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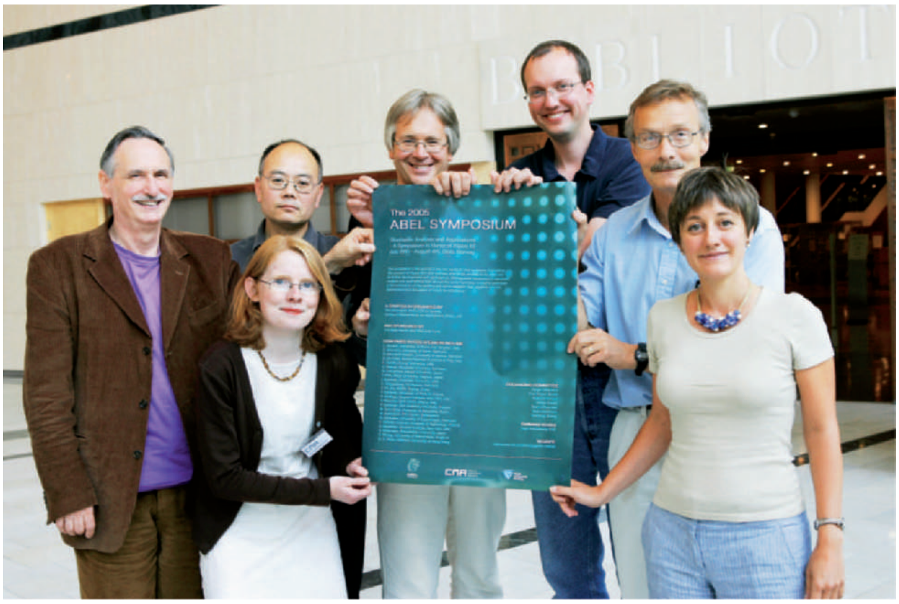

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Editors

Stochastic Analysis and Applications

The Abel Symposium 2005

Proceedings of the Second Abel Symposium, Oslo,
July 29 – August 4, 2005, held in honor of Kiyosi Itô

ABEL PRISEN
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Editors

Fred Espen Benth
Centre of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P. O. Box 1053 Blindern
0316 Oslo, Norway
e-mail: fredb@math.uio.no

Giulia Di Nunno
Centre of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P. O. Box 1053 Blindern
0316 Oslo, Norway
e-mail: giulian@math.uio.no

Tom Lindstrøm
Centre of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P. O. Box 1053 Blindern
0316 Oslo, Norway
e-mail: lindstro@math.uio.no

Bernt Øksendal
Centre of Mathematics for Applications (CMA)
Department of Mathematics
University of Oslo
P. O. Box 1053 Blindern
0316 Oslo, Norway
e-mail: oksendal@math.uio.no

Tusheng Zhang
Department of Mathematics
University of Manchester
Oxford Road
Manchester M13 9PL
United Kingdom
e-mail: tzhang@maths.man.ac.uk

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Preface to the Series

The Niels Henrik Abel Memorial Fund was established by the Norwegian government on January 1, 2002. The main objective is to honor the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics. The prize shall contribute towards raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective the board of the Abel fund has decided to finance one or two Abel Symposia each year. The topic may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level, and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The board of the Niels Henrik Abel Memorial Fund is confident that the series will be a valuable contribution to the mathematical literature.

Ragnar Winther
Chairman of the board of the Niels Henrik Abel Memorial Fund

Kiyosi Itô, at his residence in Kyoto, Japan, April 2005

Photo Credit: Søren-Aksel Sørensen

Preface

Kiyosi Itô, the founder of stochastic calculus, is one of the few central figures of the twentieth century mathematics who reshaped the mathematical world. Today stochastic calculus, also called Itô calculus, is a central research field in mathematics, with branches to several other mathematical disciplines and with many areas of application, for example physics, engineering and biology. Perhaps the most spectacular field of applications at present is economics and finance. Indeed, the Nobel Prize in Economics in 1997 was awarded to Robert Merton and Myron Scholes for their derivation of the celebrated Black–Scholes option pricing formula using stochastic calculus.

The Abel Symposium 2005 took place in Oslo, July 29th – August 4th 2005, and was organized as a tribute to Kiyosi Itô and his works on the occasion of his 90th birthday.

Distinguished researchers from all over the world were invited to present the newest developments within the exciting and fast growing field of stochastic calculus. We were happy that so many took part to this event. They were, in alphabetical order,

- Luigi Accardi, Universita' di Roma "Tor Vergata", Italy
- Sergio Albeverio, University of Bonn, Germany
- Ole E. Barndorff-Nielsen, University of Aarhus, Denmark
- Giuseppe Da Prato, Scuola Normale Superiore di Pisa, Italy
- Eugene B. Dynkin, Cornell University, USA
- David Elworthy, University of Warwick, UK
- Hans Föllmer, Humboldt University, Germany
- Masatoshi Fukushima, Kansai University, Japan
- Takeyuki Hida, Meijo University, Nagoya, Japan
- Yaozhong Hu, University of Kansas, USA
- Ioannis Karatzas, Columbia University, USA
- Claudia Klüppelberg, Technical University of Munich, Germany
- Torbjörn Kolsrud, KTH, Sweden
- Paul Malliavin, University of Paris VI, France

- Henry P. McKean, Courant Institute of Mathematical Science, New York, USA
- Shige Peng, Shandong University, China
- Yuri A. Rozanov, CNR, Milano, Italy
- Paavo Salminen, Åbo Akademy University, Finland
- Marta Sanz-Solé, Univeristy of Barcelona, Spain
- Martin Schweizer, ETH Zürich, Switzerland
- Michael Sørensen, Univeristy of Copenhagen, Denmark
- Esko Valkeila, Helsinki University of Technology, Finland
- Srinivasa Varadhan, Courant Institute of Mathematical Science, New York, USA
- Shinzo Watanabe, Ritsumeikan University, Japan
- Tusheng Zhang, University of Manchester, England
- Xianyin Zhou, Chinese University of Hong Kong

In addition there were many other international experts both attending and presenting valuable contributions to the conference. We are grateful to all for making this symposium so successful.

