Ergebnisse der Mathematik
und ihrer Grenzgebiete

Volume 34

3. Folge

A Series of Modern Surveys
in Mathematics

Michael Struwe

Variational Methods

Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems

Fourth Edition

Springer

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ISBN 978-3-540-74012-4 e-ISBN 978-3-540-74013-1

DOI 10.1007/978-3-540-74013-1

Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge / A Series of Modern Surveys in Mathematics ISSN 0071-1136

Library of Congress Control Number: 2008923744

Mathematics Subject Classification (2000): 58E05, 58E10, 58E12, 58E30, 58E35, 34C25, 34C35, 35A15, 35K15, 35K20, 35K22, 58F05, 58F22, 58G11

Typesetting: by the author using a Springer TEX macro package
Production: LE-TEX Jelonek, Schmidt & Vöckler GbR, Leipzig
Cover design: WMX Design GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

Preface to the Fourth Edition

Almost twenty years after conception of the first edition, it was a challenge to prepare an updated version of this text on the Calculus of Variations. The field has truely advanced dramatically since that time, to an extent that I find it impossible to give a comprehensive account of all the many important developments that have occurred since the last edition appeared. Fortunately, an excellent overview of the most significant results, with a focus on functional analytic and Morse theoretical aspects of the Calculus of Variations, can be found in the recent survey paper by Ekeland-Ghoussoub [1]. I therefore have only added new material directly related to the themes originally covered.

Even with this restriction, a selection had to be made. In view of the fact that flow methods are emerging as the natural tool for studying variational problems in the field of Geometric Analysis, an emphasis was placed on advances in this domain. In particular, the present edition includes the proof for the convergence of the Yamabe flow on an arbitrary closed manifold of dimension $3 \leq m \leq 5$ for initial data allowing at most single-point blow-up. Moreover, we give a detailed treatment of the phenomenon of blow-up and discuss the newly discovered results for backward bubbling in the heat flow for harmonic maps of surfaces.

Aside from these more significant additions, a number of smaller changes have been made throughout the text, thereby taking care not to spoil the freshness of the original presentation. References have been updated, whenever possible, and several mistakes that had survived the past revisions have now been eliminated. I would like to thank Silvia Cingolani, Irene Fonseca, Emmanuel Hebey, and Maximilian Schultz for helpful comments in this regard. Moreover, I am indebted to Gilles Angelsberg, Ruben Jakob, Reto Müller, and Melanie Rupflin, for carefully proof-reading the new material.

Zürich, July 2007 *Michael Struwe*

Preface to the Third Edition

The Calculus of Variations continues to be an area of very rapid growth. Variational methods are indispensable as a tool in mathematical physics and geometry.

Results on Ginzburg-Landau type variational problems inspire research on the related Seiberg-Witten functional on a Kähler surface and invite speculations about possible applications in topology (Ding-Jost-Li-Peng-Wang [1]).

Variational methods are applied in cosmology, as in the recent work of Fortunato-Giannoni-Masiello [1] and Giannoni-Masiello-Piccione [1] on geodesics in Lorentz manifolds and gravitational lenses.

Applications to Hamiltonian dynamics now include a proof of the Seifert conjecture on brake orbits (Giannoni [1]) and results on homoclinic and heteroclinic solutions (Coti Zelati-Ekeland-Séré [1], Rabinowitz [1], Séré [1]) with interesting counterparts in the field of semilinear elliptic equations (Coti-Zelati-Rabinowitz [1], Rabinowitz [13]).

The Calculus of Variations also has advanced on a more technical level. Campa-Degiovanni [1], Corvellec-Degiovanni-Marzocchi [1], Degiovanni-Marzocchi [1], Ioffe [1], and Ioffe-Schwartzman [1] have extended critical point theory to functionals on metric spaces, with applications, for instance, to quasilinear elliptic equations (Arioli [1], Arioli-Gazzola [1], Canino-Degiovanni [1]).

Bolle [1] has proposed a new approach to perturbation theory, as treated in Section II.7 of this monograph. Numerous applications are studied in Bolle-Ghoussoub-Tehrani [1].

The method of parameter dependence as in Sections I.7 and II.9 has found further striking applications in Chern-Simons theory (Struwe-Tarantello [1]) and independently for a related problem in mean field theory (Ding-Jost-Li-Wang [1]). Inspired by these results, Wang-Wei [1] were able to solve a problem in chemotaxis with a similar structure. Jeanjean [1] and Jeanjean-Toland [1] have discovered an abstract setting where parameter dependence may be exploited.

Ambrosetti [1], Ambrosetti-Badiale-Cingolani [1], and Ambrosetti-Badiale [1], [2] have found new applications of variational methods in bifurcation theory, refining the classical results of Böhme [1] and Marino [1]. In Ambrosetti-Garcia Azorero-Peral [1] these ideas are applied to obtain precise existence results for conformal metrics of prescribed scalar curvature close to a constant, which shed new light on the work of Bahri-Coron [1], [2], Chang-Yang [1] quoted in Section III.4.11.

The field of critical equations as in Chapter III has been particularly active.

Concentration profiles for Palais-Smale sequences as in Theorem III.3.1 have been studied in more detail by Rey [1] and Flucher [1].

Quite surprisingly, results analogous to Theorem III.3.1 have been discovered also for sequences of solutions to critical semilinear wave equations (Bahouri-Gérard [1]).

For the semilinear elliptic equations of critical exponential growth related to the Moser-Trudinger inequality on a planar domain the patterns for existence and non-existence results are strikingly analogous to the higher dimensional case (Adimurthi [1], Adimurthi-Srikanth-Yadava [1]), and, on a macroscopic scale, quantization phenomena analogous to Theorem III.3.1 are observed for concentrating solutions of semilinear equations with exponential growth (Brezis-Merle [1], Li-Shafrir [1]). However, results of Struwe [17] and Ogawa-Suzuki [1] on the one hand and an example by Adimurthi-Prashanth [1] on the other suggest that there may be many qualitatively distinct types of blow-up behavior for Palais-Smale sequences in this case. Still, Theorem III.3.1 remains valid for solutions (Adimurthi-Struwe [1]) and also the analogue of Theorem III.3.4 has been obtained (Struwe [25]). The many similarities and subtle differences to the critical semilinear equations in higher dimensions make this field particularly attractive for further study.

References have been updated and a small number of mistakes have been rectified. I am indepted to Gerd Müller, Paul Rabinowitz, and Henry Wente for their comments.

Zürich, July 1999 *Michael Struwe*

Preface to the Second Edition

During the short period of five years that have elapsed since the publication of the first edition a number of interesting mathematical developments have taken place and important results have been obtained that relate to the theme of this book.

First of all, as predicted in the Preface to the first edition, Morse theory, indeed, has gone through a dramatic change, influenced by the work by Andreas Floer on Hamiltonian systems and in particular, on the Arnold conjecture. There are now also excellent accounts of these developments and their ramifications; see, in particular, the monograph by Matthias Schwarz [1]. The book by Hofer-Zehnder [2] on Symplectic Geometry shows that variational methods and, in particular, Floer theory have applications that range far beyond the classical area of analysis.

Second, as a consequence of an observation by Stefan Müller [1] which prompted the seminal work of Coifman-Lions-Meyer-Semmes [1], Hardy spaces and the space BMO are now playing a very important role in weak convergence results, in particular, when dealing with problems that exhibit a special (determinant) structure. A brief discussion of these results and some model applications can be found in Section I.3.

