Potential Theory in Applied Geophysics

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Remembering

Late Prof. P.K. Bhattacharyya
(1920–1967)
Ex Professor of Geophysics
Department of Geology and Geophysics
Indian Institute of Technology
Kharagpur, India

and

Late Prof. Amalendu Roy
(1924–2005)
Ex Deputy Director
National Geophysical Research Institute
Hyderabad, India

Two Great Teachers of Geophysics

Preface

Two professors of Geophysics Late Prof. Prabhat Kumar Bhattacharyya and Late Prof. Amalendu Roy developed the courses on Potential Theory and Electromagnetic Theory in 1950s for postgraduate students of geophysics in the Department of Geology and Geophysics, Indian Institute of Technology, Kharagpur, India. The courses had gone through several stages of additions and alterations from time to time updating during the next 5 decades. Prof. Bhattacharyya died in 1967 and Prof. Amalendu Roy left this department in the year 1961. These subjects still remained as two of the core subjects in the curriculum of M.Sc level students of geophysics in the same department. Inverse theory joined in these core courses much later in late seventies and early eighties. Teaching potential theory and electromagnetic theory for a period of 9 years in M.Sc and predoctoral level geophysics in the same department enthused me to write a monograph on potential theory bringing all the pedagogical materials under one title "Potential Theory in Applied Geophysics". I hope that the book will cater some needs of the postgraduate students and researchers in geophysics. Since many subjects based on physical sciences have some common areas, the students of Physics, Applied Mathematics. Electrical Engineering, Electrical Communication Engineering, Acoustics, Aerospace Engineering etc may find some of the treatments useful for them in preparation of some background in Potential Theory. Every discipline of science has its own need, style of presentation and coverage. This book also has strong bias in geophysics although it is essentially a monograph on mathematical physics. While teaching these subjects, I felt it a necessity to prepare a new book on this topic to cater the needs of the students. Rapid growth of the subject Potential Theory within geophysics prompted me to prepare one more monograph with a strong geophysics bias. The areal coverages are different with at the most 20 to 30% overlap. Every book has a separate identity. Students should go through all the books because every author had his own plans and programmes for projecting his angle of vision.

This book originated mainly from M.Sc level class room teaching of three courses viz. Field Theory – I (Potential Theory), Field Theory -II (Electromagnetic Theory) and Inverse theory in the Department of Geology and Geophysics, I.I.T., Kharagpur, India. The prime motivation behind writing this book was to prepare a text cum reference book on Field Theory (Scalar and Vector Potentials and Inversion of Potential Fields).This book has more detailed treatments on electrical and electromagnetic potentials. It is slightly biased towards electrical methods. The content of this book is structured as follows:

In Chap. 1 a brief introduction on vector analysis and vector algebra is given keeping the undergraduate and postgraduate students in mind. Because important relations in vector analysis are used in many chapters.

In Chap. 2 I have given some introductory remarks on fields and their classifications, potentials, nature of a medium, i.e. isotropic or anisotropic, one, two and three dimensional problems, Dirichlet, Neumann and mixed boundary conditions, tensors, differential and integral homogenous and inhomogenous equations with homogenous and inhomogenous boundary conditions, and an idea about domain of geophysics where treatments are based on potential theory.

In Chap. 3 I briefly discussed about the nature of gravitational field. Newton's law of gravitation, gravitational fields and potentials for bodies of simpler geometric shapes, gravitational field of the earth and isostasy and guiding equations for any treatment on gravitional potentials.

In Chap. 4 Electrostatics is briefly introduced. It includes Coulomb's law, electrical permittivity and dielectrics, electric displacement, Gauss's law of total normal induction and dipole fields. Boundary conditions in electrostatics and electrostatic energy are also discussed.

In Chap. 5 besides some of the basics of magnetostatic field, the similarities and dissimilarities of the magnetostatic field with other inverse square law fields are highlighted. Both rotational and irrotational nature of the field, vector and scalar potentials and solenoidal nature of the field are discussed. All the important laws in magnetostatics, viz Coulomb's law, Faraday's law, Biot and Savart's law, Ampere's force law and circuital law are discussed briefly. Concept of magnetic dipole and magnetostatic energy are introduced here. The nature of geomagnetic field and different types of magnetic field measurements in geophysics are highlighted.

