L^2(0,1)$ of initial states

[1] This can be done computing directly on the Fourier representation of the solution or, by the energy method, i.e. multiplying the wave equation by ϕ_t and integrating with repesct to x. After integration by parts this yields $dE(t)/dt = 0$.

of (2.1) and may be expressed in terms of the Fourier coefficients (a_n) and (b_n) as

$$E_\phi(\phi_0, \phi_1) = \frac{1}{4}\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2). \tag{2.3}$$

Assume now that we observe the motion of the string at one of its points. To fix ideas, suppose we know the values of the velocity ϕ_t and the tension ϕ_x at some point $x = \xi$ in a time interval $(0, T)$. Let us define the *observation function*

$$\varPhi(\phi_0, \phi_1, \xi, T) = \frac{1}{4}\int_0^T |\phi_t(t, \xi)|^2 dt + \frac{1}{4}\int_0^T |\phi_x(t, \xi)|^2 dt.$$

Let us note that for $T = 2M$ with $M \in \mathbb{N}$ it holds

$$\varPhi(\phi_0, \phi_1, \xi, T) = ME_\phi(\phi_0, \phi_1). \tag{2.4}$$

Indeed, from the formula (2.2) we have

$$\phi_t(t, \xi) = \sum_{n=1}^{\infty}(-n\pi a_n \sin n\pi t + b_n \cos n\pi t)\sin n\pi\xi,$$

$$\phi_x(t, \xi) = \sum_{n=1}^{\infty}(n\pi a_n \cos n\pi t + b_n \sin n\pi t)\cos n\pi\xi$$

and then, in view of the 2-periodicity of the functions $\sin n\pi t$ and $\cos n\pi t$ and their orthogonality properties,

$$\int_0^{2M} |\phi_t(t, \xi)|^2 dt = M\int_0^2 |\phi_t(t, \xi)|^2 dt = M\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\sin^2 n\pi\xi, \tag{2.5}$$

$$\int_0^{2M} |\phi_x(t, \xi)|^2 dt = M\int_0^2 |\phi_x(t, \xi)|^2 dt = M\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\cos^2 n\pi\xi. \tag{2.6}$$

Therefore, in view of (2.3), (2.5) and (2.6) we obtain (2.4).

Clearly, the function $\varPhi(\phi_0, \phi_1, \xi, T)$ is increasing in T, so, if $2 \leq T \leq 2M$ with $M \in \mathbb{N}$ we obtain

$$\varPhi(\phi_0, \phi_1, \xi, 2) \leq \varPhi(\phi_0, \phi_1, \xi, T) \leq \varPhi(\phi_0, \phi_1, \xi, 2M),$$

or equivalently,

$$E_\phi(\phi_0, \phi_1) \leq \varPhi(\phi_0, \phi_1, \xi, T) \leq ME_\phi(\phi_0, \phi_1).$$

That means that, for all $\xi \in [0, 1]$ and $T \geq 2$, the norms defined by E_ϕ and $\varPhi(\cdot, \xi, T)$ are equivalent. That is, it is possible to estimate the energy of the solution ϕ from the measurements of ϕ_t, ϕ_x made at point ξ during a time interval of length at least two. In particular, when $T = 2$ those two norms coincide:

$$E_\phi(\phi_0, \phi_1) = \Phi(\phi_0, \phi_1, \xi, 2).$$

When $\xi = 0$ or $\xi = 1$, the observation function Φ is simpler. For instance, for $x = 0$ it becomes

$$\Phi(\phi_0, \phi_1, 0, T) = \frac{1}{4}\int_0^T |\phi_x(t, 0)|^2 dt,$$

since $\phi_t(t, 0) \equiv 0$.

Accordingly, at the boundary points, the observation of the tension ϕ_x of the string during a time-interval of length twice the length of the string, suffices to fully recover the total energy of the vibration.