Moreover, variational problems depending on some real parameter in certain cases have been shown to admit rather surprising a-priori bounds on critical points, with numerous applications. Some examples will be given in Sections I.7 and II.9.

Other developments include the discovery of Hamiltonian systems with no periodic orbits on some given energy hypersurface, due to Ginzburg and Herman, and the discovery, by Chang-Ding-Ye, of finite time blow-up for the evolution problem for harmonic maps of surfaces, thus completing the results in Sections II.8, II.9 and III.6, respectively.

A beautiful recent result of Ye concerns a new proof of the Yamabe theorem in the case of a locally conformally flat manifold. This proof is presented in detail in Section III.4 of this new edition.

In view of their numerous and wide-ranging applications, interest in variational methods is very strong and growing. Out of the large number of recent publications in the general field of the calculus of variations and its applications some 50 new references have been added that directly relate to one of the themes in this monograph.

Owing to the very favorable response with which the first edition of this book was received by the mathematical community, the publisher has suggested that a second edition be published in the Ergebnisse series. It is a pleasure to thank all the many mathematicians, colleagues, and friends who

have commented on the first edition. Their enthusiasm has been highly inspiring. Moreover, I would like to thank, in particular, Matts Essen, Martin Flucher and Helmut Hofer for helpful suggestions in preparing this new edition.

All additions and changes to the first edition were carefully implemented by Suzanne Kronenberg, using the Springer TeX-Macros package, and I gratefully acknowledge her help.

Zürich, June 1996 *Michael Struwe*

Preface to the First Edition

It would be hopeless to attempt to give a complete account of the history of the calculus of variations. The interest of Greek philosophers in isoperimetric problems underscores the importance of "optimal form" already in ancient cultures; see Hildebrandt-Tromba [1] for a beautiful treatise of this subject. While variational problems thus are part of our classical cultural heritage, the first modern treatment of a variational problem is attributed to Fermat, see Goldstine [1; p. 1]. Postulating that light follows a path of least possible time, in 1662 Fermat was able to derive the laws of refraction, thereby using methods which may already be termed analytic.

With the development of the Calculus by Newton and Leibniz, the basis was laid for a more systematic development of the calculus of variations. The brothers Johann and Jakob Bernoulli and Johann's student Leonhard Euler, all from the city of Basel in Switzerland, were to become the "founding fathers" (Hildebrandt-Tromba [1; p. 21]) of this new discipline. In 1743 Euler [1] submitted "A method for finding curves enjoying certain maximum or minimum properties", published in 1744, the first textbook on the calculus of variations. In an appendix to this book Euler [1; Appendix II, p. 298] expresses his belief that "every effect in nature follows a maximum or minimum rule" (see also Goldstine [1; p. 106]), a credo in the universality of the calculus of variations as a tool. The same conviction also shines through Maupertuis' [1] work on the famous "least action principle", also published in 1744. (In retrospect, however, it seems that Euler was the first to observe this important principle. See for instance Goldstine [1; p. 67 f. and p. 101 ff.] for a more detailed historical account.) Euler's book was a great source of inspiration for generations of mathematicians following.

Major contributions were made by Lagrange, Legendre, Jacobi, Clebsch, Mayer, and Hamilton to whom we owe what we now call "Euler-Lagrange equations", the "Jacobi differential equation" for a family of extremals, or "Hamilton-Jacobi theory".

The use of variational methods was not at all limited to one-dimensional problems in the mechanics of mass-points. In the 19^{th} century variational methods also were employed for instance to determine the distribution of an electrical charge on the surface of a conductor from the requirement that the energy of the associated electrical field be minimal ("Dirichlet's principle"; see Dirichlet [1] or Gauss [1]) or were used in the construction of analytic functions (Riemann [1]).

However, none of these applications was carried out with complete rigor. Often the model was confused with the phenomenon that it was supposed to describe and the fact (?) that for instance in nature there always exists an equilibrium distribution for an electrical charge on a conducting surface was taken as sufficient evidence for the corresponding mathematical problem to have a solution. A typical reasoning reads as follows:

"In any event therefore the integral will be non-negative and hence there must exist a distribution (of charge) for which this integral assumes its minimum value," (Gauss [1; p. 232], translation by the author).

However, towards the end of the 19$^{\text{th}}$ century progress in abstraction and a better understanding of the foundations of the calculus opened such arguments to criticism. Soon enough, Weierstrass [1; pp. 52–54] found an example of a variational problem that did not admit a minimum solution. Weierstrass challenged his colleagues to find a continuously differentiable function $u\colon [-1,1] \to \mathbb{R}$ minimizing the integral

$$I(u) = \int_{-1}^{1} \left| x \frac{d}{dx} u \right|^2 \, dx$$

subject (for instance) to the boundary conditions $u(\pm 1) = \pm 1$. Choosing

$$u_\varepsilon(x) = \frac{\arctan(\frac{x}{\varepsilon})}{\arctan(\frac{1}{\varepsilon})}, \ \varepsilon > 0,$$

as a family of comparison functions, Weierstrass was able to show that the infinium of I in the above class was 0; however, the value 0 is not attained. (See also Goldstine [1; p. 371 f.].) Weierstrass' critique of Dirichlet's principle precipitated the calculus of variations into a Grundlagenkrise comparable to the crisis in set theory and logic after Russel's discovery of antinomies in Cantor's set theory or Gödel's incompleteness proof.

However, through the combined efforts of several mathematicians who did not want to give up the wonderful tool that Dirichlet's principle had been – including Weierstrass, Arzéla, Fréchet, Hilbert, and Lebesgue – the calculus of variations was revalidated and emerged from its crisis with new strength and vigor.

Hilbert's speech at the centennial assembly of the International Congress 1900 in Paris, where he proposed his famous 20 problems – two of which were devoted to questions related to the calculus of variatons – marks this newly found confidence.

In fact, following Hilbert's [1] and Lebesgue's [1] solution of the Dirichlet problem, a development began which within a few decades brought tremendous success, highlighted by the 1929 theorem of Ljusternik and Schnirelman [1] on the existence of three distinct prime closed geodesics on any compact surface of genus zero, or the 1930/31 solution of Plateau's problem by Douglas [1], [2] and Radò [1].

The Ljusternik-Schnirelman result (and a previous result by Birkhoff [1], proving the existence of one closed geodesic on a surface of genus 0) also marks the beginning of global analyis. This goes beyond Dirichlet's principle as we no longer consider only minimizers (or maximizers) of variational integrals, but instead look at all their critical points. The work of Ljusternik and Schnirelman revealed that much of the complexity of a function space is invariably reflected in the set of critical points of any variational integral defined on it, an idea whose importance for the further development of mathematics can hardly be overestimated, whose implications even today may only be conjectured, and whose applications seem to be virtually unlimited. Later, Ljusternik and Schnirelman [2] laid down the foundations of their method in a general theory. In honor of their pioneering effort any method which seeks to draw information concerning the number of critical points of a functional from topological data today often is referred to as Ljusternik-Schnirelman theory.

Around the time of Ljusternik and Schnirelman's work, another – equally important – approach towards a global theory of critical points was pursued by Marston Morse [2]. Morse's work also reveals a deep relation between the topology of a space and the number and types of critical points of any function defined on it. In particular, this led to the discovery of unstable minimal surfaces through the work of Morse-Tompkins [1], [2] and Shiffman [1], [2]. Somewhat reshaped and clarified, in the 50's Morse theory was highly successful in topology (see Milnor [1] and Smale [1]). After Palais [1], [2] and Smale [2] in the 60's succeeded in generalizing Milnor's constructions to infinite-dimensional Hilbert manifolds – see also Rothe [1] for some early work in this regard – Morse theory finally was recognized as a useful (and usable) instrument also for dealing with partial differential equations.

