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The Geometry of Infinite-Dimensional Groups

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To our Teachers:

to Vladimir Igorevich Arnold
and to the memory of Peter Slodowy

Preface

The aim of this monograph is to give an overview of various classes of infinite-dimensional Lie groups and their applications, mostly in Hamiltonian mechanics, fluid dynamics, integrable systems, and complex geometry. We have chosen to present the unifying ideas of the theory by concentrating on specific types and examples of infinite-dimensional Lie groups. Of course, the selection of the topics is largely influenced by the taste of the authors, but we hope that this selection is wide enough to describe various phenomena arising in the geometry of infinite-dimensional Lie groups and to convince the reader that they are appealing objects to study from both purely mathematical and more applied points of view. This book can be thought of as complementary to the existing more algebraic treatments, in particular, those covering the structure and representation theory of infinite-dimensional Lie algebras, as well as to more analytic ones developing calculus on infinite-dimensional manifolds.

This monograph originated from advanced graduate courses and mini-courses on infinite-dimensional groups and gauge theory given by the first author at the University of Toronto, at the CIRM in Marseille, and at the Ecole Polytechnique in Paris in 2001–2004. It is based on various classical and recent results that have shaped this newly emerged part of infinite-dimensional geometry and group theory.

Our intention was to make the book concise, relatively self-contained, and useful in a graduate course. For this reason, throughout the text, we have included a large number of problems, ranging from simple exercises to open questions. At the end of each section we provide bibliographical notes, trying to make the literature guide more comprehensive, in an attempt to bring the interested reader in contact with some of the most recent developments in this exciting subject, the geometry of infinite-dimensional groups. We hope that this book will be useful to both students and researchers in Lie theory, geometry, and Hamiltonian systems.

It is our pleasure to thank all those who helped us with the preparation of this manuscript. We are deeply indebted to our teachers, collaborators, and friends, who influenced our view of the subject: V. Arnold, Ya. Brenier, H. Bursztyn, Ya. Eliashberg, P. Etingof, V. Fock, I. Frenkel, D. Fuchs, A. Kirillov, F. Malikov, G. Misiołek, R. Moraru, N. Nekrasov, V. Ovsienko, C. Roger, A. Rosly, V. Rubtsov, A. Schwarz, G. Segal, M. Semenov-Tian-Shansky, A. Shnirelman, P. Slodowy, S. Tabachnikov, A. Todorov, A. Veselov, F. Wagemann, J. Weitsman, I. Zakharevich, and many others. We are particularly grateful to Alexei Rosly, the joint projects with whom inspired a large part, in particular the "application chapter," of this book, and who made numerous invaluable remarks on the manuscript. We thank the participants of the graduate courses for their stimulating questions and remarks. Our special thanks go to M. Peters and the Springer team for their invariable help and to D. Kramer for careful editing of the text.

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Introduction

What is a group? Algebraists teach that this is supposedly a set with two operations that satisfy a load of easily-forgettable axioms...

V.I. Arnold "On teaching mathematics" [20]

Today one cannot imagine mathematics and physics without Lie groups, which lie at the foundation of so many structures and theories. Many of these groups are of infinite dimension and they arise naturally in problems related to differential and algebraic geometry, knot theory, fluid dynamics, cosmology, and string theory. Such groups often appear as symmetries of various evolution equations, and their applications range from quantum mechanics to meteorology. Although infinite-dimensional Lie groups have been investigated for quite some time, the scope of applicability of a general theory of such groups is still rather limited. The main reason for this is that infinite-dimensional Lie groups exhibit very peculiar features.

Let us look at the relation between a Lie group and its Lie algebra as an example. As is well known, in finite dimensions each Lie group is, at least locally near the identity, completely described by its Lie algebra. This is achieved with the help of the exponential map, which is a local diffeomorphism from the Lie algebra to the Lie group itself. In infinite dimensions, this correspondence is no longer so straightforward. There may exist Lie groups that do not admit an exponential map. Furthermore, even if the exponential map exists for a given group, it may not be a local diffeomorphism. Another pathology in infinite dimensions is the failure of Lie's third theorem, stating that every finite-dimensional Lie algebra is the Lie algebra attached to some finite-dimensional Lie group. In contrast, there exist infinite-dimensional Lie algebras that do not correspond to any Lie group at all.

In order to avoid such pathologies, any version of a general theory of infinite-dimensional Lie groups would have to restrict its attention to certain classes of such groups and study them separately. For example, one might consider the class of Banach Lie groups, i.e., Lie groups that are locally modeled on Banach spaces and behave very much like finite-dimensional Lie groups. For Banach Lie groups the exponential map always exists and is a local diffeomorphism. However, restricting to Banach Lie groups would already exclude the important case of diffeomorphism groups, and so on. This is why the attempts to develop a unified theory of infinite-dimensional differential geometry, and hence, of infinite-dimensional Lie groups, are still far from reaching greater generality.

In the present book, we choose a different approach. Instead of trying to develop a general theory of such groups, we concentrate on various examples of infinite-dimensional Lie groups, which lead to a realm of important applications.

The examples we treat here mainly belong to three general types of infinite-dimensional Lie groups: groups of diffeomorphisms, gauge transformation groups, and groups of pseudodifferential operators. There are numerous interrelations between various groups appearing in this book. For example, the group of diffeomorphisms of a compact manifold acts naturally on the group of currents over this manifold. When this manifold is a circle, this action gives rise to a deep connection between the representation theory of the Virasoro algebra and the Kac–Moody algebras. In the geometric setting of this book, this relation manifests itself in the correspondence between the coadjoint orbits of these groups.

Another strand connecting various groups considered below is the theme of the "ladder" of current groups. We regard the passage from finite-dimensional Lie groups (i.e., "current groups at a point") to loop groups (i.e., current groups on the circle), and then to double loop groups (current groups on the two-dimensional torus) as a "ladder of groups." On the side of dynamical systems this is revealed in the passage from rational to trigonometric and to elliptic Calogero–Moser systems. The passage from ordinary loop groups to double loop groups also serves as the starting point of a "real–complex correspondence" discussed in the chapter on applications of groups. There we study moduli spaces of flat or integrable connections on real and complex surfaces using the geometry of coadjoint orbits of these two types of groups.