It is natural to raise the question of whether the same happens at the internal observation points ξ. Accordingly, consider a weaker observation function:

$$\Psi(\phi_0, \phi_1, \xi, T) = \frac{1}{4}\int_0^T |\phi_x(t, \xi)|^2 dt.$$

We already know that, when $\xi = 0$ or $\xi = 1$ this function defines a norm in the space of initial data, equivalent to the energy-norm. The following questions arise naturally: *does the function Ψ define a norm in $H_0^1(0,1) \times L^2(0,1)$? If so, is that norm equivalent to the energy?*

Assume $T = 2$, then in view of (2.6) it holds

$$\Psi(\phi_0, \phi_1, \xi, 2) = \frac{1}{4}\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\cos^2 n\pi\xi. \tag{2.7}$$

Formula (2.7) is very similar to (2.3) and, clearly,

$$\sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2)\cos^2 n\pi\xi \le \sum_{n=1}^{\infty}(n^2\pi^2 a_n^2 + b_n^2),$$

and then

$$\Psi(\phi_0, \phi_1, \xi, 2) \le E_\phi(\phi_0, \phi_1).$$

However, the converse inequality is not true whatever $\xi \in (0,1)$ is. Indeed, the converse inequality would require a lower bound of the form

$$|\cos n\pi\xi| \ge C, \tag{2.8}$$

for every $n \in \mathbb{N}$. But this inequality is false for all $\xi \in (0,1)$. Indeed, if ξ is a rational number that can be expressed as

$$\xi = \frac{2p+1}{2q}, \qquad p, q \in \mathbb{Z}, \tag{2.9}$$

then, when $n = qk$ with k odd

$$\cos n\pi\xi = \cos\frac{(2p+1)\,k}{2}\pi = 0.$$

Thus, in this case, $\cos n\pi\xi = 0$ for an infinite number of values of n and consequently, inequality (2.8) cannot be true. That means that the function $\Psi(\cdot, \xi, 2)$ is not even a norm in $H_0^1(0,1) \times L^2(0,1)$.

On the other hand, when the number ξ cannot be expressed in the form (2.9) all the numbers $\cos n\pi\xi$ are different from zero. This implies that the function $\Psi(\cdot, \xi, 2)$ does define a norm in $H_0^1(0,1) \times L^2(0,1)$. But this norm is necessarily weaker than the energy.

In fact, inequality (2.8) is equivalent to the existence of a positive number α such that, for all $k, n \in \mathbb{Z}$,

$$\left| n\pi\xi - \frac{2k+1}{2}\pi \right| \geq \alpha.$$

That is

$$|(2\xi)\,n - (2k+1)| \geq \alpha_0 := \frac{2\alpha}{\pi}.$$

This rational approximation property of the number 2ξ is false for all $\xi \in (0,1)$. We will discuss this issue in detail in Chapter 3.

But, for certain values of ξ weaker inequalities may be obtained. Indeed, for instance, if 2ξ may be expanded in continuous fraction $[0, c_1, c_2, ...]$ with bounded sequence (c_n) then there exists a constant C_ξ such that

$$|(2\xi)\,n - (2k+1)| \geq C_\xi / n,$$

and this is the best lower bound one may expect. This implies that

$$|\cos n\pi\xi| \geq C_\xi / n$$

and therefore

$$\Psi(\phi_0, \phi_1, \xi, 2) \geq C_\xi \sum_{n=1}^{\infty} (a_n^2 + \frac{b_n^2}{n^2\pi^2}) = C_\xi ||\phi_0||^2_{L^2(0,1)} + ||\phi_1||^2_{H^{-1}(0,1)}.$$

Summarizing, for the values of ξ indicated above, it holds

$$C_\xi \left(||\phi_0||^2_{L^2(0,1)} + ||\phi_1||^2_{H^{-1}(0,1)} \right) \leq \Psi(\phi_0, \phi_1, \xi, 2) \leq ||\phi_0||^2_{H_0^1(0,1)} + ||\phi_1||^2_{L^2(0,1)}.$$

This is the best result we may obtain. Accordingly, for interior points $\xi \in (0,1)$, the information contained in $\Psi(\phi_0, \phi_1, \xi, 2)$ does not suffice to recover the whole energy of the string and only weaker norms may be recovered (the $[L^2(0,1) \times H^{-1}(0,1)]$-nom in the particular case above with a loss of one derivative in $L^2(0,1)$, both for ϕ and ϕ_t). This is also the case when considering other kind of observation functions, e.g.,

$$\int_0^T |\phi(t,\xi)|^2 dt.$$

As we shall see in the following chapters, this is the typical situation when addressing the problem of observability for the vibrations of a network of strings. Typicallly, one can recover only weaker energies from measurements made at some points of the strings, even if at those points both the velocity and the tension are measured[2].