However, applications of Morse theory seemed somewhat limited in view of prohibitive regularity and non-degeneracy conditions to be met in a variational problem, conditions which – by the way – were absent in Morse's original work. Today, inspired by the deep work of Conley [1], Morse theory seems to be turning back to its origins again. In fact, a Morse-Conley theory is emerging which one day may provide a tool as universal as Ljusternik-Schnirelman theory and still offer an even better resolution of the relation between the critical set of a functional and topological properties of its domain. However, in spite of encouraging results, for instance by Benci [4], Conley-Zehnder [1], Jost-Struwe [1], Rybakowski [1], [2], Rybakowski-Zehnder [1], Salamon [1], and – in particular – Floer [1], a general theory of this kind does not yet exist.

In these notes we want to give an overview of the state of the art in some areas of the calculus of variations. Chapter I deals with the classical direct methods and some of their recent extensions. In Chapters II and III we discuss minimax methods, that is, Ljusternik-Schnirelman theory, with an emphasis on some limiting cases in the last chapter, leaving aside the issue of Morse theory whose face is currently changing all too rapidly.

Examples and applications are given to semilinear elliptic partial differential equations and systems, Hamiltonian systems, nonlinear wave equations, and problems related to harmonic maps of Riemannian manifolds or surfaces of prescribed mean curvature. Although our selection is of course biased by the interests of the author, an effort has been made to achieve a good balance between different areas of current research. Most of the results are known;

some of the proofs have been reworked and simplified. Attributions are made to the best of the author's knowledge. No attempt has been made to give an exhaustive account of the field or a complete survey of the literature.

General references for related material are Berger-Berger [1], Berger [1], Chow-Hale [1], Eells [1], Nirenberg [1], Rabinowitz [11], Schwartz [2], Zeidler [1]; in particular, we recommend the recent books by Ekeland [2] and Mawhin-Willem [1] on variational methods with a focus on Hamiltonian systems and the forthcoming works of Chang [7] and Giaquinta-Hildebrandt. Besides, we mention the classical textbooks by Krasnoselskii [1] (see also Krasnoselskii-Zabreiko [1]), Ljusternik-Schnirelman [2], Morse [2], and Vainberg [1]. As for applications to Hamiltonian systems and nonlinear variational problems, the interested reader may also find additional references on a special topic in these fields in the short surveys by Ambrosetti [2], Rabinowitz [9], or Zehnder [1].

The material covered in these notes is designed for advanced graduate or Ph.D. students or anyone who wishes to acquaint himself with variational methods and possesses a working knowledge of linear functional analysis and linear partial differential equations. Being familiar with the definitions and basic properties of Sobolev spaces as provided for instance in the book by Gilbarg-Trudinger [1] is recommended. However, some of these prerequisites can also be found in the appendix.

In preparing this manuscript I have received help and encouragement from a number of friends and colleagues. In particular, I wish to thank Proff. Herbert Amann and Hans-Wilhelm Alt for helpful comments concerning the first two sections of Chapter I. Likewise, I am indebted to Prof. Jürgen Moser for useful suggestions concerning Section I.4 and to Proff. Helmut Hofer and Eduard Zehnder for advice on Sections I.6, II.5, and II.8, concerning Hamiltonian systems.

Moreover, I am grateful to Gabi Hitz, Peter Bamert, Jochen Denzler, Martin Flucher, Frank Josellis, Thomas Kerler, Malte Schünemann, Miguel Sofer, Jean-Paul Theubet, and Thomas Wurms for going through a set of preliminary notes for this manuscript with me in a seminar at ETH Zürich during the winter term of 1988/89. The present text certainly has profited a great deal from their careful study and criticism.

Special thanks I also owe to Kai Jenni for the wonderful typesetting of this manuscript with the TEX text processing system.

I dedicate this book to my wife Anne.

Zürich, January 1990 *Michael Struwe*

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Glossary of Notations

V, V^*	generic Banach space with dual V^*
$\|\cdot\|$	norm in V
$\|\cdot\|_*$	induced norm in V^*, often also denoted $\|\cdot\|$
$\langle\cdot,\cdot\rangle: V \times V^* \to \mathbb{R}$	dual pairing, occasionally also used to denote scalar product in $\mathbb{R}^n$
E	generic energy functional
DE	Fréchet derivative
$\mathrm{Dom}(E)$	domain of E
$\langle v, DE(u)\rangle = DE(u)v = D_vE(u)$	directional derivative of E at u in direction v
$L^p(\Omega;\mathbb{R}^n)$	space of Lebesgue-measurable functions $u: \Omega \to \mathbb{R}^n$ with finite L^p-norm $$\|u\|_{L^p} = \Big(\int_\Omega \|u\|^p \, dx\Big)^{1/p},\ 1 \le p < \infty$$
$L^\infty(\Omega;\mathbb{R}^n)$	space of Lebesgue-measurable and essentially bounded functions $u: \Omega \to \mathbb{R}^n$ with norm $$\|u\|_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} \|u(x)\|$$
$H^{m,p}(\Omega;\mathbb{R}^n)$	Sobolev space of functions $u \in L^p(\Omega;\mathbb{R}^n)$ with $\|\nabla^k u\| \in L^p(\Omega)$ for all $k \in \mathbb{N}_0^n, \|k\| \le m$, with norm $\|u\|_{H^{m,p}} = \sum_{0\le\|k\|\le m} \|\nabla^k u\|_{L^p}$
$H_0^{m,p}(\Omega;\mathbb{R}^n)$	completion of $C_0^\infty(\Omega;\mathbb{R}^n)$ in the norm $\|\cdot\|_{H^{m,p}}$; if Ω is bounded an equivalent norm is given by $\|u\|_{H_0^{m,p}} = \sum_{\|k\|=m} \|\nabla^k u\|_{L^p}$
$H^{-m,q}(\Omega;\mathbb{R}^n)$	dual of $H_0^{m,p}(\Omega;\mathbb{R}^n)$, where $\frac{1}{p} = \frac{1}{q} = 1$; q is omitted, if $p = q = 2$
$D^{m,p}(\Omega;\mathbb{R}^n)$	completion of $C_0^\infty(\Omega;\mathbb{R}^n)$ in the norm $\|u\|_{D^{m,p}} = \sum_{\|k\|=m} \|\nabla^k u\|_{L^p}$

$C^{m,\alpha}(\Omega;\mathbb{R}^n)$	space of m times continuously differentiable functions $u\colon \Omega \to \mathbb{R}^n$ whose mth order derivatives are Hölder continuous with exponent $0 \leq \alpha \leq 1$
$C_0^\infty(\Omega;\mathbb{R}^n)$	space of smooth functions $u\colon \Omega \to \mathbb{R}^n$ with compact support in Ω
$\operatorname{supp}(u) = \overline{\{x \in \Omega\ ;\ u(x) \neq 0\}}$	support of a function $u\colon \Omega \to \mathbb{R}^n$
$\Omega' \subset\subset \Omega$	the closure of Ω' is compact and contained in Ω
$\llcorner$	restriction of a measure
$\mathcal{L}^n$	Lebesgue measure on $\mathbb{R}^n$
$B_\rho(u;V) = \{v \in V\ ;\ \|u - v\| < \rho\}$	open ball of radius ρ around $u \in V$; in particular, if $V = \mathbb{R}^n$, then $B_\rho(x_0) = B_\rho(x_0;\mathbb{R}^n)$, $B_\rho = B_\rho(0)$
Re	real part
Im	imaginary part
c, C	generic constants
Cross-references	$(N.x.y)$ refers to formula (x,y) in Chapter N $(x.y)$ within Chapter N refers to formula $(N.x.y)$

Chapter I

The Direct Methods in the Calculus of Variations

Many problems in analysis can be cast into the form of functional equations $F(u) = 0$, the solution u being sought among a class of admissible functions belonging to some Banach space V.