Most of main objects studied in the book can be summarized in the table below.

In Chapter II, in a sense, we are moving horizontally, along the first row of this table. We study affine and elliptic groups, their orbits and geometry, as well as the related Calogero–Moser systems. We also describe in this chapter many Lie groups and Lie algebras outside the scope of this table: groups of diffeomorphisms, the Virasoro group, groups of pseudodifferential operators. In the appendices one can find the Krichever–Novikov algebras, $\mathfrak{gl}_\infty$, and other related objects.

In Chapter III we move vertically in this table and mostly focus on the current groups and on their parallel description in topological and holomorphic contexts. While affine and elliptic Lie groups correspond to the base dimension

Base dimension	Real / topological theory	Complex / holomorphic theory
1	affine (or, loop) groups (orbits $\sim$ monodromies over a circle)	elliptic (or, double loop) groups (orbits $\sim$ holomorphic bundles over an elliptic curve)
2	flat connections over a Riemann surface (Poisson structures)	holomorphic bundles over a complex surface (holomorphic Poisson structures)
3	connections over a threefold (Chern–Simons functional, singular homology, classical linking)	partial connections over a complex threefold (holomorphic Chern–Simons functional, polar homology, holomorphic linking)

1, either real or complex, in dimension 2 we describe the spaces of connections on real or complex surfaces, as well as the symplectic and Poisson structures on the corresponding moduli spaces. (In the table the main focus of study is mentioned in the parentheses of the corresponding block.) In dimension 3 the study of the Chern–Simons functional and its holomorphic version leads one to the notions of classical and holomorphic linking, and to the corresponding homology theories. (Although we confined ourselves to three dimensions, one can continue this table to dimension 4 and higher, which brings in the Yang–Mills and many other interesting functionals; see, e.g., [85].)

Note that the objects (groups, connections, etc.) in each row of this table usually dictate the structure of objects in the row above it, although the "interaction of the rows" is different in the real and complex cases. Namely, in the real setting, the lower-dimensional manifolds appear as the boundary of real manifolds of one dimension higher. For the complex case, the low-dimensional complex varieties arise as divisors in higher-dimensional ones; see details in Chapter III.

Overview of the content. Here are several details on the contents of various chapters and sections.

In Chapter I, we recall some notions and facts from Lie theory and symplectic geometry used throughout the book. Starting with the definition of a Lie group, we review the main related concepts of its Lie algebra, the adjoint and coadjoint representations, and introduce central extensions of Lie groups and algebras. We then recall some notions from symplectic geometry, including Arnold's formulation of the Euler equations on a Lie group, which are the equations for the geodesic flow with respect to a one-sided invariant metric on the group. This setting allows one to describe on the same footing many finite- and infinite-dimensional dynamical systems, including the classical Euler equations for both a rigid body and an ideal fluid, the Korteweg–de Vries equation, and the equations of magnetohydrodynamics. Finally, the preliminaries cover the Marsden–Weinstein Hamiltonian reduction, a method often used to describe complicated Hamiltonian systems starting with a simple one on a nonreduced space, by "dividing out" extra symmetries of the system.

Chapter II is the main part of this book, and can be viewed as a walk through the zoo of the various types of infinite-dimensional Lie groups. We tried to describe these groups by presenting their definitions, possible explicit constructions, information on (or, in some cases, even the complete classification of) their coadjoint orbits. We also discuss relations of these groups to various Hamiltonian systems, elaborating, whenever possible, on important constructions related to integrability of such systems. The table of contents is rather self-explanatory.

We start this chapter by introducing the loop group of a compact Lie group, one of the most studied types of infinite-dimensional groups. In Section 1, we construct its universal central extension, the corresponding Lie algebra (called the affine Kac–Moody Lie algebra), and classify the corresponding coadjoint orbits. We also return to discuss the relation of this Lie algebra to the Landau–Lifschitz equation and the Calogero–Moser integrable system in the later sections.

In Section 2 we turn to the group of diffeomorphisms of the circle and its Lie algebra of smooth vector fields. Both the group and the Lie algebra admit universal central extensions, called the Virasoro–Bott group and the Virasoro algebra respectively. It turns out that the coadjoint orbits of the Virasoro–Bott group can be classified in a manner similar to that for the orbits of the loop groups. The Euler equation for a natural right-invariant metric on the Virasoro–Bott group is the famous Korteweg–de Vries (KdV) equation, which describes waves in shallow water. Furthermore, the Euler nature of the KdV helps one to show that this equation is completely integrable.

Section 3 is devoted to various diffeomorphism groups and, in particular, to the group of volume-preserving diffeomorphisms of a compact Riemannian manifold M. The Euler equations on this group are the Euler equations for an ideal incompressible fluid filling M. Enlarging the group of volume-preserving diffeomorphisms by either smooth functions or vector fields on M gives the Euler equations of gas dynamics or of magnetohydrodynamics, respectively. We also mention some results on the Riemannian geometry of diffeomorphism groups and discuss the relation of the latter to the the Marsden–Weinstein symplectic structure on the space of immersed curves in $\mathbb{R}^3$.

Section 4 deals with the group of pseudodifferential symbols (or operators) on the circle. It turns out that this group can be endowed with the structure of a Poisson Lie group, where the corresponding Poisson structures are given by the Adler–Gelfand–Dickey brackets. The dynamical systems naturally corresponding to this group are the Kadomtsev–Petviashvili hierarchy, the higher n-KdV equations, and the nonlinear Schrödinger equation.

Section 5 returns to the loop groups "at the next level": here we deal with their generalizations, elliptic Lie groups and the corresponding Lie algebras. These groups are extensions of the groups of double loops, i.e., the groups of smooth maps from a two-dimensional torus to a finite-dimensional complex

Lie group. The central extension of such a group relies on the choice of complex structure on this torus (i.e., on the choice of the underlying elliptic curve). The coadjoint orbits of the elliptic Lie groups can be classified in terms of holomorphic principal bundles over the elliptic curve.