The present volume combines both papers from the invited speakers and contributions by the presenting lecturers. We are happy that so many sent their papers for publication in this proceedings, making it a valuable account of the research frontiers in stochastic analysis. Our gratitude is also directed to all the referees that put time and effort in reading the manuscripts.

A special feature of this volume is given by the Memoirs that Kiyosi Itô himself wrote for this occasion. We all thank him for these valuable pages which mean so much to both young and established researchers in the field.

We also thank the Abel Foundation, through the Norwegian Mathematical Society, and the Centre of Mathematics for Applications (CMA) at the University of Oslo, for their financial support and for their help with the preparation and organization of the symposium. Our special thanks go to Inga Bårdshaug Eide and Helge Galdal for their help with all practical matters before and during the conference.

Last but not least we are indebted to Sergio Albeverio for his scientific advice in the organization of the program.

Oslo, November 2006

Fred Espen Benth
Giulia Di Nunno
Tom Lindstrøm
Bernt Øksendal
Tusheng Zhang

Contents

Memoirs of My Research on Stochastic Analysis

Kiyosi Itô

Professor Emeritus, Kyoto University, Kyoto, 606-8501 Japan

It is with great honor that I learned of the 2005 Oslo Symposium on Stochastic Analysis and Applications, which is devoted to my work and its further developments. I would like to thank the symposium organizers for their tireless efforts in organizing this successful symposium and for providing me with the opportunity to present some memoirs of my research on stochastic analysis, which, I hope, will be of some interest to the participants.

My doctoral thesis published in 1942 [1] was on a decomposition of the sample path of the continuous time stochastic process with independent increments, now called the Lévy–Itô decomposition of the Lévy process. In the 1942 article written in Japanese [2] and the extended 1951 version that appeared in the Memoirs of the American Mathematical Society [3], I succeeded in unifying Lévy's view on stochastic processes and Kolmogorov's approach to Markov processes and created the theory of stochastic differential equations and the related stochastic calculus. As beautifully presented in a recent book by Daniel Stroock [11], *Markov Processes from K. Itô's Perspective,* my conception behind those works was to take, in a certain sense, a Lévy process as a tangent to the Markov process. The above mentioned papers are reprinted in *Kiyosi Itô Selected Papers* edited by Stroock and Varadhan [8], where the editors' introduction and my own foreword explain in some detail the circumstances leading to their development.

From 1954 to 1956, I was a Fellow at the Institute for Advanced Study at Princeton University, where Salomon Bochner and William Feller, both great mathematicians, were among the faculty members. In the preceding year, while still at Kyoto University, I had written a paper on stationary random distributions [4], using a Laurent Schwartz's extension of Bochner's theorem to a positive definite distribution representing it by a slowly increasing measure. As I learned from Bochner in Princeton, this had essentially already been obtained by Bochner himself by other means.

Feller had just finished his works on the most general one-dimensional diffusion process, especially representing its local generator as

$$\mathcal{G} = \frac{d}{dm}\frac{d}{ds}$$

by means of a canonical scale function s and a speed measure m. I learned about these from Henry McKean, a graduate student of Feller, while I explained my previous work to McKean. There was once an occasion when McKean tried to explain to Feller my work on the stochastic differential equations along with the above mentioned idea of tangent. It seemed to me that Feller did not fully understand its significance, but when I explained Lévy's local time to Feller, he immediately appreciated its relevance to the study of the one-dimensional diffusion. Indeed, Feller later gave us a conjecture that the Brownian motion on $[0, \infty)$ with an elastic boundary condition could be constructed from the reflecting barrier Brownian motion by killing its local time $\mathbf{t}(t, 0)$ at the origin by an independent exponentially distributed random time, which was eventually substantiated in my joint paper with McKean [9] published in 1963 in the Illinois Journal of Mathematics.

After my return to Kyoto from Princeton, McKean visited Kyoto in 1957–1958, and our intensive collaboration continued until our joint book *Diffusion Processes and Their Sample Paths* appeared from Springer in 1965 [10]. This coincides with the period when Dynkin and Hunt formulated the general theory of strong Markov processes along with their transformations by additive functionals and the associated probabilistic potential theory. The Kyoto probability seminars attracted many young probabilists in Japan; S. Watanabe, H. Kunita and M. Fukushima were among my graduate students. The primary concern of the seminar participants including myself was to fully understand the success of the study of one-dimensional diffusions and to look for its significant extensions to more general Markov processes. Let me mention some of the later developments of a different character that grew out of this exciting seminar atmosphere.

A popular saying by Feller goes as follows: A one-dimensional diffusion traveler X_t makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m appearing in the generator $\mathcal{G}$ of X_t. This was substantiated in my joint book with McKean in the following fashion. Given a one-dimensional standard Brownian motion X_t which corresponds to $ds = dx,\ dm = 2dx$, consider its local time $\mathbf{t}(t, x)$ at $x \in R^1$ and the additive functional defined by

$$A_t = \int_{R^1} \mathbf{t}(t, x) m(dx).$$

Then the time changed process X_{τ_t} by means of the inverse τ_t of A_t turns out to be the diffusion governed by the generator $\dfrac{d^2}{dmdx}$.

Observe that the transition function of the one-dimensional diffusion is symmetric with respect to the speed measure m and the associated Dirichlet form

$$\mathcal{E}(u,v) = -\int_{R^1} u \cdot \mathcal{G}v(x)dm(x) = \int_{R^1} \frac{du}{ds}\frac{dv}{ds}ds$$

is expressed only by the scale s, being separated from the symmetrizing measure m. Hence we are tempted to conjecture that the 0-order Dirichlet form $\mathcal{E}$ indicates the road map for the associated Markov process X_t and is invariant under the change of the symmetrizing measures m corresponding to the random time changes by means of the positive continuous additive functionals of X_t. The notion of the Dirichlet form was introduced by Beurling and Deny as a function space framework of an axiomatic potential theory in 1959, where already the road map was clearly indicated in analytical terms (the Beurling–Deny formula of the form) but the role of the symmetrizing measure m was much less clear. Being led by the above-mentioned picture of the one-dimensional diffusion path, the conjecture has been affirmatively resolved in later works by Fukushima and others (see the 1994 book by Fukushima, Takeda and Oshima *Dirichlet Forms and Symmetric Markov Processes,* [12]).