Typically, these equations are nonlinear; for instance, if the class of admissible functions is restricted by some (nonlinear) constraint.

A particular class of functional equations is the class of Euler-Lagrange equations

$$DE(u) = 0$$

for a functional E on V, which is Fréchet-differentiable with derivative DE. We say such equations are of *variational form.*

For equations of variational form an extensive theory has been developed, and variational principles play an important role in mathematical physics and differential geometry, optimal control and numerical anlysis.

We briefly recall the basic definitions that will be needed in this and the following chapters, see Appendix C for details: Suppose E is a Fréchet-differentiable functional on a Banach space V with normed dual V^* and duality pairing $\langle \cdot, \cdot \rangle : V \times V^* \to \mathbb{R}$, and let $DE : V \to V^*$ denote the Fréchet-derivative of E. Then the directional (Gateaux-) derivative of E at u in the direction of v is given by

$$\frac{d}{d\varepsilon} E(u + \varepsilon v)\Big|_{\varepsilon = 0} = \langle v, DE(u) \rangle = DE(u)\, v.$$

For such E, we call a point $u \in V$ *critical* if $DE(u) = 0$; otherwise, u is called *regular*. A number $\beta \in \mathbb{R}$ is a *critical value* of E if there exists a critical point u of E with $E(u) = \beta$. Otherwise, β is called *regular*. Of particular interest (also in the non-differentiable case) will be relative minima of E, possibly subject to constraints. Recall that for a set $M \subset V$ a point $u \in M$ is an *absolute minimizer* for E on M if for all $v \in M$ there holds $E(v) \geq E(u)$. A point $u \in M$ is a *relative minimizer* for E on M if for some neighborhood U of u in V it is absolutely E-minimizing in $M \cap U$. Moreover, in the differentiable case, we shall also be interested in the existence of *saddle points*, that is, critical points u of E such that any neighborhood U of u in V contains points v, w such that $E(v) < E(u) < E(w)$. In physical systems, saddle points appear as unstable equilibria or transient excited states.

In this chapter we review some basic methods for proving the existence of relative minimizers. Somewhat imprecisely we summarily refer to these methods as the *direct methods* in the calculus of variations. However, besides the classical lower semi-continuity and compactness method we also include the compensated compactness method of Murat and Tartar, and the concentration-compactness principle of P.L. Lions. Moreover, we recall Ekeland's variational principle and the duality method of Clarke and Ekeland.

Applications will be given to problems concerning minimal hypersurfaces, semilinear and quasi-linear elliptic boundary value problems, finite elasticity, Hamiltonian systems, and semilinear wave equations.

From the beginning it will be apparent that in order to achieve a satisfactory existence theory the notion of solution will have to be suitably relaxed. Hence, in general, the above methods will at first only yield generalized or "weak" solutions of our problems. A second step often will be necessary to show that these solutions are regular enough to be admitted as classical solutions. The regularity theory in many cases is very subtle and involves a delicate machinery. It would go beyond the scope of this book to cover this topic completely. However, for the problems that we will mostly be interested in, the regularity question can be dealt with rather easily. The reader will find this material in Appendix B. References to more advanced texts on the regularity issue will be given where appropriate.

1. Lower Semi-continuity

In this section we give sufficient conditions for a functional to be bounded from below and to attain its infimum.

The discussion can be made largely independent of any differentiability assumptions on E or structure assumptions on the underlying space of admissible functions M. In fact, we have the following classical result.

1.1 Theorem. *Let M be a topological Hausdorff space, and suppose $E : M \to \mathbb{R} \cup +\infty$ satisfies the condition of bounded compactness:*

$$\text{For any } \alpha \in \mathbb{R} \text{ the set } \quad K_\alpha = \{u \in M \; ; \; E(u) \le \alpha\} \quad \text{is compact (Heine-Borel property).} \tag{1.1}$$

Then E is uniformly bounded from below on M and attains its infimum. The conclusion remains valid if instead of (1.1) we suppose that any sub-level set K_α is sequentially compact.

Remark. Necessity of condition (1.1) is illustrated by simple examples: The function $E: [-1, 1] \to \mathbb{R}$ given by $E(x) = x^2$ if $x \neq 0$, $E(x) = 1$ if $x = 0$, or the exponential function $E(x) = \exp(x)$ on $\mathbb{R}$ are bounded from below but do not admit a minimizer. Note that the space M in the first example is compact while in the second example the function E is smooth – even analytic.

Proof of Theorem 1.1. Suppose (1.1) holds. We may assume $E \not\equiv +\infty$. Let

$$\alpha_0 = \inf_M E \geq -\infty,$$

and let (α_m) be the strictly decreasing sequence

$$\alpha_m \searrow \alpha_0 \qquad (m \to \infty) .$$

Let $K_m = K_{\alpha_m}$. By assumption, each K_m is compact and non-empty. Moreover, $K_m \supset K_{m+1}$ for all m. By compactness of K_m there exists a point $u \in \bigcap_{m \in \mathbb{N}} K_m$, satisfying

$$E(u) \leq \alpha_m, \qquad \text{for all } m.$$

Passing to the limit $m \to \infty$ we obtain that

$$E(u) \leq \alpha_0 = \inf_M E,$$

and the claim follows.

If instead of (1.1) each K_α is sequentially compact, we choose a *minimizing sequence* (u_m) in M such that $E(u_m) \to \alpha_0$. Then for any $\alpha > \alpha_0$ the sequence (u_m) will eventually lie entirely within K_α. By sequential compactness of K_α therefore (u_m) will accumulate at a point $u \in \bigcap_{\alpha > \alpha_0} K_\alpha$ which is the desired minimizer. □

Note that if $E : M \to \mathbb{R}$ satisfies (1.1), then for any $\alpha \in \mathbb{R}$ the set

$$\{u \in M \ ; \ E(u) > \alpha\} = M \setminus K_\alpha$$

is open, that is, E is *lower semi-continuous.* (Respectively, if each K_α is sequentially compact, then E will be sequentially lower semi-continuous.) Conversely, if E is (sequentially) lower semi-continuous and for some $\overline{\alpha} \in \mathbb{R}$ the set $K_{\overline{\alpha}}$ is (sequentially) compact, then K_α will be (sequentially) compact for all $\alpha \leq \overline{\alpha}$ and again the conclusion of Theorem 1.1 will be valid.

Note that the lower semi-continuity condition can be more easily fulfilled the finer the topology on M. In contrast, the condition of compactness of the sub-level sets K_α , $\alpha \in \mathbb{R}$, calls for a coarse topology and both conditions are competing. In practice, there is often a natural weak Sobolev space topology where both conditions can be simultaneously satisfied. However, there are many interesting cases where condition (1.1) cannot hold in *any* reasonable topology (even though relative minimizers may exist). Later in this chapter we shall see some examples and some more delicate ways of handling the possible loss of compactness. See Section 4; see also Chapter III.