This section also unifies several classes of the groups considered earlier in the light of an application to the Calogero–Moser systems. It turns out that the integrable types of potentials in these systems (rational, trigonometric, and elliptic ones) can be obtained, respectively, from the finite-dimensional semisimple Lie algebras, the affine algebras, and the elliptic Lie algebras by Hamiltonian reductions.

Chapter III deals with far-reaching applications of the parallelism between the affine and elliptic Lie algebras, which resembles the "real–complex" correspondence. The infinite-dimensional Lie groups we are concerned with here are groups of gauge transformations of principal bundles over real and complex surfaces. We show how the classification of coadjoint orbits of loop groups (respectively, double loop groups) can be used to study the Poisson structure on the moduli space of flat connections (respectively, semistable holomorphic bundles) over a Riemann surface (respectively, a complex surface).

The correspondence between the real and complex cases leads to somewhat surprising analogies between notions in differential topology (such as orientation, boundary, and the Stokes theorem) and those in complex algebraic geometry (a meromorphic differential form, its divisor of poles, and the Cauchy–Stokes formula). These analogies are formalized in the notion of polar homology, and their applications include the construction of a holomorphic linking number for a pair of complex curves in a complex threefold. The definition of the latter is closely related to a holomorphic version of the Chern–Simons functional.

In the appendices we mention several topics serving either as an explanation to some facts used in the main text, or as an indication of further developments. In particular, we include reminders on root systems and some important facts from the theory of compact Lie groups. Other appendices provide brief introductions and guides to the literature on the algebra $\mathfrak{gl}_\infty$, the Krichever–Novikov algebras (generalizing the Virasoro algebra and loop algebras to higher-genus Riemann surfaces), integrable systems on the moduli of flat connections, the Kähler structures on Virasoro orbits, a relation of diffeomorphism groups to optimal mass transport, the Hofer metric on the group of Hamiltonian diffeomorphisms, the Drinfeld–Sokolov reduction, as well as proofs of several statements from the main text.

Numeration system and shortcuts. We have employed a single numeration of definitions, theorems, etc. The Roman numeral in the cross-references addresses to the chapter number, while its absence indicates that the cross-references are within the same chapter.

The different sections in Chapter II can be read to a large degree independently. Furthermore, Chapter III is based on just two sections from Chapter

II: those on the affine groups (Section 1) and on the elliptic Lie groups (Sections 5). The section on polar homology is also rather independent, although motivated by the preceding exposition in Chapter III.

For a first reading we recommend the following "shortcut" through the book: After Chapter I on preliminaries, one can proceed to Sections 1, 2, and 5 of Chapter II and Sections 2 and 3 of Chapter III. The reader more interested in applications to Hamiltonian systems will find them mostly in Sections 2 through 5 of Chapter II, while for applications to moduli spaces of flat connections one may choose to proceed to Chapter III after reading only Sections 1 and 5 of Chapter II.

I

Preliminaries

In this chapter, we collect some key notions and facts from the theory of Lie groups and Hamiltonian systems, as well as set up the notations.

1 Lie Groups and Lie Algebras

This section introduces the notions of a Lie group and the corresponding Lie algebra. Many of the basic facts known for finite-dimensional Lie groups are no longer true for infinite-dimensional ones, and below we illustrate some of the pathologies one can encounter in the infinite-dimensional setting.

1.1 Lie Groups and an Infinite-Dimensional Setting

The most basic definition for us will be that of a (transformation) group.

Definition 1.1 A nonempty collection G of transformations of some set is called a (transformation) *group* if along with every two transformations $g, h \in G$ belonging to the collection, the composition $g \circ h$ and the inverse transformation g^{-1} belong to the same collection G.

It follows directly from this definition that every group contains the identity transformation e. Also, the composition of transformations is an associative operation. These properties, associativity and the existence of the unit and an inverse of each element, are often taken as the definition of an abstract group.[1]

The groups we are concerned with in this book are so-called Lie groups. In addition to being a group, they carry the structure of a smooth manifold such that both the multiplication and inversion respect this structure.

[1] Here we employ the point of view of V.I. Arnold, that every group should be viewed as the group of transformations of some set, and the "usual" axiomatic definition of a group only obscures its true meaning (cf. [19], p. 58).

Definition 1.2 A *Lie group* is a smooth manifold G with a group structure such that the multiplication $G \times G \to G$ and the inversion $G \to G$ are smooth maps.

The Lie groups considered throughout this book will usually be infinite-dimensional. So what do we mean by an infinite-dimensional manifold? Roughly speaking, an infinite-dimensional manifold is a manifold modeled on an infinite-dimensional locally convex vector space just as a finite-dimensional manifold is modeled on $\mathbb{R}^n$.

Definition 1.3 Let V, W be *Fréchet spaces*, i.e., complete locally convex Hausdorff metrizable vector spaces, and let U be an open subset of V. A map $f : U \subset V \to W$ is said to be *differentiable* at a point $u \in U$ in a direction $v \in V$ if the limit

$$Df(u;v) = \lim_{t \to 0} \frac{f(u + tv) - f(u)}{t} \tag{1.1}$$

exists. The function is said to be continuously differentiable on U if the limit exists for all $u \in U$ and all $v \in V$, and if the function $Df : U \times V \to W$ is continuous as a function on $U \times V$. In the same way, we can build the second derivative $D^2 f$, which (if it exists) will be a function $D^2 f : U \times V \times V \to W$, and so on. A function $f : U \to W$ is called *smooth* or C^∞ if all its derivatives exist and are continuous.

Definition 1.4 A *Fréchet manifold* is a Hausdorff space with a coordinate atlas taking values in a Fréchet space such that all transition functions are smooth maps.

Remark 1.5 Now one can start defining vector fields, tangent spaces, differential forms, principal bundles, and the like on a Fréchet manifold exactly in the same way as for finite-dimensional manifolds.

