Research Trends in Combinatorial Optimization

William J. Cook · László Lovász · Jens Vygen
Editors

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Dedicated to Bernhard Korte

Preface

The editors and authors dedicate this book to Bernhard Korte on the occasion of his seventieth birthday. We, the editors, are happy about the overwhelming feedback to our initiative to honor him with this book and with a workshop in Bonn on November 3–7, 2008. Although this would be a reason to look back, we would rather like to look forward and see what are the interesting research directions today.

This book is written by leading experts in combinatorial optimization. All papers were carefully reviewed, and eventually twenty-three of the invited papers were accepted for this book.

The breadth of topics is typical for the field: combinatorial optimization builds bridges between areas like combinatorics and graph theory, submodular functions and matroids, network flows and connectivity, approximation algorithms and mathematical programming, computational geometry and polyhedral combinatorics.

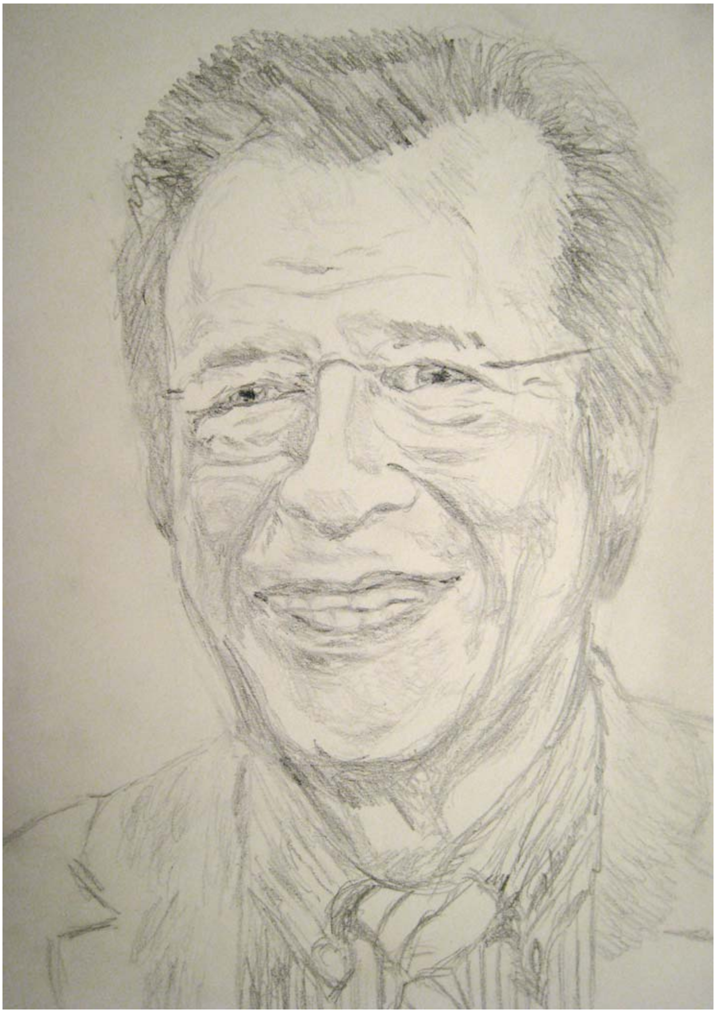
All these topics are related, and they are all addressed in this book. Combinatorial optimization is also known for its numerous applications. To limit the scope, however, this book is not primarily about applications, although some are mentioned at various places.

Most papers in this volume are surveys that provide an excellent overview of an active research area, but this book also contains many new results. Highlighting many of the currently most interesting research directions in combinatorial optimization, we hope that this book constitutes a good basis for future research in these areas.

We owe sincere thanks to all authors for their valuable contributions. We also thank all referees for carefully reviewing the papers and making many suggestions for improvements. Special thanks go to Ina Prinz for her portrait and cover design, and to Klaus Radke for technical help. Moreover, we thank Springer-Verlag for the efficient cooperation. Last, but not least, our most important thanks go to Bernhard Korte, without whom the field of combinatorial optimization would not be the same.

Atlanta, Budapest, and Bonn,
June 2008

William Cook
László Lovász
Jens Vygen



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1

On the Location and p-Median Polytopes

Mourad Baïou and Francisco Barahona

Summary. We revisit classical systems of linear inequalities associated with location problems and with the p-median problem. We present an overview of the cases for which these linear systems define integral polytopes. We also give polynomial time algorithms to recognize these cases.

1.1 Introduction

Facility location and p-median are among the most well-studied problems in combinatorial optimization. They are both NP-hard, so there is not much hope of having a complete polyhedral characterization of them. The linear programming relaxations that we use have been known since the 60's and have been the basis for many heuristics, branch and bound algorithms, and approximation algorithms. Despite all this work, very little is known about special cases where these formulations give integral polytopes, and also there are not many special cases where the associated polytope has been completely characterized. We have found a characterization of the graphs for which these linear relaxations define polytopes with all extreme points being integral. Here we present an overview of all these cases. We also give polynomial time algorithms to recognize these classes of graphs. Our characterization shows the basic structures that a graph contains when the polytope has fractional extreme points.

We first deal with location problems, we show that the linear relaxation gives an integral polytope if and only the graph does not contain a certain type of "odd" cycles. Then we deal with the p-median problem. We show that there are five configurations that should be forbidden in order to have an integral polytope. Here the proof consists of three parts as follows. First we show the result for the so-called Y-free graphs. We denote by Y some basic configuration in the graph. The result on Y-free graphs is used to start an induction proof for oriented graphs. These are directed graphs where between any two nodes u and v, at most one of the arcs (u, v) and (v, u) exists. Here the induction is done on the number of Y configurations. The third part consists of extending our result to general directed graphs. Here the induction is done on the number of pairs of nodes u and v such that both (u, v) and (v, u) exist. The initial step of the induction is given by the result on oriented graphs.

This paper is organized as follows. Section 1.2 contains some definitions. Section 1.3 deals with location problems. Section 1.4 covers the p-median problem. In Sect. 1.5 we give an algorithm to recognize the graphs defined in Sect. 1.4. Section 1.6 is devoted to some extensions.