In 1965, M. Motoo and S. Watanabe wrote a paper [13] in which they made a profound analysis of the structure of the space of square integrable martingale additive functionals of a Hunt Markov process. In the meantime, the Doob–Meyer decomposition theorem of submartingales was completed by P.A. Meyer. These two works merged into a paper by H. Kunita and S. Watanabe which appeared in the Nagoya Mathematical Journal in 1967 [14] and a series of papers by P.A. Meyer in the Strasbourg Seminar Notes in 1967 [15], where the stochastic integral was defined for a general semi-martingale, and the stochastic calculus I initiated in 1942 and 1951 was revived in a new general context. Since then, various researchers including myself also became more concerned about the stochastic calculus and stochastic differential equations.

My joint paper [9] with McKean in 1963 gave a probabilistic construction of the Brownian motion on $[0, \infty)$ subjected to the most general boundary condition whose analytic study had been established by Feller under some restrictions. Our methods involved the probabilistic idea originated in Lévy about the local time and excursions away from 0. In 1970, the idea was extended in my paper in the Proceedings of the Sixth Berkeley Symposium [7], where I considered a general standard Markov process X_t for which a specific one point a is regular for itself. A Poisson point process taking values in the space U of excursions around point a was then associated, and its characteristic measure (a σ-finite measure on U) together with the stopped process obtained from X_t by the hitting time of a was shown to uniquely determine the law of the given process X_t. This approach may be considered as an infinite dimensional analogue to a part of the decomposition of the Lévy process I studied in 1942, and may have revealed a new aspect in the study of Markov processes.

The one-dimensional diffusion theory is still important as a basic prototype of Markov processes. Besides my joint book [10] with McKean, I also gave a comprehensive account of the Feller generator as a generalized second order differential operator in Section 6 of my *Lectures on Stochastic Processes* at the Tata Institute of Fundamental Research, Bombay, 1960 [6]. The second part of my book *Stochastic Processes* [5] written in Japanese and published in 1957 contains a detailed description of the Feller generator and, in addition, of the boundary behaviors of the solutions of the associated homogeneous equation

$$(\lambda - \mathcal{G})u = 0, \qquad \lambda > 0,$$

in an analytical way together with their probabilistic implications. I had sent the Japanese original of [5] at the time of its publication to Eugene B. Dynkin and it was translated into Russian by A.D. Wentzell in 1960 (Part I) and in 1963 (Part II). In 1959 Shizuo Kakutani at Yale University, noting the importance of my description of the one dimensional diffusions, advised Yuji Itô, at that time one of his graduate students, to produce a translation of the second part into English, which was then distributed among a limited circle of mathematicians around Yale University as a typewritten mimeograph. I am very glad to hear that a full English translation of the book [5] by Yuji Itô is now being prepared for publication by the American Mathematical Society under the title *Essentials of Stochastic Processes.*

Finally, let me extend my deepest gratitude to the symposium organizers and participants for honoring my 90th birthday with your work on stochastic analysis. I also wish to thank you again for allowing me to present these memoirs to you here, and I very much look forward to studying all the papers presented at this symposium.

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Itô Calculus and Quantum White Noise Calculus

Luigi Accardi[1] and Andreas Boukas[2]

[1] Centro Vito Volterra, Università di Roma Tor Vergata via Columbia, 2–00133 Roma, Italy. accardi@Volterra.mat.uniroma2.it, http://volterra.mat.uniroma2.it

[2] Department of Mathematics and Natural Sciences, American College of Greece, Aghia Paraskevi, Athens 15342, Greece, andreasboukas@acgmail.gr

Summary. Itô calculus has been generalized in white noise analysis and in quantum stochastic calculus. Quantum white noise calculus is a third generalization, unifying the two above mentioned ones and bringing some unexpected insight into some old problems studied in different fields, such as the renormalization problem in physics and the representation theory of Lie algebras. The present paper is an attempt to explain the motivations of these extensions with emphasis on open challenges.

The last section includes a result obtained after the Abel Symposium. Namely that, after introducing a new renormalization technique, the RHPWN Lie algebra includes (in fact we will prove elsewhere that this inclusion is an identification) a second quantized version of the extended Virasoro algebra, i.e. the Virasoro–Zamolodchikov $*$–Lie algebra w_∞, which has been widely studied in string theory and in conformal field theory.

1 Introduction

The year 2005 marks Kiyosi Itô's 90th birthday and, with it, the 63th birthday of stochastic calculus. The present Abel Symposium, devoted to the celebration of these events, offers to all mathematicians an important occasion to meditate on this important development in their discipline whose influence is going to follow the times of history, even in a period when the pace of scientific development has reached a level in which most papers have a life time of less than one year.

The applications of Itô's work have been so many, ranging from physics to biology, from logistics and operation research to engineering, from meteorology to mathematical finance, . . ., that an exhaustive list is impossible.

From the mathematical point of view it is someteimes underestimated the fact that Itô calculus, with its radical innovation of the two basic operations of calculus – differentiation and integration – has been one of the few real conceptual breakthroughs in the development of classical analysis after Newton.

Itô laid down the foundations of stochastic calculus in his 1942 thesis [Itô42a, Itô42b] and the first systematic exposition of these ideas in English language appeared almost ten years later in [Itô51] and preceeded of about 15 years the now classical monograph [ItôMcKn65]. This gave rise to an impetuous development which has seen as protagonists several of the participants to the present conference and which will be reviewed by them.

My talk will take the move from one of the basic achievements of this development, completed in the late 1960's, and which led to the mathematical substantiation of a limpid and intuitive picture of the structure of a classical stochastic process indexed by the real line (interpreted as time) and with values in $\mathbb{R}^d$ (interpreted as a generalized phase space).