In applications, the conditions of the following special case of Theorem 1.1 can often be checked more easily.

1.2 Theorem. *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fullfilled:*
(1°) $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.
(2°) For any $u \in M$, any sequence (u_m) in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \le \liminf_{m\to\infty} E(u_m) \ .$$

Then E is bounded from below on M and attains its infimum in M.

The concept of minimizing sequences offers a direct and (apparently) constructive proof.

Proof. Let $\alpha_0 = \inf_M E$ and let (u_m) be a minimizing sequence in M, that is, satisfying $E(u_m) \to \alpha_0$. By coerciveness, (u_m) is bounded in V. Since V is reflexive, by the Eberlein-Šmulian theorem (see Dunford-Schwartz [1; p. 430]) we may assume that $u_m \rightharpoonup u$ weakly for some $u \in V$. But M is weakly closed, therefore $u \in M$, and by weak lower semi-continuity

$$E(u) \le \liminf_{m\to\infty} E(u_m) = \alpha_0 \ . \qquad \square$$

Examples. An important example of a sequentially weakly lower semi-continous functional is the norm in a Banach space V. Closed and convex subsets of Banach spaces are important examples of weakly closed sets. If V is the dual of a separable normed vector space, Theorem 1.2 and its proof remain valid if we replace weak by weak*-convergence.

We present some simple applications.

Degenerate Elliptic Equations

1.3 Theorem. *Let Ω be a bounded domain in $\mathbb{R}^n$, $p \in [2, \infty[$ with conjugate exponent q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, and let $f \in H^{-1,q}(\Omega)$, the dual of $H_0^{1,p}(\Omega)$, be given. Then there exists a weak solution $u \in H_0^{1,p}(\Omega)$ of the boundary value problem*

$$-\nabla \cdot (|\nabla u|^{p-2} \nabla u) = f \qquad \text{in } \Omega \tag{1.2}$$

$$u = 0 \qquad \text{on } \partial\Omega \tag{1.3}$$

in the sense that u satisfies the equation

$$(1.4) \qquad \int_\Omega (\nabla u|\nabla u|^{p-2}\nabla\varphi - f\varphi)dx = 0 \;, \qquad \forall\varphi \in C_0^\infty(\Omega) \;.$$

Proof. Note that the left part of (1.4) is the directional derivative of the C^1-functional

$$E(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,dx \;-\; \int_\Omega fu\,dx$$

on the Banach space $V = H_0^{1,p}(\Omega)$ in direction φ; that is, problem (1.2), (1.3) is of variational form.

Note that $H_0^{1,p}(\Omega)$ is reflexive. Moreover, E is coercive. In fact, we have

$$\begin{aligned} E(u) &\geq \frac{1}{p}\|u\|^p_{H_0^{1,p}} \;-\; \|f\|_{H^{-1,q}}\;\|u\|_{H_0^{1,p}} \geq \frac{1}{p}\left(\|u\|^p_{H_0^{1,p}} - c\|u\|_{H_0^{1,p}}\right) \\ &\geq c^{-1}\|u\|^p_{H_0^{1,p}} \;-\; C. \end{aligned}$$

Finally, E is (sequentially) weakly lower semi-continuous: It suffices to show that for $u_m \rightharpoonup u$ weakly in $H_0^{1,p}(\Omega)$ we have

$$\int_\Omega f\,u_m\,dx \;\;\to\;\; \int_\Omega f\,u\,dx \;.$$

Since $f \in H^{-1,q}(\Omega)$, however, this follows from the very definition of weak convergence. Hence Theorem 1.2 is applicable and there exists a minimizer $u \in H_0^{1,p}(\Omega)$ of E, solving (1.4). □

Note that for $p \geq 2$ the p-Laplacian is strongly monotone in the sense that

$$\int_\Omega \left(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\;dx \geq c\|u - v\|^p_{H_0^{1,p}} \;.$$

In particular, the solution u to (1.4) is unique.

If f is more regular, say $f \in C^{m,\alpha}(\overline{\Omega})$, we would expect the solution u of (1.4) to be more regular as well. This is true if $p = 2$, see Appendix B, but in the degenerate case $p > 2$, where the uniform ellipticity of the p-Laplace operator is lost at zeros of $|\nabla u|$, the best that one can hope for is $u \in C^{1,\alpha}(\overline{\Omega})$; see Uhlenbeck [1], Tolksdorf [2; p. 128], Di Benedetto [1].

In Theorem 1.3 we have applied Theorem 1.2 to a functional on a reflexive space. An example in a non-reflexive setting is given next.

Minimal Partitioning Hypersurfaces

For a domain $\Omega \subset \mathbb{R}^n$ let $BV(\Omega)$ be the space of functions $u \in L^1(\Omega)$ such that

$$\int_\Omega |Du| = \sup\Big\{ \int_\Omega \sum_{i=1}^n u D_i g_i \, dx \; ; \\ g = (g_1, \ldots, g_n) \in C_0^1(\Omega; \mathbb{R}^n), \; |g| \le 1 \Big\} < \infty \; ,$$

endowed with the norm

$$\|u\|_{BV} = \|u\|_{L^1} + \int_\Omega |Du| \; .$$

$BV(\Omega)$ is a Banach space, embedded in $L^1(\Omega)$, and – provided Ω is bounded and sufficiently smooth – by Rellich's theorem the injection $BV(\Omega) \hookrightarrow L^1(\Omega)$ is compact; see for instance Giusti [1; Theorem 1.19, p. 17]. Moreover, the function $u \mapsto \int_\Omega |Du|$ is lower semi-continuous with respect to L^1-convergence.

Let χ_G be the characteristic function of a set $G \subset \mathbb{R}^n$; that is, $\chi_G(x) = 1$ if $x \in G$, $\chi_G(x) = 0$ else. Also let $\mathcal{L}^n$ denote the n-dimensional Lebesgue measure.

1.4 Theorem. *Let Ω be a smooth, bounded domain in $\mathbb{R}^n$. Then there exists a subset $G \subset \Omega$ such that*

$$(1^\circ) \qquad \mathcal{L}^n(G) = \mathcal{L}^n(\Omega \setminus G) = \frac{1}{2}\mathcal{L}^n(\Omega)$$

and such that its perimeter with respect to Ω,

$$(2^\circ) \qquad P(G, \Omega) = \int_\Omega |D\chi_G| \, ,$$

is minimal among all sets satisfying (1°).

Proof. Let $M = \{\chi_G \; ; \; G \subset \Omega \text{ is measurable and satisfies } (1^\circ)\}$, endowed with the L^1-topology, and let $E : M \to \mathbb{R} \cup +\infty$ be given by

$$E(u) = \int_\Omega |Du| \; .$$

Since $\|\chi_G\|_{L^1} \le \mathcal{L}^n(\Omega)$, the functional E is coercive on M with respect to the norm in $BV(\Omega)$. Since bounded sets in $BV(\Omega)$ are relatively compact in $L^1(\Omega)$ and since M is closed in $L^1(\Omega)$, by weak lower semi-continuity of E in $L^1(\Omega)$ the sub-level sets of E are compact. The conclusion now follows from Theorem 1.1. □

The support of the distribution $D\chi_G$, where G has minimal perimeter (2°) with respect to Ω, can be interpreted as a minimal bisecting hypersurface, dividing Ω into two regions of equal volume. The regularity of the dividing hypersurface is intimately connected with the existence of minimal cones in $\mathbb{R}^n$. See Giusti [1] for further material on functions of bounded variation, sets of bounded perimeter, the area integrand, and applications.