1.2 Preliminary Definitions

A directed graph $G = (V, A)$ is called *oriented* if $(u, v) \in A$ implies $(v, u) \notin A$. For a directed graph $G = (V, A)$ and a set $W \subset V$, we denote by $\delta^+(W)$ the set of arcs $(u, v) \in A$, with $u \in W$ and $v \in V \setminus W$. Also we denote by $\delta^-(W)$ the set of arcs (u, v), with $v \in W$ and $u \in V \setminus W$. We write $\delta^+(v)$ and $\delta^-(v)$ instead of $\delta^+(\{v\})$ and $\delta^-(\{v\})$, respectively. If there is a risk of confusion we use δ^+_G and δ^-_G. A node u with $\delta^+(u) = \emptyset$ is called a *pendent* node.

A simple cycle C is an ordered sequence

$$v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p,$$

where

- v_i, $0 \le i \le p-1$, are distinct nodes,
- a_i, $0 \le i \le p-1$, are distinct arcs,
- either v_i is the tail of a_i and v_{i+1} is the head of a_i, or v_i is the head of a_i and v_{i+1} is the tail of a_i, for $0 \le i \le p-1$, and
- $v_0 = v_p$.

By setting $a_p = a_0$, we associate with C three more sets as below.

- We denote by $\hat{C}$ the set of nodes v_i, such that v_i is the head of a_{i-1} and also the head of a_i, $1 \le i \le p$.
- We denote by $\dot{C}$ the set of nodes v_i, such that v_i is the tail of a_{i-1} and also the tail of a_i, $1 \le i \le p$.
- We denote by $\tilde{C}$ the set of nodes v_i, such that either v_i is the head of a_{i-1} and also the tail of a_i, or v_i is the tail of a_{i-1} and also the head of a_i, $1 \le i \le p$.

Notice that $|\hat{C}| = |\dot{C}|$. A cycle will be called *odd* if $p + |\dot{C}|$ (or $|\tilde{C}| + |\dot{C}|$) is odd, otherwise it will be called *even*. A cycle C with $\dot{C} = \emptyset$ is a *directed* cycle. The set of arcs in C is denoted by $A(C)$.

If we do not require $v_0 = v_p$ we have a *path* P. In a similar way we define $\dot{P}$, $\hat{P}$ and $\tilde{P}$, excluding v_0 and v_p. We say that P is *odd* if $p + |\dot{P}|$ is odd, otherwise it is *even*. For the path P, the nodes $v_1, \ldots, v_{p-1}$ are called *internal*.

If G is a connected graph and there is a node u such that its removal disconnects G, we say that u is an *articulation point*. A graph is said to be *two-connected* if at least two nodes should be removed to disconnect it. For simplicity, sometimes we use z to denote the vector (x, y), i.e., $z(u) = y(u)$ and $z(u, v) = x(u, v)$. Also for $S \subseteq V \cup A$ we use $z(S)$ to denote $z(S) = \sum_{a \in S} z(a)$.

A *polyhedron* P is a set defined by a system of linear inequalities, i.e., $P = \{x \mid Ax \leq b\}$. A *face* of P is obtained by setting into equation some of these inequalities. An *extreme point* of P is given by a face that contains a unique element. In other words, some inequalities are set to equation so that this system has a unique solution. A *polytope* is a bounded polyhedron. A polyhedron is called *integral* if all its extreme points are integral.

1.3 Location Problems

Let $G = (V, A)$ be a directed graph, not necessarily connected, where each arc and each node has weight associated with it. We study a "prize collecting" version of a *location problem* (LP) as follows. A set of nodes is selected, usually called *centers*, and then each non-selected node can be assigned to a center. The weight of a node is the revenue obtained by opening a facility at that location, minus the cost of building the facility. The weight of an arc (i, j) is the revenue obtained by assigning the location i to the location j, minus the cost originated by this assignment. The goal is to maximize the sum of the weights of the selected nodes plus the sum of the weights yielded by the assignment. The linear system below defines a linear programming relaxation.

$$\max \sum w(u, v)x(u, v) + \sum w(v)y(v)$$

$$\sum_{(u,v)\in A} x(u, v) + y(u) \leq 1 \quad \forall u \in V, \tag{1}$$

$$x(u, v) \leq y(v) \quad \forall (u, v) \in A, \tag{2}$$

$$0 \leq y(v) \leq 1 \quad \forall v \in V, \tag{3}$$

$$x(u, v) \geq 0 \quad \forall (u, v) \in A. \tag{4}$$

For each node u, the variable $y(u)$ takes the value 1 if the node u is selected and 0 otherwise. For each arc (u, v) the variable $x(u, v)$ takes the value 1 if u is assigned to v and 0 otherwise. Inequalities (1) express the fact that either node u can be selected or it can be assigned to another node. Inequalities (2) indicate that if a node u is assigned to a node v then this last node should be selected. The set of integer vectors that satisfy (1)–(4) corresponds to a *transitive packing* as defined in Müller and Schulz (2002).

Let $P(G)$ be the polytope defined by (1)–(4), and let $LP(G)$ be the convex hull of $P(G) \cap \{0, 1\}^{|V|+|A|}$. Clearly

$$LP(G) \subseteq P(G).$$

Here we characterize the graphs G for which $LP(G) = P(G)$. More precisely, we show that $LP(G) = P(G)$ if and only if G does not contain an odd cycle. We also give a polynomial algorithm to recognize the graphs in this class.

The *Uncapacitated Facility Location Problem* (UFLP) is a variation where V is partitioned into V_1 and V_2. The set V_1 corresponds to the customers, and the set V_2

corresponds to the potential facilities. Each customer in V_1 should be assigned to an opened facility in V_2. This is obtained by considering $A \subseteq V_1 \times V_2$, fixing to zero the variables y for the nodes in V_1 and setting into equation the inequalities (1) for the nodes in V_1. More precisely, the linear programming relaxation for this case is

$$\min \sum c(u,v)x(u,v) + \sum d(v)y(v)$$

$$\sum_{(u,v)\in A} x(u,v) = 1 \quad \forall u \in V_1, \tag{5}$$

$$x(u,v) \le y(v) \quad \forall (u,v) \in A, \tag{6}$$

$$0 \le y(v) \le 1 \quad \forall v \in V_2, \tag{7}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A. \tag{8}$$

Here we also characterize the cases for which (5)–(8) defines an integral polytope.

We omit the proofs of several technical lemmas, the full details appear in Baïou and Barahona (2006). The facets of the uncapacitated facility location polytope have been studied in Guignard (1980), Cornuejols and Thizy (1982), Cho et al. (1983a, 1983b) and Cánovas et al. (2002). In Baïou and Barahona (2005) we gave a description of $LP(G)$ for Y-free graphs. The UFLP has also been studied from the point of view of approximation algorithms in Shmoys (1997), Chudak and Shmoys (2003), Sviridenko (2002), Byrka and Aardal (2007) and others. Other references on this problem are Cornuejols et al. (1976), Mirchandani and Francis (1990). The relationship between location polytopes and the stable set polytope has been studied in Cornuejols and Thizy (1982), Cho et al. (1983a, 1983b), De Simone and Mannino (1996), and others. It would be interesting to know if our results also have an equivalent in terms of stable set polytopes, but so far we have not found the right transformation.