For example, for a manifold M, a *tangent vector* at some point $m \in M$ is defined as an equivalence class of smooth parametrized curves $f : \mathbb{R} \to M$ such that $f(0) = m$. The set of all such equivalence classes is the *tangent space* $T_m M$ at m. The union of the tangent spaces $T_m M$ for all $m \in M$ can be given the structure of a Fréchet manifold TM, the tangent bundle of M. Now a smooth vector field on the manifold M is a smooth map $v : M \to TM$, and one defines in a similar vein the directional derivative of a function and the Lie bracket of two vector fields.

Since the dual of a Fréchet space need not be Fréchet, we define differential 1-forms in the Fréchet setting directly, as smooth maps $\alpha : TM \to \mathbb{R}$ such that for any $m \in M$, the restriction $\alpha|_{T_m M} : T_m M \to \mathbb{R}$ is a linear map. Differential forms of higher degree are defined analogously: say, a 2-form on a Fréchet manifold M is a smooth map $\beta : T^{\otimes 2} M \to \mathbb{R}$ whose restriction $\beta|_{T_m^{\otimes 2} M} : T_m^{\otimes 2} M \to \mathbb{R}$ for any $m \in M$ is bilinear and antisymmetric. The differential df of a smooth function $f : M \to \mathbb{R}$ is defined via the directional derivative, and this construction generalizes to smooth n-forms on a Fréchet manifold M to give the *exterior derivative* operator d, which maps n-forms to $(n+1)$-forms on M; see, for example, [231].

Remark 1.6 More facts on infinite-dimensional manifolds can be found in, e.g., [265, 157]. From now on, whenever we speak of an infinite-dimensional manifold, we implicitly mean a Fréchet manifold (unless we say explicitly otherwise). In particular, our infinite-dimensional Lie groups are *Fréchet Lie groups.*

Instead of Fréchet manifolds, one could consider manifolds modeled on Banach spaces. This would lead to the category of Banach manifolds. The main advantage of Banach manifolds is that strong theorems from finite-dimensional analysis, such as the inverse function theorem, hold in Banach spaces but not necessarily in Fréchet spaces. However, some of the Lie groups we will be considering, such as the diffeomorphism groups, are not Banach manifolds. For this reason we stay within the more general framework of Fréchet manifolds. In fact, for most purposes, it is enough to consider groups modeled on locally convex vector spaces. This is the setting considered by Milnor [265].

1.2 The Lie Algebra of a Lie Group

Definition 1.7 Let G be a Lie group with the identity element $e \in G$. The tangent space to the group G at its identity element is (the vector space of) the *Lie algebra* $\mathfrak{g}$ of this group G. The group multiplication on a Lie group G endows its Lie algebra $\mathfrak{g}$ with the following bilinear operation $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket* on $\mathfrak{g}$.

First note that the Lie algebra $\mathfrak{g}$ can be identified with the set of left-invariant vector fields on the group G. Namely, to a given vector $X \in \mathfrak{g}$ one can associate a vector field $\widetilde{X}$ on G by left translation: $\widetilde{X}(g) = {l_g}_* X$, where $l_g : G \to G$ denotes the multiplication by a group element g from the left, $h \in G \mapsto gh$. Obviously, such a vector field $\widetilde{X}$ is invariant under left translations by elements of G. That is, ${l_g}_* \widetilde{X} = \widetilde{X}$ for all $g \in G$. On the other hand, any left-invariant vector field $\widetilde{X}$ on the group G uniquely defines an element $\widetilde{X}(e) \in \mathfrak{g}$.

The usual Lie bracket (or commutator) $[\widetilde{X}, \widetilde{Y}]$ of two left-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ on the group is again a left-invariant vector field on G. Hence we can write $[\widetilde{X}, \widetilde{Y}] = \widetilde{Z}$ for some $Z \in \mathfrak{g}$. We define the *Lie bracket* $[X, Y]$ of two elements X, Y of the Lie algebra $\mathfrak{g}$ of the group G via $[X, Y] := Z$. The Lie bracket gives the space $\mathfrak{g}$ the *structure of a Lie algebra.*

Examples 1.8 Here are several finite-dimensional Lie groups and their Lie algebras:

- $\mathrm{GL}(n, \mathbb{R})$, the set of nondegenerate $n \times n$ matrices, is a Lie group with respect to the matrix product: multiplication and taking the inverse are smooth operations. Its Lie algebra is $\mathfrak{gl}(n,\mathbb{R}) = \mathrm{Mat}(n,\mathbb{R})$, the set of all $n \times n$ matrices.

- $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A = 1\}$ is a Lie group and a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$. Its Lie algebra is the space of traceless matrices $\mathfrak{sl}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid \operatorname{tr} A = 0\}$. This follows from the relation

$$\det(I + \epsilon A) = 1 + \epsilon \operatorname{tr} A + \mathcal{O}(\epsilon^2)\,, \qquad \text{as} \qquad \epsilon \to 0\,,$$

where I is the identity matrix.

- $\mathrm{SO}(n,\mathbb{R})$ is a Lie group of transformations $\{A:\ \mathbb{R}^n \to \mathbb{R}^n\}$ preserving the Euclidean inner product of vectors (and orientation) in $\mathbb{R}^n$, i.e. $(Au,\ Av) = (u,\ v)$ for all vectors $u, v \in \mathbb{R}^n$. Equivalently, one can define

$$\mathrm{SO}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid AA^t = I,\ \det A > 0\}.$$

The Lie algebra of $\mathrm{SO}(n)$ is the space of skew-symmetric matrices

$$\mathfrak{so}(n,\mathbb{R}) = \{A \in \mathfrak{gl}(n,\mathbb{R}) \mid A + A^t = 0\}\,,$$

as the relation

$$(I + \epsilon A)(I + \epsilon A^t) = I + \epsilon(A + A^t) + \mathcal{O}(\epsilon^2)$$

shows.

- $\mathrm{Sp}(2n,\mathbb{R})$ is the group of transformations of $\mathbb{R}^{2n}$ preserving the nondegenerate skew-product of vectors.