A related setting for the study of minimal hypersurfaces and related objects is offered by geometric measure theory. Also in this field variational principles play an important role; see for instance Almgren [1], Morgan [1], or Simon [1] for introductory material and further references.

Our next example is concerned with a parametric approach.

Minimal Hypersurfaces in Riemannian Manifolds

Let Ω be a bounded domain in $\mathbb{R}^n$, and let S be a compact subset in $\mathbb{R}^N$. Also let $u_0 \in H^{1,2}(\Omega;\mathbb{R}^N)$ with $u_0(\Omega) \subset S$ be given. Define

$$H^{1,2}(\Omega;S) = \left\{u \in H^{1,2}(\Omega;\mathbb{R}^N)\ ;\ u(\Omega) \subset S \ \text{ almost everywhere}\right\}$$

and let

$$M = \left\{u \in H^{1,2}(\Omega;S)\ ;\ u - u_0 \in H_0^{1,2}(\Omega;\mathbb{R}^N)\right\}\ .$$

Then, by Rellich's theorem, M is closed in the weak topology of $V = H^{1,2}(\Omega;\mathbb{R}^N)$. For $u = (u^1,\dots,u^N) \in H^{1,2}(\Omega;S)$ let

$$E(u) = \int_\Omega g_{ij}(u)\nabla u^i \nabla u^j\ dx\ ,$$

where $g = (g_{ij})_{1\le i,j\le N}$ is a given positively definite symmetric matrix with coefficients $g_{ij}(u)$ depending continuously on $u \in S$, and where, by convention, we tacitly sum over repeated indices $1 \le i,j \le N$. Note that since S is compact g is uniformly positive definite on S, and there exists $\lambda > 0$ such that $E(u) \ge \lambda\,\|\nabla u\|_{L^2}^2$ for $u \in H^{1,2}(\Omega;S)$. In addition, since S and Ω are bounded, we have that $\|u\|_{L^2} \le c$ uniformly, for $u \in H^{1,2}(\Omega;S)$. Hence E is coercive on $H^{1,2}(\Omega;S)$ with respect to the norm in $H^{1,2}(\Omega;\mathbb{R}^N)$.

Finally, E is lower semi-continuous in $H^{1,2}(\Omega;S)$ with respect to weak convergence in $H^{1,2}(\Omega;\mathbb{R}^N)$. Indeed, if $u_m \rightharpoondown u$ weakly in $H^{1,2}(\Omega;\mathbb{R}^N)$, by Rellich's theorem $u_m \to u$ strongly in L^2 and hence a subsequence (u_m) converges almost everywhere. By Egorov's theorem, given $\delta > 0$ there is an exceptional set Ω_δ of measure $\mathcal{L}^n(\Omega_\delta) < \delta$ such that $u_m \to u$ uniformly on $\Omega \setminus \Omega_\delta$. We may assume that $\Omega_\delta \subset \Omega_{\delta'}$ for $\delta \le \delta'$. By weak lower semi-continuity of the semi-norm on $H^{1,2}(\Omega;\mathbb{R}^N)$, defined by

$$|v|^2 = \int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla v^i \nabla v^j\ dx,$$

then

$$\begin{aligned}\int_{\Omega\setminus\Omega_\delta} & g_{ij}(u)\,\nabla u^i\nabla u^j\;dx\\ &\le \liminf_{m\to\infty}\int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u^i_m\nabla u^j_m\;dx\\ &= \liminf_{m\to\infty}\int_{\Omega\setminus\Omega_\delta} g_{ij}(u_m)\,\nabla u^i_m\nabla u^j_m\;dx\\ &\le \liminf_{m\to\infty} E(u_m)\;.\end{aligned}$$

Passing to the limit $\delta \to 0$, from Beppo Levi's theorem we obtain

$$\begin{aligned}E(u) &= \lim_{\delta\to 0}\int_{\Omega\setminus\Omega_\delta} g_{ij}(u)\,\nabla u^i\nabla u^j\;dx\\ &\le \liminf_{m\to\infty}\;E(u_m)\;.\end{aligned}$$

Applying Theorem 1.2 to E on M we obtain

1.5 Theorem. *For any boundary data $u_0 \in H^{1,2}(\Omega; S)$ there exists an E-minimal extension $u \in M$.*

In differential geometry Theorem 1.5 arises in the study of harmonic maps $u : \Omega \to S$ from a domain Ω into an N-dimensional manifold S with metric g for prescribed boundary data $u = u_0$ on $\partial\Omega$. Like in the previous example, the regularity question is related to the existence of special harmonic maps; in this case, singularities of harmonic maps from Ω into S are related to harmonic mappings of spheres into S. For further references see Eells-Lemaire [1], [2], Hildebrandt [3], Jost [2]. For questions concerning regularity see Giaquinta-Giusti [1], Schoen-Uhlenbeck [1], [2].

A General Lower Semi-continuity Result

We now conclude this short list of introductory examples and return to the development of the variational theory. Note that the property of E being lower semi-continuous with respect to some weak kind of convergence is at the core of the above existence results. In Theorem 1.6 below we establish a lower semi-continuity result for a very broad class of variational integrals, including and going beyond those encountered in Theorem 1.5, as Theorem 1.6 would also apply in the case of unbounded targets S and possibly degenerate or singular metrics g.

We consider variational integrals

$$E(u) = \int_\Omega F(x, u, \nabla u)\,dx \tag{1.5}$$

involving (vector-valued) functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

1.6 Theorem. *Let Ω be a domain in $\mathbb{R}^n$, and assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions*
(1°) $F(x,u,p) \geq \phi(x)$ for almost every x, u, p, where $\phi \in L^1(\Omega)$.
(2°) $F(x,u,\cdot)$ is convex in p for almost every x, u.
Then, if $u_m,\ u \in H^{1,1}_{loc}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$, $\nabla u_m \rightharpoonup \nabla u$ weakly in $L^1(\Omega')$ for all bounded $\Omega' \subset\subset \Omega$, it follows that

$$E(u) \leq \liminf_{m\to\infty} E(u_m) \ ,$$

where E is given by (1.5).

Notes. In the scalar case $N = 1$, weak lower semi-continuity results like Theorem 1.6 were first stated by Tonelli [1] and Morrey [1]; these results were then extended and simplified by Serrin [1], [2] who showed that for non-negative, smooth functions $F(x,u,p)$: $\Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, which are convex in p , the functional E given by (1.5) is lower semi-continuous with respect to convergence in $L^1_{loc}(\Omega)$. A corresponding result in the vector-valued case $N > 1$ subsequently was derived by Morrey [4; Theorem 4.1.1]; however, Eisen [1] not only pointed out a gap in Morrey's proof but also gave an example showing that for $N > 1$ in general, Theorem 1.6 ceases to be true without the assumption that the L^1-norms of ∇u_m are uniformly locally bounded. Theorem 1.6 is due to Berkowitz [1] and Eisen [2]. Related results can be found for instance in Morrey [4; Theorem 1.8.2], or Giaquinta [1]. Our proof is modeled on Eisen [2].