1.3.1 Decomposition

In this subsection we consider a graph $G = (V, A)$ that decomposes into two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, with $V = V_1 \cup V_2$, $V_1 \cap V_2 = \{u\}$, $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. We define G'_1 that is obtained from G_1 after replacing u by u'. We also define G'_2, obtained from G_2 after replacing u by u''. The theorem below shows that we have to concentrate on two-connected graphs.

Theorem 3.1. *Suppose that the system*

$$Az' \le b, \tag{9}$$

$$z'\left(\delta^+_{G'_1}(u')\right) + z'(u') \le 1 \tag{10}$$

describes $LP(G'_1)$. Suppose that (9) contains the inequalities (1)–(4) except for (10). Similarly suppose that

$$Cz'' \le d, \tag{11}$$

$$z''\left(\delta^+_{G'_2}(u'')\right) + z''(u'') \le 1 \tag{12}$$

describes $LP(G_2')$*. Also (11) contains the inequalities (1)–(4) except for (12). Then the system below describes an integer polytope.*

$$Az' \le b, \tag{13}$$

$$Cz'' \le d, \tag{14}$$

$$z'\left(\delta^+_{G_1'}(u')\right) + z''\left(\delta^+_{G_2'}(u'')\right) + z'(u') \le 1, \tag{15}$$

$$z'(u') = z''(u''). \tag{16}$$

We have the following corollary.

Corollary 3.2. *The polytope* $LP(G)$ *is defined by the system (13)–(16) after identifying the variables* $z'(u')$ *and* $z''(u'')$*.*

This last corollary shows that if $LP(G_1')$ and $LP(G_2')$ are defined by (1)–(4), then $LP(G)$ is also defined by (1)–(4). Thus we have to concentrate on graphs that are two-connected. A result analogous to Theorem 3.1, for the stable set polytope, has been given in Chvátal (1975).

1.3.2 Graph Transformations

First we plan to prove that if G has no odd cycle then $LP(G) = P(G)$. The proof consists of assuming that $\bar{z}$ is a fractional extreme point of $P(G)$ and arriving at a contradiction. Below we give several assumptions that can be made about $\bar{z}$ and G, they will be used in the next subsection. The proofs of the lemmas below consist of modifying the graph and the vector $\bar{z}$ so that we obtain a new extreme point associated with a new graph satisfying the assumptions below.

Lemma 3.3. *We can assume that* G *consists of only one connected component.*

Lemma 3.4. *If* $0 < \bar{z}(u, v) < \bar{z}(v)$*, we can assume that* v *is a pendent node with* $|\delta^-(v)| = 1$ *and* $\bar{z}(v) = 1$*.*

Lemma 3.5. *We can assume that* $0 < \bar{z}(u, v) < 1$ *for all* $(u, v) \in A$*.*

Lemma 3.6. *We can assume that* G *is either two-connected or it consists of a single arc.*

If the graph G consists of a single arc it is fairly easy to see that $LP(G) = P(G)$, so now we have to deal with the two-connected components. This is treated in the next subsection.

1.3.3 Treating Two-Connected Graphs

In this subsection we assume that the graph G is two-connected and it has no odd cycle. Let $\bar{z}$ be a fractional extreme point of $P(G)$, we are going to assign labels l to the nodes and arcs and define $z'(u, v) = \bar{z}(u, v) + l(u, v)\epsilon$, $z'(u) = \bar{z}(u) + l(u)\epsilon$, $\epsilon > 0$, for each arc (u, v) and each node u. We shall see that every constraint that is satisfied with equality by $\bar{z}$ is also satisfied with equality by z'. This is the required contradiction.

Given a path $P = v_0, a_0, \dots, a_{p-1}, v_p$. Assume that the label of a_0, $l(a_0)$ has the value 1 or -1. We define the *labeling procedure* as follows.

For $i = 1$ to $p - 1$ do

- If v_i is the head of a_{i-1} and it is the tail of a_i then $l(v_i) = l(a_{i-1})$, $l(a_i) = -l(a_{i-1})$.
- If v_i is the head of a_{i-1} and it is the head of a_i then $l(v_i) = l(a_{i-1})$, $l(a_i) = l(a_{i-1})$.
- If v_i is the tail of a_{i-1} and it is the head of a_i then $l(v_i) = -l(a_{i-1})$, $l(a_i) = -l(a_{i-1})$.
- If v_i is the tail of a_{i-1} and it is the tail of a_i then $l(v_i) = 0$, $l(a_i) = -l(a_{i-1})$.

Notice that the labels of v_0 and v_p were not defined.

We have to study several cases as follows.

Case 1. G contains a directed cycle $C = v_0, a_0, \dots, a_{p-1}, v_p$. Assume that the head of a_0 is v_1, set $l(v_0) = -1$, $l(a_0) = 1$ and extend the labels as above.

Case 2. G contains a cycle $C = v_0, a_0, \dots, a_{p-1}, v_p$ and $\dot{C} \neq \emptyset$. Assume $v_0 \in \dot{C}$. Set $l(v_0) = 0$, $l(a_0) = 1$ and extend the labels.

The lemma below is needed to show that for v_0, the constraints that were satisfied with equality by $\bar{z}$ remain satisfied with equality.

Lemma 3.7. *After labeling as in Cases 1 and 2 we have* $l(a_{p-1}) = -l(a_0)$.

Notice that after the first cycle has been labeled as in Cases 1 or 2, the properties below hold, we shall see that these properties hold throughout the entire labeling procedure.

Property 1 *If a node has a nonzero label, then it is the tail of at most one labeled arc.*

Property 2 *If a node has a zero label, then it is the tail of exactly two labeled arcs.*

Once a cycle C has been labeled as in Cases 1 or 2, we have to extend the labeling as follows.

Case 3. Suppose that $l(v_0) \neq 0$ for $v_0 \in C$ (v_0 is the head of a labeled arc), and there is a path $P = v_0, a_0, v_1, a_1, \dots, a_{p-1}, v_p$ in G such that:

- v_0 is the head of a_0,
- $v_p \in C$,
- $\{v_1, \dots, v_{p-1}\}$ is disjoint from C.