Exercise 1.9 Give an alternative definition of $\mathrm{Sp}(2n,\mathbb{R})$ with the help of the equation satisfied by the corresponding matrices for the following skew-product of vectors $\langle u, v\rangle := \sum_{j=1}^{n}(u_j v_{j+n} - v_j u_{j+n})$. Find the corresponding Lie algebra.

Exercise 1.10 Show that in all of Examples 1.8, the Lie bracket is given by the usual commutator of matrices: $[A, B] = AB - BA$.

The following examples are the first infinite-dimensional Lie groups we shall encounter.

Example 1.11 Let M be a compact n-dimensional manifold. Consider the set $\mathrm{Diff}(M)$ of diffeomorphisms of M. It is an open subspace of (the Fréchet manifold of) all smooth maps from M to M. One can check that the composition and inversion are smooth maps, so that the set $\mathrm{Diff}(M)$ is a Fréchet

Lie group; see [157].[2] Its Lie algebra is given by $\mathrm{Vect}(M)$, the Lie algebra of smooth vector fields on M.

Given a volume form μ on M, one can define the group of volume-preserving diffeomorphisms

$$S\mathrm{Diff}(M) := \{\phi \in \mathrm{Diff}(M) \mid \phi^*\mu = \mu\}\,.$$

It is a Lie group, since $S\mathrm{Diff}(M)$ is a closed subgroup of $\mathrm{Diff}(M)$. Its Lie algebra $S\mathrm{Vect}(M) := \{v \in \mathrm{Vect}(M) \mid \mathrm{div}(v) = 0\}$ consists of vector fields on M that are divergence-free with respect to the volume form μ.

Example 1.12 Let M be a finite-dimensional compact manifold and let G be a finite-dimensional Lie group. Set the *group of currents* on M to be $G^M = C^\infty(M, G)$, the group of G-valued functions on M. We can define a multiplication on G^M pointwise, i.e., we set $(\varphi \cdot \psi)(g) = \varphi(g)\psi(g)$ for all φ, $\psi \in G^M$. This multiplication gives G^M the structure of a (Fréchet) Lie group, as we discuss below.

Example 1.13 A slight, but important, generalization of the example above is the following: Let G be a finite-dimensional Lie group, and P a principal G-bundle over a manifold M. Denote by $\pi : P \to M$ the natural projection to the base. Define the Lie *group* $\mathrm{Gau}(P)$ *of gauge transformations* (or, simply, the *gauge group*) of P as the group of bundle (i.e., fiberwise) automorphisms: $\mathrm{Gau}(P) = \{\varphi \in \mathrm{Aut}(P) \mid \pi \circ \varphi = \pi\}$. The group multiplication is the natural composition of the bundle automorphisms. (Automorphisms of each fiber of P form a copy of the group G, and all together they define the associated bundle over M with the structure group G. The identity bundle automorphism gives the trivial section of this associated G-bundle, and the gauge transformation group consists of all smooth sections of it; see details in [265].) One can show that this is a Lie group (cf. [157]), and we denote the corresponding Lie algebra by $\mathfrak{gau}(P)$. For a topologically trivial G-bundle P, the group $\mathrm{Gau}(P)$ coincides with the current group G^M.

Exercise 1.14 Describe the Lie brackets for the Lie algebras in the last three examples.

Remark 1.15 For a Lie group G, the Lie bracket on the corresponding Lie algebra $\mathfrak{g}$, which we defined via the usual Lie bracket of left-invariant vector fields on the group, satisfies the following properties:

[2] In many analysis questions it is convenient to work with the larger space of diffeomorphisms $\mathrm{Diff}^s(M)$ of Sobolev class H^s. For $s > n/2 + 1$ these spaces are smooth Hilbert manifolds. On the other hand, the spaces $\mathrm{Diff}^s(M)$ are only topological (but not smooth) groups, since the composition of such diffeomorphisms is not smooth. Indeed, while the right multiplication $r_\phi : \psi \mapsto \psi \circ \phi$ is smooth, the left multiplication $l_\psi : \phi \mapsto \psi \circ \phi$ is only continuous, but not even Lipschitz continuous; see [95].

(*i*) it is antisymmetric in X and Y, i.e., $[X, Y] = -[Y, X]$, and
(*ii*) it satisfies the Jacobi identity:

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0\,.$$

The Jacobi identity can be thought of as an infinitesimal analogue of the associativity of the group multiplication.

1.3 The Exponential Map

Definition 1.16 The *exponential map* from a Lie algebra to the corresponding Lie group $\exp : \mathfrak{g} \to G$ is defined as follows: Let us fix some $X \in \mathfrak{g}$ and let $\widetilde{X}$ denote the corresponding left-invariant vector field. The flow of the field $\widetilde{X}$ is a map $\phi_X : G \times \mathbb{R} \to G$ such that $\frac{d}{dt}\phi_X(g, t) = \widetilde{X}(\phi_X(g, t))$ for all t and $\phi_X(g, 0) = g$. The flow ϕ_X is the solution of an ordinary differential equation, which, if it exists, is unique. In the case that the flow subgroup $\phi_X(e, .)$ exists for all $X \in \mathfrak{g}$, we define the exponential map $\exp : \mathfrak{g} \to G$ via the time-one map $X \mapsto \phi_X(e, 1)$; see Figure 1.1.

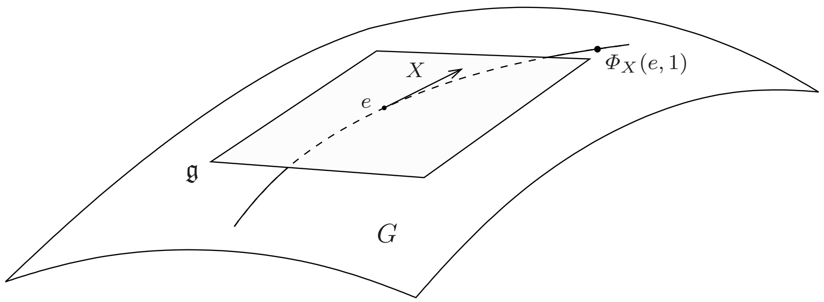

Fig. 1.1. The exponential map on the group G associates to a vector X the time-one map for the trajectory of a left-invariant vector field defined by X at $e \in G$.