Proof. We may assume that $\big(E(u_m)\big)$ is finite and convergent. Moreover, replacing F by $F - \phi$ we may assume that $F \geq 0$. Let $\Omega' \subset\subset \Omega$ be given. By weak local L^1-convergence $\nabla u_m \rightharpoonup \nabla u$, for any $m_0 \in \mathbb{N}$ there exists a sequence $(P^l)_{l \geq m_0}$ of convex linear combinations

$$P^l = \sum_{m=m_0}^{l} \alpha^l_m \nabla u_m \ , \ 0 \leq \alpha^l_m \leq 1 \ , \ \sum_{m=m_0}^{l} \alpha^l_m = 1 \ , \ l \geq m_0$$

such that $P^l \to \nabla u$ strongly in $L^1(\Omega')$ and pointwise almost everywhere as $l \to \infty$; see for instance Rudin [1; Theorem 3.13]. By convexity, for any m_0, any $l \geq m_0$, and almost every $x \in \Omega'$:

$$\begin{aligned} F\left(x, u(x), P^l(x)\right) &= F\left(x, u(x), \sum_{m=m_0}^{l} \alpha^l_m \nabla u_m(x)\right) \\ &\leq \sum_{m=m_0}^{l} \alpha^l_m F\left(x, u(x), \nabla u_m(x)\right) \ . \end{aligned}$$

Integrating over Ω' and passing to the limit $l \to \infty$, from Fatou's lemma we obtain:

$$\int_{\Omega'} F\left(x,u(x),\nabla u(x)\right)\,dx \le \liminf_{l\to\infty}\int_{\Omega'} F\left(x,u(x),P^l(x)\right)\,dx$$
$$\le \sup_{m\ge m_0}\int_{\Omega'} F\big(x,u(x),\nabla u_m(x)\big)\,dx\ .$$

Since m_0 was arbitrary, this implies that

$$\int_{\Omega'} F\big(x,u(x),\nabla u(x)\big)\,dx \le \limsup_{m\to\infty}\int_{\Omega'} F\left(x,u(x),\nabla u_m(x)\right)\,dx\,,$$

for any bounded $\Omega' \subset\subset \Omega$.

Now we need the following result (Eisen [2; p. 75]).

1.7 Lemma. *Under the hypotheses of Theorem 1.6 on F, u_m, and u there exists a subsequence (u_m) such that:*

$$F\left(x,u_m(x),\nabla u_m(x)\right) - F\left(x,u(x),\nabla u_m(x)\right) \to 0$$

in measure, locally in Ω.

Proof of Theorem 1.6 (completed). By Lemma 1.7 for any $\Omega' \subset\subset \Omega$, any $\varepsilon > 0$, and any $m_0 \in \mathbb{N}$ there exists $m \ge m_0$ and a set $\Omega'_{\varepsilon,m} \subset \Omega'$ with $\mathcal{L}^n\left(\Omega'_{\varepsilon,m}\right) < \varepsilon$ such that

$$|F\big(x,u_m(x),\nabla u_m(x)\big) - F\big(x,u(x),\nabla u_m(x)\big)| < \varepsilon \tag{1.6}$$

for all $x \in \Omega' \setminus \Omega'_{\varepsilon,m}$. Replacing ε by $\varepsilon_m = 2^{-m}$ and passing to a subsequence, if necessary, we may assume that for each m there is a set $\Omega'_{\varepsilon_m,m} \subset \Omega'$ of measure $< \varepsilon_m$ such that (1.6) is satisfied (with ε_m) for all $x \in \Omega' \setminus \Omega'_{\varepsilon_m,m}$. Hence, for any given $\varepsilon > 0$, if we choose $m_0 = m_0(\varepsilon) > |\log_2 \varepsilon|$, $\Omega'_\varepsilon = \bigcup_{m\ge m_0}\Omega'_{\varepsilon_m,m}$, this set has measure $\mathcal{L}^n(\Omega'_\varepsilon) < \varepsilon$ and inequality (1.6) holds uniformly for all $x \in \Omega' \setminus \Omega'_\varepsilon$, and all $m \ge m_0(\varepsilon)$. Moreover, for $\varepsilon < \delta$ by construction $\Omega'_\varepsilon \subset \Omega'_\delta$.

Cover Ω by disjoint bounded sets $\Omega^{(k)} \subset\subset \Omega$, $k \in \mathbb{N}$. Let $\varepsilon > 0$ be given and choose a sequence $\varepsilon^{(k)} > 0$, such that $\sum_{k\in\mathbb{N}} \mathcal{L}^n\left(\Omega^{(k)}\right)\varepsilon^{(k)} \le \varepsilon$. Passing to a subsequence, if necessary, for each $\Omega^{(k)}$ and $\varepsilon^{(k)}$ we may choose $m_0^{(k)}$ and $\Omega_\varepsilon^{(k)} \subset \Omega^{(k)}$ such that $\mathcal{L}^n\left(\Omega_\varepsilon^{(k)}\right) < \varepsilon^{(k)}$ and

$$|F\left(x,u_m(x),\nabla u_m(x)\right) - F\left(x,u(x),\nabla u_m(x)\right)| < \varepsilon^{(k)}$$

uniformly for $x \in \Omega^{(k)}\setminus\Omega_\varepsilon^{(k)}$, $m \ge m_0^{(k)}$. Moreover, we may assume that $\Omega_\varepsilon^{(k)} \subset \Omega_\delta^{(k)}$, if $\varepsilon < \delta$, for all k. Then for any $K \in \mathbb{N}$, letting $\Omega^K = \cup_{k=1}^K \Omega^{(k)}$, $\Omega_\varepsilon^K = \cup_{k=1}^K \Omega_\varepsilon^{(k)}$, we have

$$\begin{aligned}\int_{\Omega^K\setminus\Omega^K_\varepsilon} F(x,u,\nabla u)\,dx &\le \limsup_{m\to\infty}\int_{\Omega^K\setminus\Omega^K_\varepsilon} F\left(x,u,\nabla u_m\right)\,dx \\ &\le \limsup_{m\to\infty}\int_{\Omega^K\setminus\Omega^K_\varepsilon} F\left(x,u_m,\nabla u_m\right)\,dx+\varepsilon \\ &\le \limsup_{m\to\infty} E(u_m)+\varepsilon = \liminf_{m\to\infty} E(u_m)+\varepsilon\ .\end{aligned}$$

Letting $\varepsilon \to 0$ and then $K \to \infty$, the claim follows from Beppo Levi's theorem, since $F \ge 0$ and since $\Omega^K \setminus \Omega^K_\varepsilon$ increases when $\varepsilon \downarrow 0$, followed by $K \uparrow \infty$. □

Proof of Lemma 1.7. We basically follow Eisen [2]. Suppose by contradiction that there exist $\Omega' \subset\subset \Omega$ and $\varepsilon > 0$ such that, letting

$$\Omega_m = \{x \in \Omega'\ ;\ |F\left(x,u_m,\nabla u_m\right) - F\left(x,u,\nabla u_m\right)| \ge \varepsilon\}\ ,$$

there holds

$$\liminf_{m\to\infty} \mathcal{L}^n(\Omega_m) \ge 2\varepsilon\ .$$

The sequence (∇u_m), being weakly convergent, is uniformly bounded in $L^1(\Omega')$. In particular,

$$\mathcal{L}^n\{x \in \Omega'\ ;\ |\nabla u_m(x)| \ge l\} \le l^{-1}\int_{\Omega'}|\nabla u_m|\,dx \le \frac{C}{l} \le \varepsilon\ ,$$

if $l \ge l_0(\varepsilon)$ is large enough. Setting $\tilde\Omega_m := \{x \in \Omega_m\ ;\ |\nabla u_m(x)| \le l_0(\varepsilon)\}$ therefore there holds

$$\liminf_{m\to\infty} \mathcal{L}^n\left(\tilde\Omega_m\right) \ge \varepsilon.$$