In Chap. 6 most of the elementary ideas and concepts of direct current flow field are discussed. Equation of continuity, boundary conditions, different electrode configurations, depth of penetration of direct current and nature of the DC dipole fields are touched upon.

In Chap. 7 solution of Laplace equation in cartisian, cylindrical polar and spherical polar coordinates using the method of separation of variables are discussed in great details. Bessel's Function, Legendre's Polynomials, Associated Legendre's Polynomial and Spherical Harmonics are introduced. Nature of a few boundary value problems are demonstrated.

In Chap. 8 advanced level boundary value problems in direct current flow field are given in considerable details. After deriving the potentials in different layers for an N-layered earth the nature of surface and subsurface kernel functions in one dimensional DC resistivity field are shown. Solution of Laplace and nonlaplace equations together, solution of these equations using Frobeneous power series, solution of Laplace equations for a dipping contact and anisotropic medium are given.

In Chap. 9 use of complex variables and conformal transformation in potential theory has been demonstrated.A few simple examples of transformation in a complex domain are shown. Use of Schwarz-Christoffel method of conformal transformation in solving two dimensional potential problem of geophysical interest are discussed in considerable detail. A brief introduction is given on elliptic integrals and elliptic functions.

In Chap. 10 Green's theorem, it's first, second and third identities and corollaries of Green's theorem and Green's equivalent layers are discussed. Connecting relation between Green's theorem and Poisson's equation, estimation of mass from gravity field measurement, total normal induction in gravity field, two dimensional nature of the Green's theorem are given.

In Chap. 11 use of electrical images in solving simpler one dimensional potential problems for different electrotrode configurations are shown along with formation of multiple images.

In Chap. 12 after an elaborate introduction on electromagnetic waves and its application in geophysics, I have discussed about a few basic points on Electromagnetic waves, elliptic polarization, mutual inductance, Maxwell's equations, Helmholtz electromagnetic wave equations, propagation constant, skin depth, perturbation centroid frequency, Poynting vector, boundary conditions in electromagnetics, Hertz and Fitzerald vector potentials and their connections with electric and magnetic fields.

In Chap. 13 I have presented the simplest boundary value problems in electromagnetic wave propagations through homogenous half space. Boundary value problems in electromagnetic wave propagations, Plane wave propagation through layered earth (magnetotellurics), propagation of em waves due to vertical oscillating electric dipole, vertical oscillating magnetic dipole, horizontal oscillating magnetic dipole, an infinitely long line source are discussed showing the nature of solution of boundary value problems using the method of separation of variables. Electromagnetic response in the presence of conducting cylindrical and spherical inhomogeneities in an uniform field are discussed. Principle of electrodynamic similitude has been defined.

In Chap. 14 I have discussed the basic definition of Green's function and some of its properties including it's connection with potentials and fields, Fredhom's integral equations and kernel function and it's use for solution of Poisson.s equation.A few simplest examples for solution of potential problems are demonstrated. Basics of dyadics and dyadic Green;s function are given.

In Chap. 15 I have discussed the entry of numerical methods in potential theory. Finite difference, finite element and integral equation methods are mostly discussed. Finite difference formulation for surface and borehole geophysics in DC resistivity domain and for surface geophysics in plane wave electromagnetics (magnetotellurics) domain are discussed. Finite element formulation for surface geophysics in DC resistivity domain using Rayleigh-Ritz energy minimization method, finite element formulation for surface geophysics in magnetotellurics using Galerkin's method, finite element formulation for surface geophysics in magnetotellurics using advanced level elements, Galerkin's method and isoparametric elements are discussed. Integral equation method for surface geophysics in electromagnetics is mentioned briefly.

In Chap. 16 I have discussed on the different approaches of analytical continuation of potential field based on the class lecture notes and a few research papers of Prof. Amalendu Roy. In this chapter I have discussed the use of (a) harmonic analysis for downward continuation, (b) Taylor's series expansion and finite difference grids for downward continuation, (c) Green's theorem and integral equation in upward and downward continuation, (d) Integral equation and areal averages for downward continuation, (e) Integral equation and Lagrange's interpolation formula for analytical continuation.