The sample space of a *generic process* of this type is identified to a space of $\mathbb{R}^d$–valued functions, interpreted as trajectories of a dynamical system, and each trajectory is canonically decomposed into a sum of two parts: a *regular* (bounded variation) part, corresponding to the drift in the stochastic equation and a *pure fluctuation* term, corresponding to the martingale part in the stochastic equation. The former part is handled with classical, *Newtonian*, calculus; the latter with Itô calculus. The picture is completed by the Kunita–Watanabe *martingale representation theorem* [KunWat67], which characterizes the *generic martingales* as stochastic integrals with respect to some stationary, independent increment process and by the *Lévy–Itô decomposition* of a stationary, independent increment process (Z_t):

$$Z_t = mt + \sigma B_t + X_t$$

where m is a constant, B_t is a Brownian motion and X_t is a compound Poisson process, i.e. an integral

$$X_T = \int_0^T dt \int P_{u,t} d\beta(u)$$

of independent Poisson processes $P_{u,t}$ with intensity of jumps equal to u, with respect to a measure $d\beta(u)$, called the **Levy measure** and with support in $\mathbb{R} \setminus \{0\}$.

The early generalizations of Itô calculus had gone in the direction of extending it to more general *state spaces* thus passing from $\mathbb{R}^d$ to manifolds or to infinite dimensions or both. Another, less developed extension was from vector valued to operator valued classical stochastic processes [Skor84]. However these extensions did not change the basic conceptual framework of the theory.

The situation changed in the past 30 years when *three qualitative innovations* appeared. This drastically enlarged not only the conceptual status of Itô calculus, and more generally of stochastic analysis, but also its technical apparatus. The traditional bridges between probability, classical analysis and combinatorics became an intricate network including practically every field of mathematics, from operator theory to graph theory, from Hopf algebras to group representations, ...

The traditional applications to the classical world (physics, information, communications, engineering, finance, ...) have now been expanded to the corresponding sectors in the quantum world thus bringing a remedy to the historical paradox according to which the mathematical discipline, dealing with the laws of chance, was not powerful enough to include into its framework the most advanced physical theory, quantum mechanics, in which chance enters in a much more intrinsic way than in any other physical theory.

These innovations begun with two, initially quite separated and independent, lines of research: *white noise analysis*, (1975) and *quantum stochastic calculus* (1982) and found their unification, starting from 1993, in *quantum white noise calculus*.

The rate of progression of these events, as well as the merging of different generalizations into a single, unified picture, has been so swift that, even for those who actively participated in the construction of these developments, it is quite hard to follow all the new ideas and to embrace the whole landscape in a single eyesight.

It is precisely on this broad picture that the present paper will be focused. Not only details, but also several important achievements, will be omitted from the exposition, in the attempt to convey an idea of some of the exciting new perspectives of *quantum stochastic analysis*.

The first attempts to go beyond the Itô calculus framework and to include processes which, although much more singular, were frequently used in the physics and engineering literature, was Hida *white noise theory*, first proposed in his Carleton lectures of 1975 [Hida75, Hida92].

The second conceptual generalization of Itô calculus took place in 1982 when Hudson and Parthasarathy developed their *quantum stochastic calculus* [HuPa82a, HuPa84c]. In it for the first time, the noises themselves (i.e. the martingales driving the stochastic differential equations) were no longer classical additive independent increment processes but *quantum independent increment processes*. This was the first quantum generalization of Itô calculus and opened the way to all subsequent ones. The culmination of the theory is the determination of the structure of those stochastic equations which admit a unitary solution. The reason why this result has fundamental implications both for quantum mechanics and for classical probability, will be explained starting from Section (7).

The Hudson–Parthasarathy theory inspired, directly or indirectly, most of the developments of quantum probability for the decade after its appearance. Its importance can be compared to the original Itô paper and the multiplicity of investigations it motivated was surveyed in [Partha92].

But the story does not end here: a third conceptual generalization, motivated by the stochastic limit of quantum theory, was developed between 1993 and 1995 and can be described as the unification of the white noise and the quantum stochastic approach: the non triviality of this unification will be clear starting from Section (12) of the present exposition. In particular this third step threw a new and unexpected light on the *microscopic structure of quantum, hence in particular classical, stochastic equations* as a consequence of:

(i) the discovery of the Hamiltonian structure of the (classical and quantum) stochastic differential equations
(ii) the discovery of the *translation code* between white noise and stochastic differential equations. This required the development of the *theory of distributions on the standard simplex* [AcLuVo99] which is the mathematical counterpart of the *time consecutive principle* of the stochastic limit of quantum theory.

However the main point of the new development was not so much the deeper understanding of the structure of classical and quantum stochastic calculus, but the possibilities it opened of further extensions, which cannot be obtained with the traditional tools of stochastic analysis. In fact the *white noise extension of the Itô table* opened the way to the nonlinear generalizations of Itô calculus to which is devoted the second part of the present report.

The beautiful landscape emerging from the simplest of these extensions, i.e. the one dealing with the second power of white noise, and the subsequent, totally unexpected unification of the five Meixner classes as classical subprocesses (algebraically: Cartan $*$–sub algebras) respectively of the first and second order white noise, rose strong hopes that this hierarchy could be extended from the second powers of white noise to its higher powers. This would lead to a new, interesting class of infinitely divisible processes (for a short while there was even the hope to obtain a new parametrization of all these processes).

This hope however collided with the wall of the *no go theorems* described in the last part of the present paper. Although negative results, these theorems are very interesting because they have revealed an hitherto unknown phenomenon relating stochastic analysis to two different fields, each of which has been the object of a huge literature outside probability theory namely:

(i) the representation theory of infinite dimensional Lie algebras
(ii) renormalization theory.

These two theories are at the core of contemporary theoretical physics and the fact that some developments, motivated by quantum white noise analysis, could bring new insight and new results in such a fundamental issue, which resisted decades of efforts from the best minds of theoretical physics, is an indication that this direction is deep and worth being pursued. For this reason while the first part of the present paper consists in an exposition of already established results, in the second part emphasis has been laid on the formulation of the problems facing the construction of a satisfactory theory of the higher powers of white noise. This has led to the introduction of some new notions, such as *Fock representation of a Lie algebra*, which are going to play an essential role in the development of the theory.