Example 1.17 For each of the finite-dimensional Lie groups considered in Example 1.8, the exponential map is given by the usual exponential map for matrices:

$$\exp : A \mapsto \exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n\,.$$

Remark 1.18 The definition of the exponential map relies on the existence and uniqueness of solutions of certain first-order differential equations. In general, solutions of differential equations in Fréchet spaces might not be unique.[3] However, the differential equation in the definition of the exponential map is of special type, which secures the solution's uniqueness upon fixing its initial condition. Namely, let $\phi : \mathbb{R} \to G$ be a smooth path in the Lie group G. Its derivative $\phi'(t) := \frac{d}{dt}\phi(t)$ is a tangent vector to the group G at the point $\phi(t)$. Translate this vector back to the identity via left multiplication by $\phi^{-1}(t)$. The corresponding element of the Lie algebra $\mathfrak{g}$ is denoted by $\phi^{-1}(t)\phi'(t)$ and is called the *left logarithmic derivative* of the path ϕ.

Now consider a Lie algebra element $X \in \mathfrak{g}$. By definition of the exponential map, the curve $\phi(t) = \exp(tX)$ satisfies the differential equation $\phi'(t) = \phi(t)X$ with the initial condition $\phi(0) = e$. So for all solutions of this differential equation, the left logarithmic derivative is given by the constant curve $X \in \mathfrak{g}$. Now the uniqueness of the exponential map is implied by the following Exercise.

Exercise 1.19 Show that two smooth paths ϕ, $\psi : \mathbb{R} \to G$ have the same left logarithmic derivative for all $t \in \mathbb{R}$ if and only if they are translations of each other by some constant element $g \in G$: $\phi(t) = g\,\psi(t)$ for all $t \in \mathbb{R}$. (Hint: see, e.g., [265].)

Remark 1.20 As far as the existence is concerned, the exponential map exists for all finite-dimensional Lie groups and more generally for Lie groups modeled on Banach spaces, as follows from the general theory of differential equations. However, there may exist infinite-dimensional Lie groups that do not admit an exponential map. Moreover, even in the cases in which the exponential map of an infinite-dimensional group exists, it can exhibit rather peculiar properties; see the examples below.

Example 1.21 For the diffeomorphism group $\mathrm{Diff}(M)$ the exponential map $\exp : \mathrm{Vect}(M) \to \mathrm{Diff}(M)$ has to assign to each vector field on M the time-one map for its flow. However, for a noncompact M this map may not exist: the corresponding vector field may not be complete. Indeed, for example, for the vector field $\xi = x^2\partial/\partial x$ on the real line $M = \mathbb{R}$, the time-one map of the flow is not defined on the whole of $\mathbb{R}$: the corresponding flow sends some points to infinity for the time less than 1! Fortunately, for compact manifolds M and smooth vector fields, the time-one maps of the corresponding flows, and hence the exponential maps, are well defined.

Note that the group of diffeomorphisms of a noncompact manifold is not complete, and hence it is not a Lie group in our sense. It is an important open problem to find a Lie group that is modeled on a complete space and does not admit an exponential map.

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[3] For instance, the initial value problem $u(x, 0) = f(x)$ for the equation $u_t(x, t) = u_x(x, t)$ with $x \in [0, 1]$ has wave-type solutions $u(x, t) = f(x + t)$. For nonzero t such a solution $u(x, t)$ for $x \in [0, 1]$ depends on the extension of $f(x)$ to the segment $[-t, 1 - t]$. Due to arbitrariness in the choice of a smooth extension of f from $[0, 1]$ to $\mathbb{R}$, the solution to this initial value problem is not unique.

Let us return to the current group G^M, where the exponential map exists and can be used to give this group the structure of a Fréchet Lie group. Namely, the space $\mathfrak{g}^M = C^\infty(M, \mathfrak{g})$ endowed with the topology of uniform convergence is a Fréchet space. Moreover, the map $\exp : \mathfrak{g} \to G$ can be used to define a map $\widetilde{\exp} : \mathfrak{g}^M \to G^M$ pointwise. In a sufficiently small neighborhood of $0 \in \mathfrak{g}^M$, the map $\widetilde{\exp}$ is bijective. Thus it can be used to define a local system of open neighborhoods of the identity in G^M. We can use left translation to transfer this system to any point in G^M and thus define a topology on the group G^M. Again using the exponential map, we can define coordinate charts on G^M. This definition implies that multiplication and inversion in G^M are smooth maps. So G^M is an infinite-dimensional Lie groups (see, e.g., [157] for more details).

From the construction of the Lie group structure on G^M, it is clear that its Lie algebra is the current algebra $\mathfrak{g}^M$, and that the exponential map $\mathfrak{g}^M \to G^M$ is the map $\widetilde{\exp}$ described above. Note, however, that $\widetilde{\exp}$ is not, in general, surjective, even if $\exp : \mathfrak{g} \to G$ is surjective. As an example, take the manifold M to be the circle S^1 and G to be the group $\mathrm{SU}(2)$.

Exercise 1.22 Show that the map

$$\theta \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

for $\theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ defines an element in G^{S^1} that does not belong to the image of the exponential map $\widetilde{\exp} : \mathfrak{g}^{S^1} \to G^{S^1}$.

In contrast to the exponential map in the case of the current group G^M, the exponential map $\exp : \mathrm{Vect}(M) \to \mathrm{Diff}(M)$ for the diffeomorphism group of a compact M is not, in general, even locally surjective already for the case of a circle.