In Chap. 17 I have discussed a few points on Inversion of Potential field data. In that I covered the following topics briefly, e.g., (a) singular value decomposition(SVD), (b) least squares estimator,(c) ridge regression estimator, (d) weighted ridge regression estimator, (e) minimum norm algorithm for an underdetermined problem, (f) Bachus Gilbert Inversion, (g) stochastic inversion, (h) Occam's inversion, (i) Global optimization under the following heads, (i) Montecarlo Inversion (ii) simulated annealing, (iii) genetic algorithm, (j) artificial neural network, (k) joint inversion.The topics are discussed briefly.Complete discussion on these subjects demands a separate book writing programme. Many more topics do exist besides whatever have been covered.

This book is dedicated to the name of Late Prof.P.K.Bhattacharyya and Late Prof. Amalendu Roy, our teachers, and both of them were great teachers and scholars in geophysics in India. Prof.Amalendu Roy has seen the first draft of the manuscript. I regret that Prof. Amalendu Roy did not survive to see the book in printed form. I requested him for writing the chapter on "Analytical Continuation of Potential Field Data" He however expressed his inability because of his poor health condition. He expired in December 2005 at the age of 81 years. I am grateful to our teacher late Prof. P. K. Bhattacharya whose inspiring teaching formed the basis of this book. He left a group of student to pursue research in future to push forward his ideas.

Towards completion of this monograph most of my students have lot of contributions in one form or the other. My students at doctoral level Dr. O. P. Rathi, Chief Geophysicist, Coal India Limited, Ranchi, India, Dr. D. J. Dutta, Senior Geophysicist, Schlumberger Well Surveying Corporation, Teheran, Iran, Dr. A. K. Singh, Scientist,Indian Institute of Geomagnetism, Mumbai, India, Dr. C. K. Rao, Scientist, Indian Institute of Geomagnetism, Mumbai, India, Dr. N. S. R. Murthy, Infosys, Bangalore, India, and masters level Dr. P. S. Routh, Assistant Professor of Geophysics, University of Boise at Idaho, USA, Dr. Anupama Venkata Raman, Geophysicist, Exxon, Houston, Texas, USA, Dr. Anubrati Mukherjee, Schlumberger, Mumbai, India, Dr. Mallika Mullick, Institute of Man and Enviroment, Kolkata, India, Mr. Priyank Jaiswal, Graduate Student, Rice University, Houston Texas, USA, Mr. Souvik Mukherjee, Ex Graduate student, University of Utah,Salt Lake City, USA Mrs. Tanima Dutta, graduate student Stanford University, USA have contributions towards development of this volume. My classmate Dr. K. Mallick of National Geophysical Research Institute, Hyderabad have some contribution in this volume. In the References I have included the works of all the scientists whose contributions have helped me in developing this manuscript. Those works are cited in the text.

The author is grateful to Dr. P. S. Routh, my son in law, for critically going through some of the chapters of this book and making some useful comments. I am grateful to my elder daughter Dr. Baishali Roy, senior geophysicist, Conoco Phillip, Houston, Texas, USA, for computer drafting of many diagrams of this book, collecting some reference materials and purchasing a few books for me, needed to write this monograph. I am grateful to my younger daughter Miss Debanjali Roy, research student, University of Miami, Florida, USA for collecting some literatures for me from the University library. I am grateful to Mr. Priyank Jaiswal, graduate student Rice University, Houston Texas for making arrangement for my visit to Rice University Library. I am grateful to Mr. M. Venkat at Katy, Texas for offering me car ride upto Rice University Library for an extended period. I am grateful to Mr. Subhobroto Sarkar, senior computer engineer, Dell,Salt lake Kolkata for his help in scanning the diagrams. His all round help in providing softwares to computer maintenance is gratefully acknowledged. I am grateful to Ms Lilly Chakraborty and Mr. Sudipta Saha of Printek Point, Technology Market, IIT, Kharagpur for typing the first draft of the manuscript. Second draft of the manuscript was typed jointly by Mr. Dilip Kumar Manna, Technology Cooperative Stores, IIT, Kharagpur and Mr. Rana Roy and his associates at High Tech Point, Jadavpur University Calcutta. I am grateful to Mr. S. P. Hazra, Department of Mining Engineering, Mr. Tapan Sarkar, Department of Geology and Geophysics and Mr. Mukti Ram Bose, Department of Electrical Communication Engineering and Radar Centre, all from IIT, Kharagpur for drafting many diagrams of the book. I am grateful to my wife for her patience and tolerating the troubles she faced for bringing home considerable amount of work and using a part of home as office space.