All these developments show that Itô calculus shares, with the richest and deepest mathematical theories, the germs of its radical innovation. Historical experience shows that these innovations often occur in directions which are quite unexpected for the experts of the field and this sometimes generates a feeling of extraneousness.

An instructive example is given by the theory of elliptic functions, originated from a deep intuition of Abel and initially developed within a purely analytical context, but now stably settled in a purely algebraic and geometrical framework.

The story we are going to tell shows that Itô calculus gives another important example in this direction.

2 Plan of the Present Paper

The goal of the present section is twofold: (i) to give a more analytical outline of the content of the present paper; (ii) to catch this occasion to say a few words about the motivations and the inner logic underlying the developments described here as well as about their connections with other sectors of quantum probability which could not be dealt with for reasons of space.

Section (3) defines the notion of quantum (Boson Fock) white noise and illustrates, in this basic particular case, one of the main ideas of quantum probability, i.e. the idea that *algebra implies statistics*. Let me just mention here that also the converse statement, i.e. that statistics implies algebra (e.g. commutation or anti commutation relations), is true and it lies at a deeper level. The first result in this direction was proved by von Waldenfels in the Bose and Fermi case [voWaGi78, voWa78] and about 20 years later, with the introduction of the notion of interacting Fock space [AcLuVo97b], this principle became a quite universal principle of probability theory and opened the way to the program of a full algebraic classification of probability measures. This is a quite interesting direction, and is also deeply related to the main topic of the present paper, stochastic and white noise calculus, but we will not discuss this connection and we refer the interested reader to [AcBo98, AcKuSt02, AcKuSt05a].

Section (4) describes another important new idea of quantum probability, i.e. the notion of *quantum decomposition of a classical random variable* (or stochastic process). This idea is illustrated in the important particular case of classical white noise and extended, in Section (6), to the Poisson noise.

The two above mentioned decompositions are at the root of Hudson–Parthasarathy's quantum extension of classical Itô calculus, briefly outlined in Section (6).

Section (7) briefly describes the classical Schrödinger and Heisenberg equations as a preparation to their stochastic and white noise versions.

The algebraic form of a classical stochastic process is described in Section (8). This leads to a reformulation, explained in Section (10), of classical stochastic differential equations, that makes quite transparent their equivalence to stochastic versions of the classical Schrödinger or of Heisenberg equations.

In Sections (9), (10) it is briefly outlined how this reformulation is nothing but a stochastic analogue of Koopman's algebraization of the theory of classical, deterministic dynamical systems.

Combining the content of Section (8) with the quantum decomposition of classical white and Poisson noise, described in Sections (4), (6), one arrives, in Section (11), to the full quantum versions of the stochastic Schrödinger and Heisenberg equations, which are the main object of study of the Hudson–Parthasarathy theory.

These equations are not of Hamiltonian type and they were developed by Hudson and Parthasarathy on the basis of a purely mathematical analogy with the classical Itô calculus. Hence their connection with the Hamiltonian equations of quantum physics was obscure and the early applications of these equations to physical problems, proposed by Barchielli [Barc88], Belavkin [Bela86a], Gardiner and Collet [GaCo85], ... , were built on a purely phenomenological basis. This led to some misgivings among physicists on the meaning of these models and their relations to the fundamental laws of quantum mechanics.

On the other hand, combining the main results of Hudson and Parthasarathy (construction of unitary Markovian cocycles) with the quantum Feynman–Kac formula of [Ac78b] we see that, by quantum conditioning of a stochastic Heisenberg evolution $X_0 \mapsto U_t X_0 U_t^*$ on the time zero algebra, one obtains a quantum Markov semigroup (P^t):

$$E_{0]}\left(U_t X_0 U_t^*\right) = P^t(X_0) \tag{2.1}$$

just as the analogue classical conditioning leads to a classical Markov semigroup. It was also known, since the early results of Pauli and van Hove, that quantum Markov semigroups (P^t) (and the associated master equations, which are the quantum analogue of the Chapman–Kolmogorov equations) can arise as appropriate time–scaling limits of reduced Heisenberg evolutions. The time–scaling being the same one used in classical stochastic homogenization (i.e. $t \to t/\lambda^2$), and known in the physical literature as *van Hove* or $1/\lambda^2$–scaling, and the limit being taken for $\lambda \to 0$. Since (in this particular context) the physical operation of *reducing an Heisenberg evolution to a subsystem*, used in these papers, is mathematically equivalent to conditioning on the time zero algebra, the above statement can be rewritten as:

$$\lim_{\lambda \to 0} E_{0]}\left(U^{(\lambda)}_{t/\lambda^2} X_0 U^{(\lambda)*}_{t/\lambda^2}\right) = P^t(X_0) \tag{2.2}$$

Comparing (2.1) with (2.2) it was therefore natural to conjecture that also the unconditioned limits,

$$\lim_{\lambda \to 0} U^{(\lambda)}_{t/\lambda^2} X_0 U^{(\lambda)*}_{t/\lambda^2} = U_t X_0 U_t^* \tag{2.3}$$

$$\lim_{\lambda\to 0} U^{(\lambda)}_{t/\lambda^2} = U_t \tag{2.4}$$

of the original Schrödinger and Heisenberg evolutions should exist, for some (at those times unspecified) topology, and satisfy some quantum stochastic Schrödinger and Heisenberg equations of Hudson–Parthasarathy type.

This conjecture was formulated by Frigerio and Gorini immediately after the development of quantum stochastic calculus [FrGo82a] and was proved a few years later by Accardi, Frigerio and Lu [AcFrLu87].

This result marked the beginning of the stochastic limit of quantum theory. It proved that *quantum stochastic differential equations arise as physically meaningful scaling and limiting procedures from the fundamental laws of quantum mechanics, expressed in terms of Hamiltonian equations.* This produced, among other things, a microscopic interpretation not only of the coefficients of the stochastic equations, but also of the fine structure of the driving martingales (quantum noises).