When the observation is made on a larger set, say on some interval $\omega \subset (0,1)$, then the total energy can be recovered. Indeed, consider the observation function

$$\int_0^T \int_\omega |\phi_x(t,x)|^2 dxdt.$$

Assume that $T = 2$. Then

$$\begin{aligned}\int_0^2 \int_\omega |\phi_x(t,x)|^2 dxdt &= \int_\omega \int_0^2 |\phi_x(t,x)|^2 dtdx \\ &\geq \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2) \int_\omega \sin^2 n\pi x \; dx.\end{aligned} \tag{2.10}$$

But, for any $\omega \subset (0,1)$ there exists a constant $C_\omega > 0$ such that

$$\int_\omega \sin^2 n\pi x \; dx \geq C_\omega$$

for every $n \in \mathbb{N}$. Therefore,

$$C_\omega \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2) \leq \int_0^2 \int_\omega |\phi_x(t,x)|^2 dxdt \leq |\omega| \sum_{n=1}^{\infty} (n^2\pi^2 a_n^2 + b_n^2),$$

that is

$$4C_\omega E_\phi \leq \int_0^2 \int_\omega |\phi_x(t,x)|^2 dxdt \leq 4|\omega| E_\phi.$$

Using the d'Alembert formula for the representation of the solutions of the wave equation, one may improve the estimate above on the time needed for this estimate to be true. Namely, the property

$$C_1 E_\phi \leq \int_0^T \int_\omega |\phi_x(t,x)|^2 dxdt \leq C_2 E_\phi,$$

holds for any $T > 2\text{dist}\{\omega, \{0,1\}\}$, for some positive constants C_1 and C_2. The time $2\text{dist}\{\omega, \{0,1\}\}$ is actually the characteristic one and it is in agreement

[2] As we shall see, there is a case in which this does not happen and the whole energy may be recovered: For tree-like networks when the tension is measured in all the external nodes except at most one.

with the Geometric Control Condition (GCC) mentioned in the introduction that indicates that, for the observability inequality to hold, all rays should enter the observation region in the given observation time.

But for networks of strings, observing on a subinterval of one of the strings will not help. This allows recovering the information on the string where the observation is being made but will only yield weaker measurements on the other ones.

2.2 Networks of Strings

2.2.1 Elements on Graphs

A graph G is a pair $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is a set, whose elements are called *vertices* of G, and $\mathcal{E}$ is a family of non-ordered pairs $\mathbf{v}, \mathbf{w}$ of vertices, which we will denote by $\widehat{\mathbf{vw}}$. The elements of $\mathcal{E}$ are called *edges* of G with vertices $\mathbf{v}, \mathbf{w}$. When the graph G does not contain edges of the form $\widehat{\mathbf{vv}}$ it is said that the graph is *simple*[3].

A *path* between the vertices $\mathbf{v}$ and $\mathbf{w}$ of a graph G is a set of edges of the form

$$\widehat{\mathbf{v}\mathbf{v}_1}, \widehat{\mathbf{v}_1\mathbf{v}_2}, ..., \widehat{\mathbf{v}_{m-1}\mathbf{v}_m}, \widehat{\mathbf{v}_m\mathbf{w}}.$$

If all the edges forming a path are different, it is said that the path is *simple*; if all the vertices $\mathbf{v}_1, ..., \mathbf{v}_m$ are different, the path is called *elementary*.

A *closed path* is a path between a vertex and itself. An elementary closed path is called a *cycle*. When the graph G does not contain cycles it is said that G is a *tree*.

Graphs with a finite number of vertices are called *finite*. In this book we shall be concerned only with finite graphs.