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I hope students of physical sciences may find some pages of their interest.

September, 2007

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1

Elements of Vector Analysis

Since foundation of potential theory in geophysics is based on scalar and vector potentials, a brief introductory note on vector analysis is given. Besides preliminaries of vector algebra, gradient divergence and curl are defined. Gauss's divergence theorem to convert a volume integral to a surface integral and Stoke's theorem to convert a surface integral to a line integral are given. A few well known relations in vector analysis are given as ready references.

1.1 Scalar & Vector

In vector analysis, we deal mostly with scalars and vectors.

Scalars: A quantity that can be identified only by its magnitude and sign is termed as a scalar. As for example distance temperature, mass and displacement are scalars.

Vector: A quantity that has both magnitude, direction and sense is termed as a vector. As for example: Force, field, velocity etc are vectors.

1.2 Properties of Vectors

(i) Sign of a vector. If $\mathrm{A\vec{B}}$ is vector $\vec{\mathrm{V}}$ then $\mathrm{B\vec{A}}$ is a vector $-\vec{\mathrm{V}}$

ii) The sum of two vectors (Fig. 1.1)

$$\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{AC}}. \tag{1.1}$$

Here

$$\overrightarrow{\mathrm{AB}} + \overrightarrow{\mathrm{BC}} = \overrightarrow{\mathrm{BC}} + \overrightarrow{\mathrm{AB}}. \tag{1.2}$$

iii) The difference of two vectors

$$\vec{A} - \vec{B} = \vec{A} + \left(-\vec{B}\right). \tag{1.3}$$

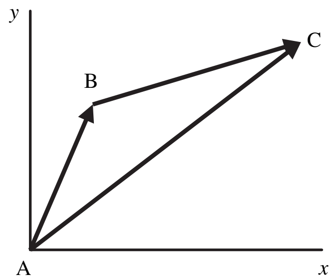

Fig. 1.1. Shows the resultant of two vectors

iv)

$$\vec{A} = b\vec{C} \tag{1.4}$$

i.e., the product of a vector and a scalar is a vector.

v) Unit Vector:
A unit vector is defined as a vector of unit magnitude along the three mutually perpendicular directions $\vec{i}, \vec{j}, \vec{k}$. Components of a vector along the x, y, z directions in a Cartesian coordinate are

$$\vec{A} = \vec{i}A_x + \vec{j}A_y + \vec{k}A_z. \tag{1.5}$$

vi) Vector Components: Three scalars A_x, A_y, and A_z are the three components in a cartisian coordinate system (Fig. 1.2). The magnitude of the vector A is $|A| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$.
When A makes specific angles α, β and γ with the three mutually perpendicular directions x, y and z, cosines of these angles are respectively given by (Fig. 1.3)

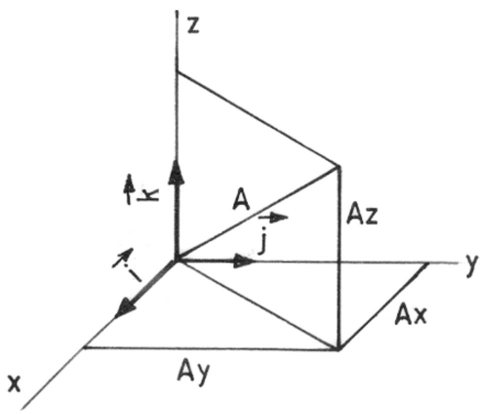

Fig. 1.2. Shows the three components of a vector in a Cartesian coordinate system