Several years later Accardi, Lu and Volovich [AcLuVo93] realized that in fact *stochastic differential equations (both classical and quantum) are themselves Hamiltonian equations* but not of usual type: they are *white noise Hamiltonian equations.* The identification of these two classes of equations required the development of new mathematical techniques such as the notion of *causal normal order* and the strictly related *time consecutive principle* and *theory of distributions on the standard simplex* (cf. [AcLuVo02] for a discussion of these notions).

The inclusion: *classical and quantum SDE* $\subseteq$ *WN Hamiltonian equations* is a consequence of this development and is described in Sections (12), (13). These few pages condensate a series of developments which took place in several years and in several papers. The interested reader is referred to [AcLuVo99] (the first attempt to systematize the impetuous development of the previous years) and to the more recent expositions [Ayed05] (thesis of Wided Ayed) and the papers [AcAyOu03, AcAyOu05a, AcAyOu05b]. The last of this papers deals with another one of the several interesting developments born from the stochastic limit of quantum theory which, for lack of space, are not discussed in the present paper, namely the module generalization of white noise calculus and the qualitatively new structure of the quantum noises emerging from it (the reader, interested in the first and main physical example of this new structure, is referred to [AcLuVo97b]).

Even more condensed is the description, in Sections (14), (15), (16), (17), of the renormalized square of WN. This is because the survey paper [AcBou04c] is specifically devoted to this subject and the interested reader can find there the necessary information.

On the contrary, since most of the material in Sections from (18) to (22) has not yet been published, we tried to give all the necessary definitions even if proofs had to be omitted for reasons of space.

The general problem, concerning the renormalized higher powers of WN, is formulated in Section (21) with the related no–go theorems. As explained in

Section (22), this problem is also related with an old open problem of classical probability, i.e. the infinite divisibility of the odd powers of a standard Gaussian random variable.

Further investigations are needed to understand the effective impact of these no–go theorems. Do they really close the hope of a general theory of higher powers of white noise? Our feeling is that the answer to this question is *no*! This hope is supported by the following considerations. The no–go theorems heavily depend on:

(i) the choice of a renormalization procedure;
(ii) the fact that we restrict our attention to a very special representation, i.e. the Fock one.

A way out of this conundrum has to be looked for in the relaxation of one of these assumption, i.e. one has to look for either new renormalization procedures or different representations. Both ways are now under investigation and raise challenging but fascinating mathematical problems.

The last section of the present paper refers to a development that took place after the end of the Abel Symposium and which shows that the idea to look for different types of renormalization procedures turned out particularly fruitful and brought to the fore a connection between the renormalized higher powers of white noise and the Virasoro algebra which promises to be as rich of developments as the connection between the renormalized square of white noise and the Meixner classes.

3 Fock Scalar White Noise (WN)

Definition 1. *The standard d–dimensional Fock scalar White Noise (WN) is defined by a quadruple*

$$\{\mathcal{H}, b_t, b_t^+, \Phi\}; \quad t \in \mathbb{R}^d$$

where $\mathcal{H}$ is a Hilbert space, $\Phi \in \mathcal{H}$ a unit vector called the (Fock) vacuum, and b_t, b_t^+ are operator valued distributions (for an explanation of this notion see the comment at the end of the present section and the discussion in [AcLuVo02], Section (2.1)) with the following properties.

The vectors of the form

$$b_{t_n}^+ \cdots b_{t_1}^+ \Phi \tag{3.1}$$

called the number vectors are well defined in the distribution sense and total in $\mathcal{H}$.

b_t is the adjoint of b_t^+ on the linear span of the number vectors

$$(b_t^+)^+ = b_t \tag{3.2}$$

Weakly on the same domain and in the distribution sense:

$$[b_s, b_t^+] := b_s b_t^+ - b_t^+ b_s = \delta(t - s) \tag{3.3}$$

where, here and in the following, the symbol $[\,\cdot\,,\,\cdot\,]$ *will denote the commutator:*

$$[A, B] := AB - BA$$

Finally b_t *and* Φ *are related by the Fock property (always meant in the distribution sense):*

$$b_t\Phi = 0 \tag{3.4}$$

The unit vector Φ *determines the expectation value*

$$\langle\Phi, X\Phi\rangle =: \langle X\rangle \tag{3.5}$$

which is well defined for any operator X *acting on* $\mathcal{H}$ *and with* Φ *in its domain.*

Remark. In the Fock case *algebra implies statistics* in the sense that the algebraic rules (3.3), (3.2), (3.4) uniquely determine the restriction of the expectation value (3.5) on the polynomial algebra generated by b_t and b_t^+. This is because, with the notation

$$X^\varepsilon = \begin{cases} X\,, \ \varepsilon = -1 \\ X^*\,, \ \varepsilon = +1 \end{cases} \tag{3.6}$$

the Fock prescription (3.4) implies that the expectation value

$$\langle b_{t_n}^{\varepsilon_n} \cdots b_{t_1}^{\varepsilon_1}\rangle \tag{3.7}$$

of any monomial in b_t and b_t^+ is zero whenever either n is odd or $b_{t_1}^{\varepsilon_1} = b_{t_1}$ or $b_{t_n}^{\varepsilon_n} = b_{t_n}^+$. If neither of these conditions is satisfied, then there is a $k \in \{2, \ldots, n\}$ such that the expectation value (3.7) is equal to

$$\langle b_{t_n}^{\varepsilon_n} \cdots b_{t_1}^{\varepsilon_1}\rangle = \langle b_{t_n}^{\varepsilon_n} \cdots b_{t_{k+1}}^{\varepsilon_{k+1}} [b_{t_k}, b_{t_{k-1}}^+ \cdots b_{t_1}^+]\rangle \tag{3.8}$$

Using the derivation property of the commutator $[b_{t_k}\,,\,\cdot\,]$ (i.e. (7.4)) one then reduces the expectation value (3.8) to a linear combination of expectation values of monomials of order less or equal than $n-2$. Iterating one sees that only the scalar term can give a nonzero contribution.