Let us suppose that G is a finite graph with N vertices and M edges:

$$\mathcal{V} = \{\mathbf{v}_1, ..., \mathbf{v}_N\}, \qquad \mathcal{E} = \{\mathbf{e}_1, ..., \mathbf{e}_M\}.$$

The multiplicity $m(\mathbf{v})$ of the vertex $\mathbf{v}$ is the number of edges that meet at $\mathbf{v}$:

$$m(\mathbf{v}) := \mathrm{card}\{\mathbf{e} \in \mathcal{E}: \quad \mathbf{v} \in \mathbf{e}\}.$$

We also define the sets

$$\mathcal{V}_{\mathcal{S}} := \{\mathbf{v} \in \mathcal{V}: \quad m(\mathbf{v}) = 1\}, \quad \mathcal{V}_{\mathcal{M}} := \mathcal{V} \setminus \mathcal{V}_{\mathcal{S}},$$

where $\mathcal{V}_{\mathcal{S}}$ is the set of those vertices that belong to a single edge, the *exterior* ones, while $\mathcal{V}_{\mathcal{M}}$ contains the remaining vertices, the *interior* ones, i.e., those that belong to more than one edge.

[3] Sometimes the term graph is used only for simple graphs, that is, for those that do not have edges with equal vertices. Non-simple graphs are then called *pseudographs*.

For a vertex $\mathbf{v}$ we denote by

$$I_{\mathbf{v}} := \{i : \mathbf{v} \in \mathbf{e}_i\},$$

the set of indices of all those edges of G which are incident to $\mathbf{v}$. If the vertex $\mathbf{v}_j$ is exterior, $I_{\mathbf{v}_j}$ contains a single index; it will be denoted by $i(j)$ and, if this does not lead to misunderstanding, simply by i.

In this book we consider only simple finite graphs whose vertices are points of a plane. The edges of the graph are viewed as rectilinear segments joining some of those points. The length of the segment corresponding to the edge $\mathbf{e}_i$ is called length of $\mathbf{e}_i$ and is denoted by ℓ_i.

We will also assume that the edges of the graphs may meet only at the vertices of G. Such graphs are known as *planar graphs*.

On every edge of G we choose an orientation (that is, one of the vertices has been chosen as the initial one). Then $\mathbf{e}_i$ may be parametrized as a function of its arc length by means of the functions $x_i : [0, \ell_i] \to \mathbf{e}_i$.

We define the incidence matrix of G

$$\varepsilon_{ij} = \begin{cases} \text{-1} & \text{if } x_i(0) = \mathbf{v}_j, \\ +1 & \text{if } x_i(\ell_i) = \mathbf{v}_j. \end{cases}$$

Let us denote by L the sum of the lengths of all the edges of the graphs, the *length of the graph*. To indicate to which graph it corresponds, we shall write, if necessary, L_G.

Given functions $u^i : [0, \ell_i] \to \mathbb{R}$, $i = 1, ..., M$, we will denote by $\bar{u} : G \to \mathbb{R}$ the function defined for $\mathbf{x} \in \mathbf{e}_i$ by

$$\bar{u}(\mathbf{x}) = u^i(x_i^{-1}(\mathbf{x})).$$

In this case, we will say that $\bar{u}$ is a function defined on the graph G with components u^i. Frequently, we will indicate this fact just by writing $\bar{u} = (u^1, ..., u^M)$. In particular, the vector with null components will be denoted by $\bar{0}$.

2.2.2 Equations of Motion for Networks

Now we consider a planar network of elastic strings that undergo small perpendicular vibrations. At rest, the network coincides with a planar graph G contained in that plane.

Let us suppose that the function $u^i = u^i(t, x) : \mathbb{R} \times [0, \ell_i] \to \mathbb{R}$, describes the transversal displacement in time t of the string that coincides at rest with the edge $\mathbf{e}_i$. Then, for every $t \in \mathbb{R}$, the functions u^i, $i = 1, ..., M$, define a function $\bar{u}(t)$ on G with components $u^i : \mathbb{R} \times [0, \ell_i] \to \mathbb{R}$ given by $u^i(t, x) = u^i(t, x_i(x))$. This function allows to identify the network with its rest graph; in this sense, the vertices of G will be called nodes and the vertices, strings.

As a model of the motion of the network we assume that the displacements u^i satisfy the following non-homogeneous system