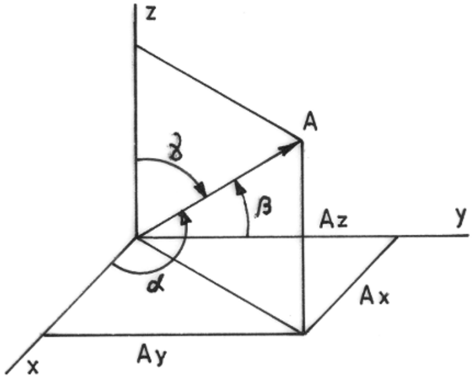

Fig. 1.3. Shows the direction cosines of a vector

$$\cos\alpha = \frac{Ax}{A}, \cos\beta = \frac{A_y}{A} \; and \; \cos\gamma = \frac{A_z}{A}. \tag{1.6}$$

In general cos α, cos β, cos γ are denoted as lx, ly and lz and they are known as direction cosines.

vii) Scalar product or dot product: The scalar product of two vectors is a scalar and is given by (Fig. 1.4)

$$\mathrm{A.B} = \mathrm{AB}\cos\theta \tag{1.7}$$

i.e. the product of two vectors multiplied by cosine of the angles between the two vectors. Some of the properties of dot product are

$$\begin{aligned} &\text{a)} \quad \mathrm{A.B} = \mathrm{B.A}, \\ &\text{b)} \quad \vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0 \text{ and} \\ &\text{c)} \quad \vec{\mathrm{i}}.\vec{\mathrm{i}} = \vec{\mathrm{j}}.\vec{\mathrm{J}} = \vec{\mathrm{k}}.\vec{\mathrm{k}} = 1. \end{aligned} \tag{1.8}$$

Here i, j, k are the unit vectors in the three mutually perpendicular directions.

$$\text{d)} \quad \mathrm{A.B} = \mathrm{A_xB_x} + \mathrm{A_yB_y} + \mathrm{A_zB_z}. \tag{1.9}$$

viii) Vector product or cross product:
The cross product or vector product of two vectors is a vector and its direction is at right angles to the directions of both the vectors (Fig. 1.5).

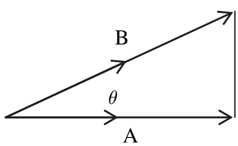

Fig. 1.4. Shows the scalar product of two vectors

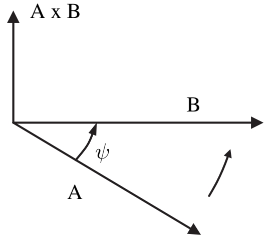

Fig. 1.5. Shows the vector product of two vectors

$$|\mathrm{A} \times \mathrm{B}| = \mathrm{AB}\sin\psi \tag{1.10}$$

where ψ is the angle between the two vectors A and B.

Some of the properties of cross product are

$$\begin{aligned}
&\text{a) } \vec{A} \times \vec{B} = -\vec{B} \times \vec{A},\\
&\text{b) } \vec{A} \times \vec{A} = 0,\\
&\text{c) } \vec{i} \times \vec{j} = \vec{k},\\
&\text{d) } \vec{j} \times \vec{k} = \vec{i},\\
&\text{e) } \vec{k} \times \vec{i} = \vec{j},
\end{aligned} \tag{1.11}$$

$$\begin{aligned}
&\text{f) } \vec{i} \times \vec{i} = 0,\\
&\text{g) } \vec{j} \times \vec{j} = 0,\\
&\text{h) } \vec{k} \times \vec{k} = 0 \quad \text{and}
\end{aligned} \tag{1.12}$$

i) $\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\,\vec{i} + (A_zB_x - A_xB_z)\,\vec{j} + (A_xB_y - A_yB_x)\,\vec{k}.$

In the matrix form, it can be written as

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \tag{1.13}$$

1.3 Gradient of a Scalar

Gradient of a scalar is defined as the maximum rate of change of any scalar function along a particular direction in a space domain. The gradient is a mathematical operation. It operates on a scalar function and makes it a vector. So the gradient has a direction. This direction coincides with the direction of the maximum slope or the maximum rate of change of any scalar function.