We set $l(a_0) = l(v_0)$ and extend the labels. Case 3 is needed so that any inequality (2) associated with v_0 that is satisfied with equality, remains satisfied with equality.

We have to see that the label $l(a_{p-1})$ is such that constraints associated with v_p that were satisfied with equality remain satisfied with equality. This is discussed in the next lemma.

Lemma 3.8. *If v_p is the head of a_{p-1} then $l(a_{p-1}) = l(v_p)$. If v_p is the tail of a_{p-1} then $l(a_{p-1}) = -l(v_p)$.*

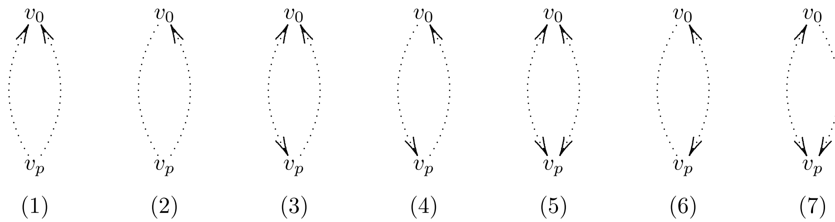

Fig. 1.1. Possible paths in C between v_0 and v_p. It is shown whether v_0 and v_p are the head or the tail of the arcs in C incident to them

Proof (cf. Baïou and Barahona 2006*).* Notice that $v_0 \notin \dot{C}$, in Fig. 1.1 we represent the possible configurations for the paths in C between v_0 and v_p. In this figure we show whether v_0 and v_p are the head or the tail of the arcs in C incident to them. These two paths are denoted by P_1 and P_2.

Consider configuration (1), these two paths should have different parity. When adding the path P, an odd cycle is created with either P_1 or P_2. So configuration (1) will not occur. The same happens with configuration (2).

Now we discuss configuration (3). These two paths should have the same parity. If v_p is the tail of a_{p-1} then P would create an odd cycle with either P_1 or P_2. If v_p is the head of a_{p-1} then P should have the same parity as P_1 and P_2. Then $l(a_{p-1}) = l(v_p)$.

The study of configuration (4) is similar. The two paths should have the same parity. If v_p is the tail of a_{p-1} then P would create an odd cycle with either P_1 or P_2. If v_p is the head of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = l(v_p)$.

For configuration (5) again the two paths should have the same parity. If v_p is the head of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = l(v_p)$. If v_p is the tail of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = -l(v_p)$.

Also in configuration (6) the paths P_1 and P_2 should have the same parity. If v_p is the tail of a_{p-1} then P would form an odd cycle with either P_1 or P_2. If v_p is the head of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = l(v_p)$.

In configuration (7) also the two paths should have the same parity. If v_p is the head of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = l(v_p)$. If v_p is the tail of a_{p-1} then P should have the same parity as P_1 and P_2, and $l(a_{p-1}) = -l(v_p)$. □

Based on this the labels are extended successively. Denote by G_l the subgraph defined by the labeled arcs. This is a two-connected graph, so for any two nodes v_0 and v_p it contains a cycle going through these two nodes. Thus we can check if Case 3 applies and extend the labels adding each time a path to the graph G_l. The two lemmas below show that Properties 1 and 2 remain satisfied.

Lemma 3.9. *Let v_p be a node with $l(v_p) \neq 0$. If v_p is the tail of an arc in G_l, then in Case 3 it cannot be the tail of a_{p-1}. Thus Property 1 remains satisfied.*

Lemma 3.10. *Let v_p be a node with $l(v_p) = 0$, thus v_p is the tail of exactly two arcs in G_l. Then in Case 3 it cannot be the tail of a_{p-1}. Therefore Property 2 remains satisfied.*

Once Case 3 has been exhausted we might have some nodes in G_l that are not pendent in G and that are only the head of labeled arcs. For such nodes we have to ensure that inequalities (1) that were satisfied as equality remain satisfied as equality. This is treated in the following.

Case 4. Suppose that v_0 is only the head of labeled arcs, ($l(v_0) \neq 0$), v_0 is not pendent. We have that $\delta^+(v_0) \neq \emptyset$ thus there is a cycle C in G_l and there is a path $P = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$ in G such that:

- $v_0 \in C$ is the tail of a_0,
- $v_p \in C$,
- $\{v_1, \ldots, v_{p-1}\}$ is disjoint from G_l.

We set $l(a_0) = -l(v_0)$ and extend the labels. We have to see that the label $l(a_{p-1})$ is such that constraints associated with v_p, that were satisfied with equality, remain satisfied with equality. This is discussed below.

Lemma 3.11. *In Case 4 we have that v_p is the tail of a_{p-1} and $l(a_{p-1}) = -l(v_p)$. Also Properties 1 and 2 continue to hold.*

To summarize, the labeling algorithm consists of the following steps.

- Step 1. Identify a cycle C in G and treat it as in Cases 1 or 2. Set $G_l = C$.
- Step 2. For as long as needed label as in Case 3. Each time add to G_l the new set of labeled nodes and arcs.
- Step 3. If needed, label as in Case 4. Each time add to G_l the new set of labeled nodes and arcs. If some new labels have been assigned in this step go to Step 2, otherwise stop.

At this point we can discuss the properties of the labeling procedure. The labels are such that any inequality (2) that was satisfied with equality by $\bar{z}$ is also satisfied with equality by z'. To see that inequalities (1) that were tight remain tight, we need two observations about G_l:

- Any node that has a nonzero label is the tail of exactly one labeled arc having the opposite label.
- If u is a node with $l(u) = 0$, then there are exactly two labeled arcs having opposite labels and whose tail is u.

Finally we give the label "0" to all nodes and arcs that are unlabeled, this completes the definition of z'. Lemma 3.5 shows that inequalities (4) will not be violated. The fact that nodes v with $\bar{z}(v) = 0$ or $\bar{z}(v) = 1$ receive a zero label, shows that inequalities (3) will not be violated. Any constraint that is satisfied with equality by $\bar{z}$ is also satisfied with equality by z', this contradicts the assumption that $\bar{z}$ is an extreme point. We can state the main result of this subsection.

Theorem 3.12. *If the graph G is two-connected and has no odd cycle then $LP(G) = P(G)$.*

This implies the following.