Remark. The **practical rule to deal with operator valued distributions** is the following: products of the form (3.7) are meant in the sense that, after multiplication by $\varphi(t_n) \cdot \ldots \cdot \varphi(t_1)$, where $\varphi_1, \ldots, \varphi_n$ are elements of an appropriate test function space (typically one chooses the space of smooth functions decreasing at infinity faster than any polynomial), and integration with respect to all variables $dt_1 \cdot \ldots \cdot dt_n$ (each of which runs over $\mathbb{R}^d$) one obtains a product of well defined operators whose products contain the vector Φ in their domains. Here and in the following we will not repeat each time when an identity has to be meant in the distribution sense.

4 Classical Real Valued White Noise

Lemma. Let b_t, b_t^+ be a Fock scalar white noise. Then

$$w_t := b_t + b_t^+ \tag{4.1}$$

is a classical real random variable valued distribution satisfying:

$$w_t = w_t^+ \tag{4.2}$$

$$[w_s, w_t] = 0; \qquad \forall s, t \tag{4.3}$$

$$\langle w_t \rangle = 0 \tag{4.4}$$

$$\langle w_s w_t \rangle = \delta(t-s) \tag{4.5}$$

$$\langle w_{t_{2n}} \dots w_{t_1} \rangle = \sum_{\{l_\alpha, r_\alpha\} \in p.p.\{1,\dots,2n\}} \prod_{\alpha=1}^{n} \langle w_{t_{l_\alpha}} w_{t_{r_\alpha}} \rangle \tag{4.6}$$

moreover all odd moments vanish and $p.p.\{1, \dots, 2n\}$ denotes the set of all pair partitions of $\{1, \dots, 2n\}$.

Remark. The self–adjointness condition (4.2) and the commutativity condition (4.3) mean that (w_t) is (isomorphic to) a classical real valued process. Conditions (4.4) and (4.5) mean respectively that (w_t) is mean zero and δ–correlated. Finally (4.6), which follows from (3.4) and from the same arguments used to deduce the explicit form of (3.7), shows that the classical process (w_t) is Gaussian.

Definition 2. *The process (w_t) satisfying (4.2),...,(4.5) (one can prove its uniqueness up to stochastic equivalence) is called the standard d–dimensional classical real valued White Noise (WN). The identity (4.1) is called the quantum decomposition of the classical d–dimensional white noise.*

Remark. Notice that, for the classical process (w_t), it is not true that algebra implies statistics: this becomes true only using the quantum decomposition (4.1) combined with the Fock prescription (3.4).

Remark. In the case $d = 1$, integrating the classical WN one obtains the classical Brownian motion with zero initial condition:

$$W_t = B_t + B_t^+ = \int_0^t ds(b_s^+ + b_s) \tag{4.7}$$

Notice that (4.7) gives the q–decomposition of the classical BM just as (4.1) gives the q–decomposition of the classical WN.

From now on we will only consider the case $d = 1$.

5 Classical Subprocesses Associated to the First Order White Noise

An important generalization of the quantum decomposition (4.1) of the classical white noise is the identity:

$$p_t(\lambda) = b_t + b_t^+ + \lambda b_t^+ b_t; \quad \lambda \geq 0 \tag{5.1}$$

which can be shown to define (in the sense of vacuum distribution) a 1–parameter family of classical real valued distribution processes (i.e. $p_t(\lambda) = p_t(\lambda)^+$ and $[p_s(\lambda), p_t(\lambda)] = 0$). In fact this classical process can be identified, up to a time rescaling, to the compensated scalar valued standard classical Poisson noise with intensity $1/\lambda$ and the identity (5.1) gives a q–decomposition of this process.

Integrating (5.1), in analogy with (4.7), one obtains the standard compensated Poisson processes. Notice that the critical value

$$\lambda = 0$$

corresponds to the classical WN while any other value

$$\lambda \neq 0$$

gives a Poisson noise. As a preparation to the discussion of Section (17) notice that $\lambda = 0$ is the only critical point, i.e. a point where the vacuum distribution changes and that these two classes of stochastic processes exactly coincide with the first two Meixner classes.

6 The Hudson–Parthasarathy Quantum Stochastic Calculus

In the previous sections we have seen that, integrating the densities

$$w_t = b_t + b_t^+$$

$$p(\lambda)_t = b_t + b_t^+ + \lambda b_t^+ b_t$$

one obtains the stochastic differentials (random measures) as WN integrals

$$dW_t = \int_t^{t+dt} w_s ds = \int_t^{t+dt} (b_s + b_s^+) ds =: dB_t^+ + dB_t$$

$$dP_t(\lambda) = \int_t^{t+dt} p_s(\lambda) ds = \int_t^{t+dt} (b_s + b_s^+ + \lambda b_s^+ b_s) ds = dB_t^+ + dB_t + \lambda dN_t$$

Starting from these one defines the classical stochastic integrals with the usual constructions.

$$\int_0^t F_s dW_s; \qquad \int_0^t F_s dP_s(\lambda)$$

The passage to q–stochastic integrals consists in separating the stochastic integrals corresponding to the different pieces. In other words, the quantum decomposition (5.1) suggests to introduce separately the stochastic integrals

$$\int_0^t F_s dB_s; \quad \int_0^t F_s dB_s^+; \quad \int_0^t F_s dN_s$$

This important development was due to Hudson and Parthasarathy and we refer to the monograph [Partha92] for an exposition of the whole theory.

7 Schrödinger and Heisenberg Equations

A *Schrödinger equation* (also called an *operator Hamiltonian equation*) is an equation of the form:

$$\partial_t U_t = -iH_t U_t; \quad U_0 = 1; \quad t \in \mathbb{R} \tag{7.1}$$

where the 1–parameter family of symmetric operators on a Hilbert space $\mathcal{H}$

$$H_t = H_t^*$$

is called the *Hamiltonian.* In the pyhsics literature one often requires the positivity of H_t. We do not follow this convenction in order to give a unified treatment of the usual Schrödinger equation and of its so–called *interaction representation form.* This approach is essential to underline the analogy with the white noise Hamiltonian equations, to be discussed in Section (12).