Let $\phi(x, y, z)$ be a scalar function of position in space of coordinate x, y, z. If the coordinates are increased by dx, dy and dz, (Fig. 1.6) then

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz. \tag{1.14}$$

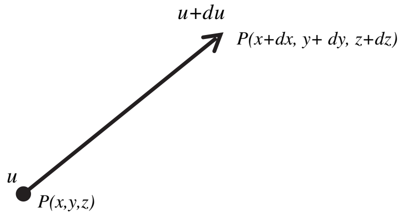

Fig. 1.6. Change of position of a scalar function in a field

If we assume the displacement to be dr, then

$$\overrightarrow{\mathrm{dr}} = \vec{\mathrm{i}}\mathrm{dx} + \vec{\mathrm{j}}\mathrm{dy} + \vec{\mathrm{k}}\mathrm{dz}. \tag{1.15}$$

In vector algebra, the differential operator ∇ is defined as

$$\vec{\nabla} = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z} \tag{1.16}$$

and the gradient of a scalar function is defined as

$$\mathrm{grad}\ \phi = \vec{\mathrm{i}}\frac{\partial\phi}{\partial \mathrm{x}} + \vec{\mathrm{j}}\frac{\partial\phi}{\partial \mathrm{y}} + \vec{\mathrm{k}}\frac{\partial\phi}{\partial \mathrm{z}}. \tag{1.17}$$

The operator ∇ also when operates on a scalar function $\phi(\mathrm{x},\mathrm{y},\mathrm{z})$, we get

$$\vec{\nabla}\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \tag{1.18}$$

where $\frac{\partial\phi}{\partial \mathrm{x}}, \frac{\partial\phi}{\partial \mathrm{y}}$ and $\frac{\partial\phi}{\partial \mathrm{z}}$ are the rates of change of a scalar function along the three mutually perpendicular directions. We can now write

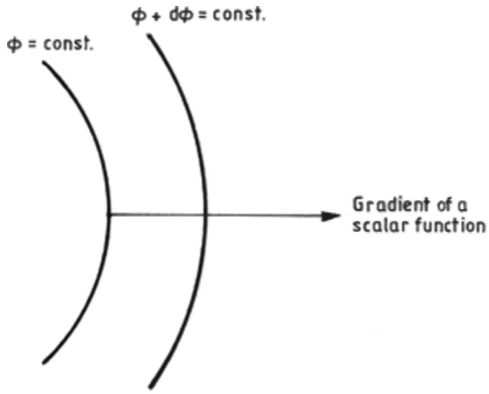

Fig. 1.7. Gradient of a scalar function, the direction of maximum rate of change of a function: Orthogonal to the equipotential lines or surface

$$d\phi = \left(\vec{i}\frac{\partial\phi}{\partial y} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)\left(\vec{i}dx + \vec{j}dy + \vec{k}dz\right) \tag{1.19}$$

$$= (\nabla\phi).\mathrm{dr}$$

where dr is along the normal of the scalar function $\phi(x, y, z) = \text{constant}$. We get the gradient of a scalar function as $d\phi = (\nabla\phi).dr = 0$, when the vector $\nabla\phi$ is normal to the surface $\phi = \text{constant}$. It is also termed as grad ϕ or the gradient of ϕ. (Fig. 1.7).

1.4 Divergence of a Vector

Divergence of a vector is a scalar or dot product of a vector operator ∇ and a vector $\vec{A}$ gives a scalar. That is

$$\nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = div\vec{A}. \tag{1.20}$$

This concept of divergence has come from fluid dynamics. Consider a fluid of density $\rho(x, y, z, t)$ is flowing with a velocity V (x, y, z, t). and let $V = v\rho.v$ is the volume. If S is the cross section of a plane surface (Fig. 1.8) then V.S is the mass of the fluid flowing through the surface in an unit time (Pipes, 1958).