Proposition 1.23 (see, e.g., [265, 301, 322]) *The exponential map* $\exp : \mathrm{Vect}(S^1) \to \mathrm{Diff}(S^1)$ *is not locally surjective.*

PROOF. First observe that any nowhere-vanishing vector field on S^1 is conjugate under $\mathrm{Diff}(S^1)$ to a constant vector field. Indeed, if $\xi(\theta) = v(\theta)\frac{\partial}{\partial\theta}$ is such a vector field, we can define a diffeomorphism $\psi : S^1 \to S^1$ via $\psi(\theta) = a\int_0^\theta \frac{dt}{v(t)}$. Here, $a \in \mathbb{R}$ is chosen such that $\psi(2\pi) = 2\pi$. Then $\psi_*(\xi \circ \psi^{-1})$ is a constant vector field on S^1.

From this observation, one can conclude that any diffeomorphism of S^1 that lies in the image of the exponential map and that does not have any fixed points is conjugate to a rigid rotation of S^1. Hence in order to see that the exponential map is not locally surjective, it is enough to construct diffeomorphisms arbitrarily close to the identity that do not have any fixed points and that are not conjugate to a rigid rotation. For this, one can take diffeomorphisms without fixed points, but which have isolated periodic points, i.e., fixed points for a certain nth iteration of this diffeomorphism. Indeed, if such a diffeomorphism ψ belonged to the image of the exponential map, so would its nth power ψ^n. Then the corresponding vector field defining the ψ^n as the time-one map would either have zeros or be nonvanishing everywhere. In the former case, the n-periodic points of ψ must actually be its fixed points, while in the latter case, the diffeomorphism ψ^n, as well as ψ, would be conjugate to a rigid rotation and hence *all* points of ψ would be n-periodic. Both cases give us a contradiction.

Explicitly, a family of such diffeomorphisms can be constructed as follows: Let us identify S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Then consider the map $\psi_{n,\epsilon} : x \mapsto x + \frac{2\pi}{n} + \epsilon \sin(nx)$. For ϵ small enough, this is indeed a diffeomorphism of S^1. Furthermore, by choosing n large and ϵ small, the diffeomorphisms $\psi_{n,\epsilon}$ can be made arbitrarily close to the identity while having no fixed points. Finally, for $\epsilon \neq 0$, $\psi_{n,\epsilon}$ cannot be conjugate to a rigid rotation. If it were conjugate to a rotation, it would have to be the rotation $\psi_{n,0}$, since $\psi^n_{n,\epsilon}(0) = 0$. But in this case, we would have $\psi^n_{n,\epsilon} = \mathrm{id}$, which is not true for $\epsilon \neq 0$. □

1.4 Abstract Lie Algebras

As we have seen in the last section, the Lie bracket of two left-invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ on a Lie group G defines a bilinear map $[\,.\,,.\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra of G that is antisymmetric in X and Y and satisfies the Jacobi identity (1.2). These properties can be taken as the definition of an abstract Lie algebra:

Definition 1.24 An *(abstract) Lie algebra* is a real or complex vector space $\mathfrak{g}$ together with a bilinear map $[\,.\,,.\,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (the Lie bracket) that is antisymmetric in X and Y and that satisfies the Jacobi identity

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0\,. \tag{1.2}$$

All the Lie algebras we have encountered so far as accompanying the corresponding Lie groups can also be regarded by themselves, i.e., as abstract Lie algebras. A famous theorem of Sophus Lie states that every finite-dimensional (abstract) Lie algebra $\mathfrak{g}$ is the Lie algebra of some Lie group G. In infinite dimensions this is no longer true in general.

Example 1.25 ([205, 207]) To illustrate the failure of Lie's theorem in an infinite-dimensional context, consider the Lie algebra of complex vector fields on the circle $\mathrm{Vect}^{\mathbb{C}}(S^1) = \mathrm{Vect}(S^1) \otimes \mathbb{C}$. Let us show that this Lie algebra cannot be the Lie algebra of any Lie group. First note that $\mathrm{Vect}^{\mathbb{C}}(S^1)$ contains as a subalgebra the Lie algebra $\mathrm{Vect}(S^1)$ of real vector fields on the circle, which is the Lie algebra of the group $\mathrm{Diff}(S^1)$.

Let G_1 denote the group $\mathrm{PSL}(2,\mathbb{R})$ and let G_k denote the k-fold covering of G_1. The group G_2 is isomorphic to $\mathrm{SL}(2,\mathbb{R})$, while for $k>2$ it is known that the groups G_k have no matrix realization. The group $\mathrm{Diff}(S^1)$ contains each G_k as a subgroup. Namely, G_k is the subgroup corresponding to the Lie subalgebra $\mathfrak{g}_k$ spanned by the vector fields

$$\frac{\partial}{\partial\theta},\ \sin(k\theta)\frac{\partial}{\partial\theta},\ \cos(k\theta)\frac{\partial}{\partial\theta}\,.$$

(Note that each $\mathfrak{g}_k$ is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$.)

Now suppose that there exists a complexification of the group $\mathrm{Diff}(S^1)$, i.e., a Lie group G corresponding to the complex Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$. Such a group G would have to contain the complexifications of all the groups G_k. However, for $k>2$ the groups G_k do not admit complexifications: the only complex groups corresponding to the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ are $\mathrm{SL}(2,\mathbb{C})$ and $\mathrm{PSL}(2,\mathbb{C})$.

More precisely, if the complex Lie group G existed, the real subgroups G_k would belong to the complex subgroups of G corresponding to complex subalgebras $\mathfrak{g}_k^{\mathbb{C}}\simeq\mathfrak{sl}(2,\mathbb{C})$. But these complex subgroups have to be isomorphic either to $\mathrm{SL}(2,\mathbb{C})$, which contains only $\mathrm{SL}(2,\mathbb{R})=G_2$, or to $\mathrm{PSL}(2,\mathbb{C})$, which contains only $\mathrm{PSL}(2,\mathbb{R})=G_1$. Thus the complex group G containing all G_k cannot exist, and hence there is no Lie group for the Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$.