Theorem 3.13. *If G is a graph with no odd cycle, then $LP(G) = P(G)$.*

Theorem 3.14. *For graphs with no odd cycle, the uncapacitated facility location problem is polynomially solvable.*

1.3.4 Odd Cycles

In this subsection we study the effect of odd cycles in $P(G)$. Let C be an odd cycle. We can define a fractional vector $(\bar{x}, \bar{y}) \in P(G)$ as follows:

$$\bar{y}(u) = 0 \quad \text{for all nodes } u \in \dot{C}, \tag{17}$$

$$\bar{y}(u) = 1/2 \quad \text{for all nodes } u \in C \setminus \dot{C}, \tag{18}$$

$$\bar{x}(a) = 1/2 \quad \text{for } a \in A(C), \tag{19}$$

$$\bar{y}(v) = 0 \quad \text{for all other nodes } v \notin C, \tag{20}$$

$$\bar{x}(a) = 0 \quad \text{for all other arcs.} \tag{21}$$

Below we show a family of inequalities that separate the vectors defined above from $LP(G)$. We call them *odd cycle* inequalities.

Lemma 3.15. *The following inequalities are valid for $LP(G)$.*

$$\sum_{a \in A(C)} x(a) - \sum_{v \in \hat{C}} y(v) \leq \frac{|\tilde{C}| + |\hat{C}| - 1}{2} \tag{22}$$

for every odd cycle C.

These inequalities are $\{0, 1/2\}$-Chvatal–Gomory cuts, using the terminology of Caprara and Fischetti (1996). A separation algorithm can be obtained from the results of Caprara and Fischetti (1996). In Baïou and Barahona (2006) we gave an alternative separation algorithm.

Now we can present the following result.

Theorem 3.16. *Let G be a directed graph, then $LP(G) = P(G)$ if and only if G does not contain an odd cycle.*

Proof (cf. Baïou and Barahona 2006*).* If G contains and odd cycle C, then we can define a vector $(\bar{x}, \bar{y}) \in P(G)$ as in (17)–(21). We have

$$\sum_{a \in A(C)} \bar{x}(a) - \sum_{v \in \hat{C}} \bar{y}(v) = \frac{|\tilde{C}| + |\hat{C}|}{2}.$$

Lemma 3.15 shows that $\bar{z} \notin LP(G)$.

Then the theorem follows from Theorem 3.13. □

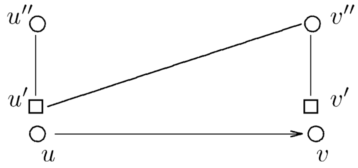

Fig. 1.2.

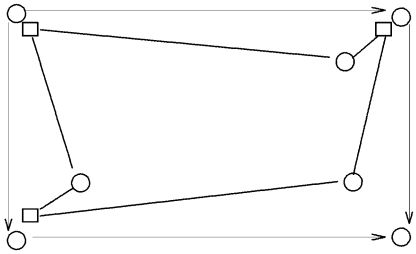

Fig. 1.3. An odd cycle in G and the corresponding cycle in H. The nodes of H close to a node $u \in G$ correspond to u' or u''

1.3.5 Detecting Odd Cycles

Now we study how to recognize the graphs G for which $LP(G) = P(G)$. We start with a graph G and several transformations are needed.

The first transformation consists of building an undirected graph $H = (N, E)$. For every node $u \in G$ we have the nodes u' and u'' in N, and the edge $u'u'' \in E$. For every arc $(u, v) \in G$ we have an edge $u'v'' \in E$. See Fig. 1.2.

Consider a cycle C in G, we build a cycle C_H in H as follows.

- If (u, v) and (u, w) are in C, then the edges $u'v''$ and $u'w''$ are taken.
- If (u, v) and (w, v) are in C, then the edges $u'v''$ and $v''w'$ are taken.
- If (u, v) and (v, w) are in C, then the edges $u'v''$, $v''v'$, and $v'w''$ are taken.

On the other hand, a cycle in H corresponds to a cycle in G. Thus there is a one to one correspondence among cycles of G and cycles of H. Moreover, if the cycle in H has cardinality $2q$, then $q = |\dot{C}| + |\tilde{C}|$, where C is the corresponding cycle in G. Therefore an odd cycle in G corresponds to a cycle in H of cardinality $2(2p + 1)$ for some positive integer p. See Fig. 1.3.

In other words, finding an odd cycle in G reduces to finding a cycle of cardinality $2(2p + 1)$, for some positive integer p, in the bipartite graph H.

For this question, a linear time algorithm was given in Yannakakis (1985), a simple $O(|V||A|^2)$ has been given in Conforti and Rao (1987).

1.3.6 Uncapacitated Facility Location

Now we assume that V is partitioned into V_1 and V_2, $A \subseteq V_1 \times V_2$, and we deal with the system

$$\sum_{(u,v)\in A} x(u,v) = 1 \quad \forall u \in V_1, \tag{23}$$

$$x(u,v) \le y(v) \quad \forall (u,v) \in A, \tag{24}$$

$$0 \le y(v) \le 1 \quad \forall v \in V_2, \tag{25}$$

$$x(u,v) \ge 0 \quad \forall (u,v) \in A. \tag{26}$$

If the variables x and y are constrained to be integer, then we have the uncapacitated facility location problem (UFLP). We denote by $\Pi(G)$ the polytope defined by (23)–(26). Notice that $\Pi(G)$ is a face of $P(G)$. Let $\bar{V}_1$ be the set of nodes $u \in V_1$ with $|\delta^+(u)| = 1$. Let $\bar{V}_2$ be the set of nodes in V_2 that are adjacent to a node in $\bar{V}_1$. It is clear that the variables associated with nodes in $\bar{V}_2$ should be fixed, i.e., $y(v) = 1$ for all $v \in \bar{V}_2$. Let us denote by $\bar{G}$ the subgraph induced by $V \setminus \bar{V}_2$. In this section we prove that $\Pi(G)$ is an integer polytope if and only if $\bar{G}$ has no odd cycle.

Let us first assume that $\bar{G}$ has no odd cycle. As before, we suppose that $\bar{z}$ is a fractional extreme point of $\Pi(G)$. The analogues of Lemmas 3.3, 3.4 and 3.5 apply here. Thus we can assume that we deal with a connected component G'. Lemma 3.4 implies that any node in $\bar{V}_2$ is not in a cycle of G'. Therefore G' has no odd cycle and $P(G')$ is an integer polytope. Since $\Pi(G')$ is a face of $P(G')$, we have a contradiction.