Let us assume a small parallelepiped of dimension dx, dy and dz. Mass of the fluid flowing through the face F_1 per unit time is $V_y dx\, dz = (\rho v)_y$ dx dz$(S = dxdz)$.

Fluids going out of the face F_2 is

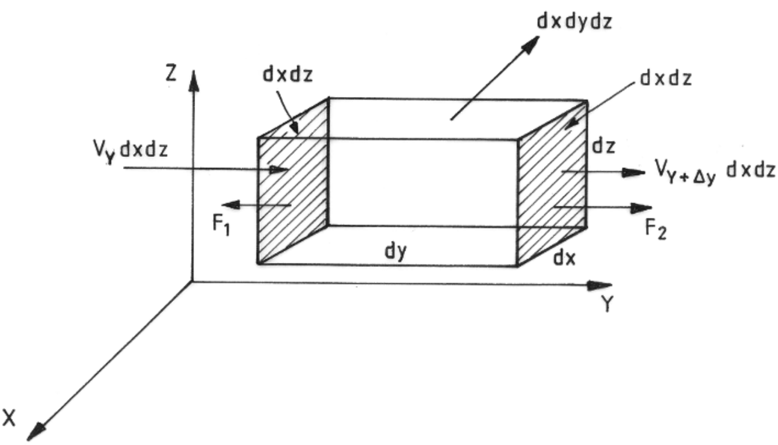

Fig. 1.8. Inflow and out flow of fluid through a parallelepiped to show the divergence of a vector

$$\mathrm{V_{y+dy}dx\ dz} = \left(\mathrm{Vy} + \frac{\partial \mathrm{Vy}}{\partial \mathrm{y}}\mathrm{dy}\right)\mathrm{dx\ dz}. \tag{1.21}$$

Hence the net increase of mass of the fluid per unit time is

$$\mathrm{V_y dx\ dz} - \left(V_y + \frac{\partial V_y}{\partial y}\right) dxdz = \frac{\partial V_y}{\partial y} dxdydz. \tag{1.22}$$

Considering the increase of mass of fluid per unit time entering through the other two pairs of faces, we obtain

$$-\left(\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}\right) dxdydz = -(\nabla.\mathrm{V})\mathrm{dx\ dy\ dz} \tag{1.23}$$

as the total increase in mass of fluid per unit time. According to the principle of conservation of matter, this must be equal to the rate of increase of density with time multiplied by the volume of the parallelepiped.

Hence

$$-(\nabla.\mathrm{V})\mathrm{dx\ dy\ dz} = \left(\frac{\partial \rho}{\partial \mathrm{t}}\right)\mathrm{dx\ dy\ dz}. \tag{1.24}$$

Therefore

$$\nabla.\mathrm{V} = -\frac{\partial \rho}{\partial \mathrm{t}}. \tag{1.25}$$

This is known as the equation of continuity in a fluid flow field. This concept is also valid in other fields, viz. direct current flow field, heat flow field etc. Divergence represents the flow outside a volume whether it is a charge or a mass. Divergence of a vector is a dot product between the vector operator ∇ and a vector V and ultimately it generates a scalar.

1.5 Surface Integral

Consider a surface as shown in the (Fig. 1.9). The surface is divided into the representative vectors $\mathrm{ds_1, ds_2, ds_3}$.... etc (Pipes, 1958).

Let $\mathrm{V_1}$ be the value of the vector function of position $\mathrm{V_1(x, y, z)}$ at $\mathrm{ds_i}$.

Then

$$\underset{\substack{\Delta s \to 0 \\ n \to \infty}}{Lim} \sum_{i=1}^{n} V_i dS_i = \iint V.dS. \tag{1.26}$$

The sign of the integral depends on which face of the surface is taken positive. If the surface is closed, the outward normal is taken as positive.