Hence also for $\Omega^M = \bigcup_{m\ge M}\tilde\Omega_m$ we have

$$\mathcal{L}^n(\Omega^M) \ge \varepsilon\ ,$$

uniformly in $M \in \mathbb{N}$. Moreover, $\Omega' \supset \Omega^M \supset \Omega^{M+1}$ for all M and therefore $\Omega^\infty := \bigcap_{M\in\mathbb{N}} \Omega^M \subset \Omega'$ has $\mathcal{L}^n(\Omega^\infty) \ge \varepsilon$. Finally, neglecting a set of measure zero and passing to a subsequence, if necessary, we may assume that $F(x,z,p)$ is continuous in (z,p), that $u_m(x)$, $u(x)$, $\nabla u_m(x)$ are unambiguously defined and finite while $u_m(x) \to u(x)$ as $m \to \infty$ at every point $x \in \Omega^\infty$.

Note that every point $x \in \Omega^\infty$ by construction belongs to infinitely many of the sets $\tilde\Omega_m$. Choose such a point x. Relabeling, we may assume $x \in \bigcap_{m\in\mathbb{N}}\tilde\Omega_m$.

By uniform boundedness $|\nabla u_m(x)| \le C$ there exists a subsequence $m \to \infty$ and a vector $p \in \mathbb{R}^{nN}$ such that $\nabla u_m(x) \to p \ \ (m \to \infty)$. But then by continuity

$$F\big(x, u_m(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

while also

$$F\big(x, u(x), \nabla u_m(x)\big) \to F\big(x, u(x), p\big)$$

which contradicts the characterization of Ω_m given above. □

1.8 Remarks. The following observations may be useful in applications.
(1°) Theorem 1.6 also applies to functionals involving higher (mth-) order derivatives of a function u by letting $U = (u, \nabla u, \dots, \nabla^{m-1} u)$ denote the $(m-1)$-jet of u. Note that convexity is only required in the highest-order derivatives $P = \nabla^m u$.
(2°) If (u_m) is bounded in $H^{1,1}(\Omega')$ for any $\Omega' \subset\subset \Omega$, by Rellich's theorem and repeated selection of subsequences there exists a subsequence (u_m) which converges strongly in $L^1(\Omega')$ for any $\Omega' \subset\subset \Omega$.

Local boundedness in $H^{1,1}$ of a minimizing sequence (u_m) for E can be inferred from a coerciveness condition like

$$F(x, z, p) \ge |p|^\mu - \phi(x), \ \mu \ge 1, \ \phi \in L^1 \ . \tag{1.7}$$

The delicate part in the hypotheses concerning (u_m) is the assumption that (∇u_m) converges weakly in L^1_{loc}. In case $\mu > 1$ in (1.7) this is clear, but in case $\mu = 1$ the local L^1-limit of a minimizing sequence may lie in BV_{loc} instead of $H^{1,1}_{loc}$. See Theorem 1.4, for example; see also Section 3.
(3°) By convexity in p, continuity of F in (u, p) for almost every x is equivalent to the following condition, which is easier to check in applications:

> $F(x, \cdot, \cdot)$ is continuous, separately in $u \in \mathbb{R}^N$ and $p \in \mathbb{R}^{nN}$, for almost every $x \in \Omega$.

Indeed, for any fixed x, u, p and all $e \in \mathbb{R}^{nN}, |e| = 1, \ \alpha \in [0,1]$, letting $q = p + \alpha e, \ p_+ = p + e, \ p_- = p - e$ and writing $F(x, u, p) = F(u, p)$ for brevity, by convexity we have

$$F(u, q) = F(u, \alpha p_+ + (1-\alpha)p) \le \alpha F(u, p_+) + (1-\alpha) F(u, p) \ ,$$

$$F(u, p) = F(u, \frac{1}{1+\alpha} q + \frac{\alpha}{1+\alpha} p_-) \le \frac{1}{1+\alpha} F(u, q) + \frac{\alpha}{1+\alpha} F(u, p_-) \ .$$

Hence

$$\alpha \left(F(u, p) - F(u, p_+)\right) \le F(u, p) - F(u, q) \le \alpha \left(F(u, p_-) - F(u, p)\right)$$

and it follows that

$$\sup_{|q-p| \le 1} \frac{|F(u, q) - F(u, p)|}{|q - p|} \le \sup_{|q-p|=1} |F(u, q) - F(u, p)| \ .$$

Since the sphere of radius 1 around p lies in the convex hull of finitely many vectors $q_0, q_1, \ldots, q_{nN}$, by continuity of F in u and convexity in p the right-hand side of this inequality remains uniformly bounded in a neighborhood of (u,p). Hence $F(\cdot,\cdot)$ is locally Lipschitz continous in p, locally uniformly in $(u,p) \in \mathbb{R}^N \times \mathbb{R}^{nN}$. Therefore, if $u_m \to u$, $p_m \to p$ we have

$$
\begin{aligned}
|F(u_m,p_m) - F(u,p)| &\le |F(u_m,p_m) - F(u_m,p)| + |F(u_m,p) - F(u,p)| \\
&\le c|p_m - p| + o(1) \to 0 \qquad \text{as } m \to \infty,
\end{aligned}
$$

where $o(1) \to 0$ as $m \to \infty$, as desired.
(4°) In the scalar case ($N = 1$), if F is C^2 for example, the existence of a minimizer u for E implies that the *Legendre condition*

$$
\sum_{\alpha,\beta=1}^{n} F_{p_\alpha p_\beta}(x,u,p)\,\xi_\alpha \xi_\beta \ge 0, \qquad \text{for all } \xi \in \mathbb{R}^n
$$

holds at all points $(x, u = u(x), p = \nabla u(x))$, see for instance Giaquinta [1; p. 11 f.]. This condition in turn implies the convexity of F in p.

The situation is quite different in the vector-valued case $N > 1$. In this case, in general only the *Legendre-Hadamard condition*

$$
\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1} F_{p^i_\alpha p^j_\beta}(x,u,p)\xi_\alpha \xi_\beta \eta^i \eta^j \ge 0 \;, \qquad \text{for all } \xi \in \mathbb{R}^n,\ \eta \in \mathbb{R}^N
$$

will hold at a minimizer, which is much weaker then convexity (Giaquinta [1; p. 12]).

In fact, in Section 3 below we shall see how, under certain additional structure conditions on F, the convexity assumption in Theorem 1.6 can be weakened in the vector-valued case.

2. Constraints

Applying the direct methods often involves a delicate interplay between the functional E, the space of admissible functions M, and the topology on M. In this section we will see how, by means of imposing constraints on admissible functions and/or by a suitable modification of the variational problem, the direct methods can be successfully employed also in situations where their use seems highly unlikely at first.

Note that we will not consider constraints that are dictated by the problems themselves, such as physical restrictions on the response of a mechanical system. Constraints of this type in general lead to variational inequalities, and we refer to Kinderlehrer-Stampacchia [1] for a comprehensive introduction to this field. Instead, we will show how certain variational problems can be solved