Now let C be an odd cycle of $\bar{G}$. We can define a fractional vector as follows:

$$\bar{y}(v) = 1/2 \quad \text{for all nodes } v \in V_2 \cap V(C),$$
$$\bar{x}(a) = 1/2 \quad \text{for } a \in A(C),$$
$$\bar{y}(v) = 1 \quad \text{for all nodes } v \in V_2 \setminus V(C).$$

For every node $u \in V_1 \setminus V(C)$, we look for an arc $(u,v) \in \delta^+(u)$. If $\bar{y}(v) = 1$ we set $\bar{x}(u,v) = 1$. If $\bar{y}(v) = 1/2$, then there is another arc $(u,w) \in \delta^+(u)$ such that $\bar{y}(w) = 1/2$ or $\bar{y}(w) = 1$. We set $\bar{x}(u,v) = \bar{x}(u,w) = 1/2$. Finally we set $\bar{x}(a) = 0$ for each remaining arc a. This vector satisfies (23)–(26), but it violates the inequality (22) associated with C. This shows that in this case (23)–(26) does not define an integer polytope. Thus we can state our main results.

Theorem 3.17. *The system (23)–(26) defines an integral polytope if and only if $\bar{G}$ has no odd cycle.*

Theorem 3.18. *The UFLP is polynomially solvable for graphs G such that $\bar{G}$ has no odd cycle.*

This class of graphs can be recognized in polynomial time as described in Sect. 1.3.5.

1.4 The p-Median Problem

The p-median problem is closely related to the uncapacitated facility location problem. Here we need to select a specific number of centers. Formally, let $G = (V, A)$

be a directed graph, not necessarily connected. We assume that G is simple, i.e., between any two nodes u and v there is at most one arc directed from u to v. Also for each arc $(u, v) \in A$ and node $v \in V$ there is an associated cost $c(u, v)$ and $w(v)$, respectively. The *p-median problem* (pMP) consists of selecting p nodes, usually called *centers*, and then assign each non-selected node to a selected node. The goal is to select p nodes that minimize the sum of the costs of the selected nodes plus the sum of the costs yield by the assignment of the non-selected nodes. This problem has several applications such as location of bank accounts (Cornuejols et al. 1976), placement of web proxies in a computer network (Vigneron et al. 2000), semistructured data bases (Toumani 2002; Nestorov et al. 1998).

The following define an integer linear programming formulation for the pMP:

$$\min \sum_{(u,v)\in A} c(u, v)x(u, v) + \sum_{v\in V} d(v)y(v) \tag{27}$$

$$\sum_{v\in V} y(v) = p, \tag{28}$$

$$\sum_{v:(u,v)\in A} x(u, v) + y(u) = 1 \quad \forall u \in V, \tag{29}$$

$$x(u, v) \le y(v) \quad \forall (u, v) \in A, \tag{30}$$

$$0 \le y(v) \le 1 \quad \forall v \in V, \tag{31}$$

$$x(u, v) \ge 0 \quad \forall (u, v) \in A. \tag{32}$$

Denote by $P_p(G)$ the polytope defined by (28)–(32), this gives a linear programming relaxation of the pMP. Let $pMP(G)$ be the convex hull of $P_p(G) \cap \{0, 1\}^{|A|+|V|}$.

The facets of $pMP(G)$ have been studied in Avella and Sassano (2001) and de Farias (2001). In Avella and Sassano (2001), new facets have been presented using a reduction to the stable set problem in the *intersection graph of* G. The intersection graph of G is defined as follows: its nodes are the arcs of G and there is an edge between two nodes (u, v) and (w, t) if $u = w$ or $v = w$. If we associate the cost $c(u, v)$ with each node (u, v) of the intersection graph, then the p-median problem in G, when the cost associated with the nodes of G is zero, is equivalent to find a stable set with minimum weight of cardinality $|V| - p$ in the intersection graph of G. In de Farias (2001), other class of facets have been presented in the class of bipartite graphs.

In this section we characterize all directed graphs such that $P_p(G) = pMP(G)$. To state our main result we need some definitions.

In Fig. 1.4, we show four directed graphs and for each of them a fractional extreme point of $P_p(G)$. The numbers near the nodes correspond to the variables y, all the arcs variables are equal to $\frac{1}{2}$.

Definition 4.1. *A simple cycle C is called a Y-cycle if for every $v \in \hat{C}$ there is an arc $(v, \bar{v})$, where $\bar{v}$ is in $V \setminus \dot{C}$.*

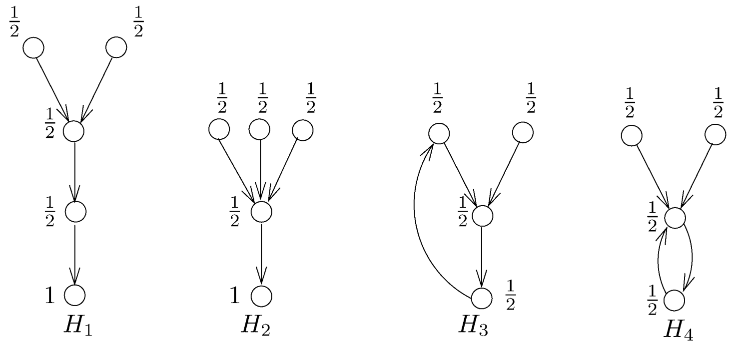

Fig. 1.4. Fractional extreme points of $P_p(G)$

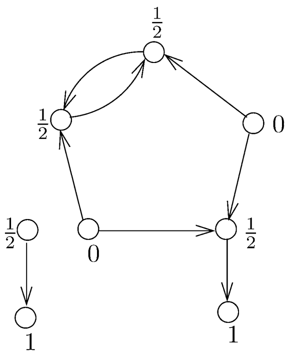

Fig. 1.5. An odd Y-cycle with an arc outside the cycle

In Fig. 1.5 we show a fractional extreme point of $P_p(G)$ different from those given in Fig. 1.4. It consists of an odd Y-cycle with an arc having both of its endnodes outside the cycle. The values reported near each node represent the node variables, the arc variables are all equal to $\frac{1}{2}$. These values form a fractional extreme point of $P_p(G)$, with $p = 4$.

The theorem below is the main result of this section. It shows that the configurations in Figs. 1.4 and 1.5 are the only configurations that should be forbidden in order to have an integral polytope.