When H_t is a self–adjoint operator independent of t, the solution of equation (7.1) exists and is a 1–parameter group of unitary operators:

$$U_t \in Un(\mathcal{H}); \; U_s U_t = U_{s+t}; \; U_0 = 1; \; U_t^* = U_t^{-1} = U_{-t}; \; s, t \in \mathbb{R}$$

Conversely every 1–parameter group of unitary operators is the solution of equation (7.1) for some self–adjoint operator $H_t = H$ independent of t.

An *Heisenberg equation*, associated to equation (7.1), is

$$\partial_t X_t = \delta_t(X_t); \qquad X_0 = X \in \mathcal{B}(\mathcal{H}) \tag{7.2}$$

where δ_t has the form

$$\delta_t(X_t) := -i[H_t, X_t]; \qquad X_0 = X \in \mathcal{B}(\mathcal{H}) \tag{7.3}$$

One can prove that δ_t is a $*$–derivation, i.e. a linear operator on an appropriate subspace of the algebra $\mathcal{B}(\mathcal{H})$ of all the bounded operators on $\mathcal{H}$, also called *the algebra of observables*, satisfying (on this subspace):

$$\delta_t(ab) = \delta_t(a)b + a\delta_t(b) \tag{7.4}$$

$$\delta_t^*(a) := \delta_t(a^*)^* = \delta_t(a)$$

Not all $*$–derivations δ_t on subspaces (or sub algebras) of $\mathcal{B}(\mathcal{H})$ have the form (7.3). If this happens, then the $*$–derivation, δ_t, and sometimes also the Heisenberg equation, is called *inner* and its solution has the form

$$X_t = U_t X_t U_t^* \tag{7.5}$$

where U_t is the solution of the corresponding Schrödinger equation (7.1). Conversely, every solution U_t of the Schrödinger equation (7.1) defines, through (7.5), a solution of the Heisenberg equation (7.2) with δ_t given by (7.3).

Thus every Schrödinger equation is canonically associated to an Heisenberg equation. The converse is in general false, i.e. there are Heisenberg equations with no associated Schrödinger equation (equivalently: not always a derivation is inner). The simplest physically relevant examples of this situation are given by the quantum generalization of the so called *interacting particle systems* [AcKo00b] which have been widely studied in classical probability.

8 Algebraic Form of a Classical Stochastic Process

Let (X_t) be a real valued stochastic process. Define

$$j_t(f) := f(X_t)$$

In the spirit of quantum probability, we realize f as a multiplication operator on $L^2(\mathbb{R})$ and $f(X_t)$ as a multiplication operator on

$$L^2(\mathbb{R} \times \Omega, \mathcal{B}_{\mathbb{R}} \times \mathcal{F}, dx \otimes P) \equiv L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)$$

where $(\Omega, \mathcal{F}, P)$ is the probability space of the process (X_t) and $\mathcal{B}_{\mathbb{R}}$ denotes the Borel σ–algebra on $\mathbb{R}$. Sometimes we use the notation:

$$M_f \varphi(x) := f(x)\varphi(x); \qquad \varphi \in L^2(\mathbb{R})$$

The same notation will be used if $x \in \mathbb{R}$ is replaced by $(x, \omega) \in \mathbb{R} \times \Omega$.

Thus $f(X_t)$ is realized as multiplication operator on $L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P)$. With these notations, for each $t \geq 0$, j_t is a $*-$homomorphism

$$j_t : \mathcal{C}^2(\mathbb{R}) \subseteq \mathcal{B}(L^2(\mathbb{R})) \to \mathcal{B}(L^2(\mathbb{R}) \otimes L^2(\Omega, \mathcal{F}, P))$$

9 Koopman's Argument and Quantum Extensions of Classical Deterministic Dynamical Systems

The following considerations, due to Koopman, constitute the basis of the algebraic approach to dynamical systems which reduces the study of such systems to the study of 1–parameter groups of unitary operators or of

$*$–automorphisms of appropriate commutative $*$–algebras or, at infinitesimal level, to the study of appropriate Schrödinger or Heisenberg equations.

To every ordinary differential equation in $\mathbb{R}^d$

$$dx_t = b(x_t)dt; \qquad x(0) = x_0 \in \mathbb{R}^d$$

such that the initial value problem admits a unique solution for every initial data x_0 and for every $t \geq 0$: one associates the 1–parameter family of maps

$$T_t : \mathbb{R}^d \to \mathbb{R}^d$$

characterized by the property that the image of x_0 under T_t is the value of the solution at time t:

$$x_t(x_0) =: T_t x_0; \qquad T_0 = id$$

Uniqueness then implies the semigroup property:

$$T_t T_s = T_{t+s}$$

If the above properties hold not only for every $t \geq 0$, but for every $t \in \mathbb{R}$, then the system is called reversible. In this case each T_t is invertible and

$$T_t^{-1} = T_{-t}$$

Typical examples of these systems are the classical Hamiltonian systems. They have the additional property that the maps T_t preserve the Lebesgue measure (Liouville's theorem).

Abstracting the above notion to an arbitrary measure space leads to the notion of (deterministic) dynamical system:

Definition 3. *Let (S, μ) be a measure space. A classical, reversible, deterministic dynamical system is a pair:*

$$\{(S, \mu);\ (T_t)\ t \in \mathbb{R}\}$$

where $T_t : S \to S$ $(t \in \mathbb{R})$ is a 1–parameter group of invertible bi–measurable maps of (S, μ) admitting μ as a quasi–invariant measure:

$$\mu \circ T_t \sim \mu$$

The quasi–invariance of (S, μ) is equivalent to the existence of a μ–almost everywhere invertible Radon–Nikodym derivative:

$$\frac{d(\mu \circ T_t)}{d\mu} =: p_{\mu,t} \in L^1(S, \mu)$$

$$p_{\mu,t} > 0;\ \mu - \text{a.e.}; \quad \int_S p_{\mu,t}(s) d\mu(s) = 1$$