Since

$$d\vec{S} = \vec{i}\ dS_x + \vec{j}\ dS_y + \vec{k}\ dS_z, \tag{1.27}$$

we can write

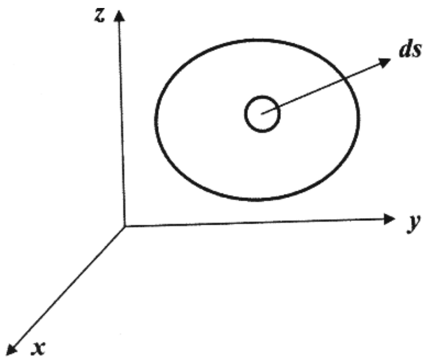

Fig. 1.9. Shows the surface integral as a vector

$$\iint_S \mathrm{V.ds} = \iint_S (\mathrm{V_x\ ds_x + V_y\ ds_y + V_z\ ds_z}). \quad (1.28)$$

Surface integral of the vector V is termed as the flux of V through out the surface.

1.6 Gauss's Divergence Theorem

Gauss's divergence theorem states that volume integral of divergence of a vector A taken over any volume V is equal to the surface integral of A taken over a closed surface surrounding the volume V, i.e.,

$$\iiint_V \left(\nabla.\vec{A}\right) dv = \iint_S \vec{A}.ds. \quad (1.29)$$

Therefore it is an important relation by which one can change a volume integral to a surface integral and vice versa. We shall see the frequent application of this theorem in potential theory.

Gauss's theorem can be proved as follows. Let us expand the left hand side of the (1.29) as

$$\begin{aligned}\iiint_V (\nabla.A)\, dv &= \iiint \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dxdydz \\ &= \iiint_V \frac{\partial A_x}{\partial x}.dxdydz + \iiint_V \frac{\partial A_y}{\partial y} dxdydz \\ &\quad + \iiint_V \frac{\partial A_z}{\partial z} dxdydz. \end{aligned} \quad (1.30)$$

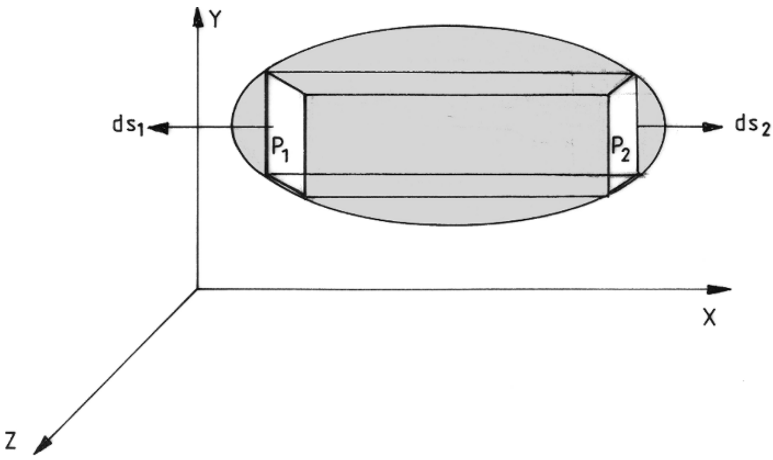

Fig. 1.10. Shows the divergence of a vector

Let us take the first integral on the right hand side. We can now integrate the first integral with respect to x, i.e., along a strip of cross section dy dz extending from P_1 to P_2.(Fig. 1.10).

We thus obtain

$$\iiint_v \frac{\partial A_x}{\partial x} dxdydz = \iint_s \left[A_x\left(x_2, y, z\right) - A_x\left(x_1, y, z\right)\right] dydz. \tag{1.31}$$

Here (x_1, y, z) and (x_2, y, z) are respectively the coordinates of P_1 and P_2. At P_1, we have

$$dy\ dz = -dS_x,$$

and at P_2

$$dy\ dz = dS_x. \tag{1.32}$$

Because the direction of the surface vectors are in the opposite direction.

Therefore

$$\iiint_v \frac{\partial A_x}{\partial x} dxdydz = \iint_s A_x\ dS_x. \tag{1.33}$$

where the surface integral on the right hand side is evaluated on the whole surface. This way we can get

$$\iiint_v \frac{\partial A_y}{\partial y} dxdydz = \iint_s A_y\ dS_y \text{ and} \tag{1.34}$$

$$\iiint_v \frac{\partial A_z}{\partial z} dxdydz = \iint_s A_z\ dS_z. \tag{1.35}$$