Theorem 4.2. *Let $G = (V, A)$ be a directed graph, then $P_p(G)$ is integral if and only if*

(i) *it does not contain as a subgraph any of the graphs H_1, H_2, H_3 or H_4 of Fig. 1.4, and*
(ii) *it does not contain an odd Y-cycle C and an arc (u, v) with neither u nor v in $V(C)$.*

The proof of this theorem consists of three parts presented in Sects. 1.4.2, 1.4.3 and 1.4.4. The last two parts are the subject of two papers, see Baïou and Barahona

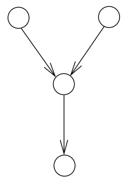

Fig. 1.6. The graph Y

(2007a, 2007b), each requires more than twenty pages. For these reasons, here we only present an overview of the proof.

In the first part of the proof, Sect. 1.4.2, we show that $P_p(G)$ is integral in Y-free graphs with no odd directed cycles. A *Y-free graph* is an oriented graph that does not contain as a subgraph the graph Y of Fig. 1.6. This class of graphs has been introduced in (Baïou and Barahona 2005).

In the second part, Sect. 1.4.3, we prove Theorem 4.2 when restricted to oriented graphs. This proof uses an induction on the number of subgraphs Y. The last part is devoted to the proof of Theorem 4.2 in general directed graphs and uses the result in oriented graphs as starting point. We will only present the sufficiency proof. The necessity proof is illustrated in Figs. 1.4 and 1.5. The fractional extreme points given in these figures can be easily extended to any graph that does not satisfy conditions (i) and (ii) of Theorem 4.2. Thus the graphs we consider do not contain as a subgraph any of the graphs H_1, H_2, H_3 or H_4 of Fig. 1.4.

1.4.1 Preliminaries

Let $G = (V, A)$ be a directed graph. Let $l : V \cup A \to \{0, -1, 1\}$ be a labeling function that associates to each node and arc of G a label 0, -1 or 1.

A vector $(x, y) \in P_p(G)$ will be denoted by z, i.e., $z(u) = y(u)$ for all $u \in V$ and $z(u, v) = x(u, v)$ for all $(u, v) \in A$. Given a vector z and a labeling function l, we define a new vector z_l from z as follows: $z_l(u) = z(u) + l(u)\epsilon,$ for all $u \in V$, and $z_l(u, v) = z(u, v) + l(u, v)\epsilon,$ for all $(u, v) \in A$, where ϵ is a sufficiently small positive scalar.

The Labeling Procedure for Even Cycles

Let $C = v_0, a_0, v_1, a_1, \ldots, a_{p-1}, v_p$ be an even cycle, not necessarily a Y-cycle.

- If C is a directed cycle, assume that v_0 is the tail of a_0, then set $l(v_0) \leftarrow 1$; $l(a_0) \leftarrow -1$. Otherwise, assume $v_0 \in \dot{C}$ and set $l(v_0) \leftarrow 0$; $l(a_0) \leftarrow 1$.
- Extend the labels as in Sect. 1.3.3.

Remark 4.3. If C is a directed even cycle, then $l(a_{p-1}) = l(v_0)$ and $\sum l(v_i) = 0$.

This remark is easy to see. The second property is given in the following lemma and it concerns non-directed cycles.

Lemma 4.4. *If C is a non-directed even cycle, then $l(a_{p-1}) = -l(a_0)$ and $\sum l(v_i) = 0$.*

We are going to deal with a vector z that is a fractional extreme point of $P_p(G)$.

Recall that the graph G we consider in Sect. 1.4.2 is Y-free and with no odd directed cycles and the graph G in Sects. 1.4.3 and 1.4.4 do not contain as a subgraph any of the graphs H_1, H_2, H_3 or H_4 of Fig. 1.4. In these graphs the following two lemmas hold:

Lemma 4.5. *We may assume that $z(u, v) > 0$ for all $(u, v) \in A$.*

Proof (cf. Baïou and Barahona 2007a*).* Let G' be the graph obtained after removing all arcs (u, v) with $\bar{z}(u, v) = 0$. The graph G' has the same properties as G. Let z' be the restriction of $\bar{z}$ on G'. Then z' is a fractional extreme point of $P_p(G')$. □

Lemma 4.6. *We may assume that $|\delta^-(v)| \leq 1$ for every pendent node v in G.*

Proof (cf. Baïou and Barahona 2007a*).* If v is a pendent node in G and $\delta^-(v) = \{(u_1, v), \ldots, (u_k, v)\}$, we can split v into k pendent nodes $\{v_1, \ldots, v_k\}$ and replace every arc (u_i, v) with (u_i, v_i). Then we define z' such that $z'(u_i, v_i) = z(u_i, v)$, $z'(v_i) = 1$, for all i, and $z'(u) = z(u)$, $z'(u, w) = z(u, w)$ for every other node and arc. Let G' be this new graph. The graph G' has the same properties as G. Moreover, it is easy to check that z' is a fractional extreme point of $P_{p+k-1}(G')$. □

1.4.2 Y-Free Graphs

In Baïou and Barahona (2005), we characterized the fractional extreme points of $P_p(G)$ for Y-free graphs. Then we showed that by adding the family of odd cycle inequalities associated with each directed odd cycle in G we obtain an integral polytope. An alternate proof of this result based on matching theory is given in Stauffer (2007).

To prove our main result we do not need the description of $pMP(G)$ in Y-free graphs. We need its description in a smaller class described by those Y-free graphs with no odd directed cycle. In this restricted class of graphs $P_p(G)$ is integral, this is a directed consequence of Theorem 14 in Baïou and Barahona (2005). Below we give a proof based on the matching polytope in bipartite graphs, which is along the same lines of the proof given in Stauffer (2007).

Theorem 4.7. *If $G = (V, A)$ is a Y-free graph with no odd directed cycle, then for any p the polytope $P_p(G)$ is integral.*

Proof. Let $G = (V, A)$ be a Y-free graph with no odd directed cycle. Assume the contrary, and let $z = (x, y)$ be an extreme fractional point of $P_p(G)$.

Using Fourier-Motzkin elimination, we obtain the following system of linear inequalities, that defines the projection of $P_p(G)$ onto the arc variables space; call it $Q_p(G)$.

$$\sum_{(u,v)\in A} x(u,v) = |V| - p, \tag{33}$$

$$x(w,u) + \sum_{v:(u,v)\in A} x(u,v) \leq 1 \quad \forall (w,u) \in A, \tag{34}$$

$$x(u,v) \geq 0 \quad \forall (u,v) \in A. \tag{35}$$

Remark that by Lemma 4.6 and the fact that G is a Y-free graph, we have that $|\delta^-(v)| \leq 1$ for all $v \in V$. Hence if we omit the orientation of the arcs in G we obtain a undirected graph $I(G) = (V, E)$, and inequalities (34) and (35) are equivalent to

$$x(\delta_{I(G)}(v)) \leq 1 \quad \forall v \in V, \tag{36}$$

$$x(e) \geq 0 \quad \forall e \in E. \tag{37}$$

Combining Lemma 4.6 and the fact that G does not contain an odd directed cycle, we obtain that $I(G)$ is a bipartite graph and hence the polytope defined by inequalities (36) and (37) is the matching polytope of a bipartite graph, so it is integral. Now by adding the equality $\sum_{e\in E} x(e) = |V| - p$ to the linear system defined by (36) and (37) the resulting polytope still integral, this is a well known property of the matching polytope, see for instance Lawler (1976). This proves that $Q_p(G)$ is integral.

To finish the proof of our theorem it suffices to see that if $z = (x, y)$ is an extreme point of $P_p(G)$, then x is an extreme point of $Q_p(G)$, which is easy to verify. □

1.4.3 Oriented Graphs

Let $G = (V, A)$ be an oriented graph that satisfies conditions (i) and (ii) of Theorem 4.2. First we study the case when G has no odd Y-cycle, and in the second case we assume that G has an odd Y-cycle.

G Does not Contain an Odd Y-Cycle

Let $t \in V$. The node t is called a *Y-node in $G = (V, A)$* if there are three different nodes u_1, u_2, w in V such that (u_1, t), (u_2, t) and (t, w) belong to A. Denote by Y_G the set of Y-nodes in G.

The proof is done by induction on the number of Y-nodes. If $|Y_G| = 0$ then, the graph is Y-free with no odd directed cycle, it follows from Theorem 4.7 in Sect. 1.4.2 that $P_p(G)$ is integral. Assume that $P_p(G')$ is integral for any positive integer p and for any oriented graph G', with $|Y_{G'}| < |Y_G|$, that satisfies condition (i) and does not contain an odd Y-cycle. Now we suppose that $z = (x, y)$ is a fractional extreme point of $P_p(G)$ and we plan to obtain a contradiction. The next lemma we need is as follows.

Lemma 4.8. *G does not contain a cycle.*

Proof (sketch, cf. Baïou and Barahona 2007a*).* The proof of this lemma is a direct application of the labeling procedure of Sect. 1.4.1. We assume that there is a cycle, using Lemma 4.6, we can derive an even Y-cycle C, and we assign labels to the nodes and arcs of C following the labeling procedure of Sect. 1.4.1. Extend the labels as follows: for each node $v \in \hat{C}$, choose an arc $(v, \bar{v})$, $\bar{v} \notin V(C)$, and assign the label $-l(v)$ to it. Assign a zero label to all remaining nodes and arcs. In the last step, using Lemma 4.5 we show that any constraint that is satisfied with equality by z is also satisfied with equality by z_l. This contradicts the fact that z is an extreme point of $P_p(G)$. □

The graph G must contain at least one Y-node t with its incident arcs (u_1, t), (u_2, t), (t, w). Using Lemma 4.8 we can prove that V can be partitioned into W_1 and W_2 so that $\{u_1, t, w\} \subseteq W_1$ and $u_2 \in W_2$, and that the only arc in G between W_1 and W_2 is (u_2, t).

Next we show that $z(t) = \frac{1}{2}$. We have that $Q(G)$, the polytope defined by (29)–(32), is a face of the polytope $P(G)$ defined by (1)–(4)) studied in Sect. 1.3. And by Theorem 3.13, we know that $P(G)$ is integral when G does not contain an odd cycle, which is the case here. Thus $Q(G)$ is also integral. The polytope $P_p(G)$ is obtained from $Q(G)$ by adding exactly one equation. A simple polyhedral fact is that if $Q(G)$ is integral, then the values of z are in $\{0, 1, \alpha, 1 - \alpha\}$, for some number $\alpha \in [0, 1]$. But since $z(t) = \frac{1}{2}$ we have that all fractional values of z are equal to $\frac{1}{2}$.

Define $p_1 = \sum_{v \in W_1} z(v)$ and $p_2 = \sum_{v \in W_2} z(v)$, so $p = p_1 + p_2$. We distinguish two cases: p_1 and p_2 are integer; and they are not.

If the numbers p_1 and p_2 are integer, we define the graphs G^1 and G^2 as follows. Let $A(W_1)$ and $A(W_2)$ be the set of arcs in G having both endnodes in W_1 and W_2, respectively. Let $G^1 = (W_1, A(W_1))$ and $G^2 = (W_2 \cup \{t', v', w'\}, A(W_2) \cup \{(u_2, t'), (t', v'), (v', w')\})$.

Let z_1 be the restriction of z to G^1. Clearly $z_1 \in P_{p_1}(G^1)$. Define z_2 as follows, $z_2(u_2, t') = z(u_2, t) = \frac{1}{2}$, $z_2(t') = \frac{1}{2}$, $z_2(t', v') = \frac{1}{2}$, $z_2(v') = \frac{1}{2}$, $z_2(v', w') = \frac{1}{2}$, $z_2(w') = 1$ and $z_2(u) = z(u)$, $z_2(u, v) = z(u, v)$ for all other nodes and arcs of G^2. We have that $z_2 \in P_{p_2+2}(G^2)$.

Both graphs G^1 and G^2 satisfy condition (i) of Theorem 4.2 and do not contain an odd Y-cycle. Moreover, $|Y_{G^1}| < |Y_G|$ and $|Y_{G^2}| < |Y_G|$. Since z_1 and z_2 are both fractional, the induction hypothesis implies that they are not extreme points of $P_{p_1}(G^1)$ and $P_{p_2+2}(G^2)$, respectively. Thus there must exist a 0-1 vector $z'_1 \in P_{p_1}(G^1)$ with $z'_1(t) = 0$ so that the same constraints that are tight for z_1 are also tight for z'_1. Also there must exist a 0-1 vector $z'_2 \in P_{p_2+2}(G^2)$ with $z'_2(t') = 0$ such that the same constraints that are tight for z_2 are also tight for z'_2. Now by combining z'_1 and z'_2 one can define a solution $z' \in P_p(G)$ that satisfies as equality each constraint that is satisfied as equality by z.

In the case where the numbers p_1 and p_2 are not integer, we use the same idea as above but applied for new graphs G^1 and G^2, where $G^1 = (W_1 \cup \{u'_1\}, (A(W_1) \setminus \{(u_1, t)\}) \cup \{(u_1, u'_1), (u'_1, t)\})$ and $G^2 = (W_2 \cup \{t', w'\}, A(W_2) \cup \{(u_2, t'), (t', w')\})$.