Lie algebra homomorphisms are defined in the usual way: A map $\rho:\mathfrak{g}\to\mathfrak{h}$ between two Lie algebras is a Lie algebra homomorphism if it satisfies $\rho([X,Y])=[\rho(X),\rho(Y)]$ for all X, $Y\in\mathfrak{g}$. We will also need another important class of maps between Lie algebras called derivations:

Definition 1.26 A linear map $\delta:\mathfrak{g}\to\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ to itself is called a *derivation* if it satisfies

$$\delta([X,Y])=[\delta(X),Y]+[X,\delta(Y)]$$

for all X, $Y\in\mathfrak{g}$.

Exercise 1.27 Define the map $\mathrm{ad}_X:\mathfrak{g}\to\mathfrak{g}$ associated to a fixed vector $X\in\mathfrak{g}$ via

$$\mathrm{ad}_X(Y)=[X,Y]\,.$$

Show that this is a derivation for any choice of X. (Hint: use the Jacobi identity.)

If a derivation of a Lie algebra $\mathfrak{g}$ can be expressed in the form ad_X for some $X\in\mathfrak{g}$, it is called an *inner derivation*; otherwise, it is called an *outer derivation* of $\mathfrak{g}$.

Exercise 1.28 Let δ be a derivation of a Lie algebra $\mathfrak{g}$, and suppose that $\exp(\delta) = \sum_{i=0}^{\infty} \frac{1}{i!}\delta^i$ makes sense (for example, suppose, the map δ is nilpotent). Show that the map $\exp(\delta)$ is an automorphism of the Lie algebra $\mathfrak{g}$.

Definition 1.29 A *subalgebra* of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subset \mathfrak{g}$ invariant under the Lie bracket in $\mathfrak{g}$. An *ideal* of a Lie algebra $\mathfrak{g}$ is a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $[X, \mathfrak{h}] \subset \mathfrak{h}$ for all $X \in \mathfrak{g}$.

The importance of ideals comes from the fact that if $\mathfrak{h} \subset \mathfrak{g}$ is an ideal, then the quotient space $\mathfrak{g}/\mathfrak{h}$ is again a Lie algebra.

Exercise 1.30 (i) Show that for an ideal $\mathfrak{h} \subset \mathfrak{g}$ the Lie bracket on $\mathfrak{g}$ descends to a Lie bracket on the quotient space $\mathfrak{g}/\mathfrak{h}$.

(ii) Show that if $\rho : \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is a homomorphism of two Lie algebras, then the kernel $\ker\rho$ of ρ is an ideal in $\mathfrak{g}$.

Definition 1.31 A Lie algebra is *simple* (respectively, *semisimple*) if it does not contain nontrivial ideals (respectively, nontrivial abelian ideals).

Any finite-dimensional semisimple Lie algebra is a direct sum of nonabelian simple Lie algebras.

A group analogue of an ideal is the notion of a normal subgroup. A subgroup $H \subset G$ of a group G is called *normal* if $gHg^{-1} \subset H$ for all $g \in G$. Exercise 1.30 translates directly to normal subgroups.

2 Adjoint and Coadjoint Orbits

Writing out a linear operator in a different basis or a vector field in a different coordinate system has a far-reaching generalization as the adjoint representation for any Lie group. In this section we define the adjoint and coadjoint representations and the corresponding orbits for an arbitrary Lie group.

2.1 The Adjoint Representation

A *representation* of a Lie group G on a vector space V is a linear action φ of the group G on V that is smooth in the sense that the map $G \times V \to V$, $(g, v) \mapsto gv$, is smooth. If V is a real vector space, (V, φ) is called a real representation, and if V is complex, it is a complex representation. (Here V is assumed to be a Fréchet space, and, often, a Hilbert space. In the latter case, the representation is said to be unitary if the inner product on V is invariant under the action of G.)

Every Lie group has two distinguished representations: the adjoint and the coadjoint representations. Since they will play a special role in this book, we describe them in more detail.

Any element $g \in G$ defines an automorphism c_g of the group G by conjugation:

$$c_g : h \in G \mapsto ghg^{-1}.$$

The differential of c_g at the identity $e \in G$ maps the Lie algebra of G to itself and thus defines an element $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$, the group of all automorphisms of the Lie algebra $\mathfrak{g}$.

Definition 2.1 The map $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{Ad}_g$ defines a representation of the group G on the space $\mathfrak{g}$ and is called the *group adjoint representation*; see Figure 2.1. The orbits of the group G in its Lie algebra $\mathfrak{g}$ are called the *adjoint orbits* of G.

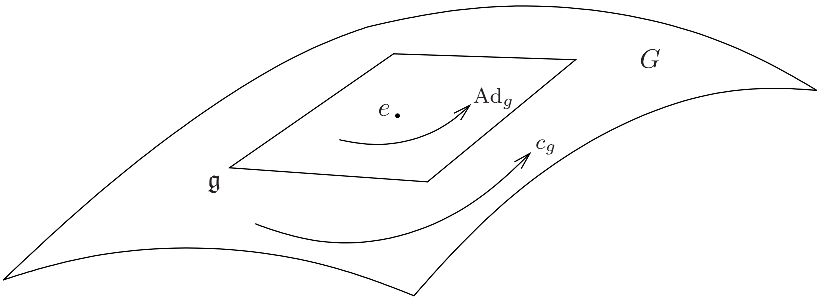

Fig. 2.1. Conjugation c_g on the group G generates the adjoint representation Ad_g on the Lie algebra $\mathfrak{g}$.

The differential of $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ at the group identity $g = e$ defines a map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, the *adjoint representation of the Lie algebra* $\mathfrak{g}$.

One can show that the bracket [,] on the space $\mathfrak{g}$ defined via

$$[X, Y] := \mathrm{ad}_X(Y)$$

coincides with the bracket (or commutator) of the corresponding two left-invariant vector fields on the group G and hence with the Lie bracket on $\mathfrak{g}$ defined in Section 1.2.

Example 2.2

- Let $g \in \mathrm{GL}(n, \mathbb{R})$ and $A \in \mathfrak{gl}(n, \mathbb{R})$. Then $\mathrm{Ad}_g A = gAg^{-1}$. Hence the adjoint orbits are given by sets of similar (i.e., conjugate) matrices in $\mathfrak{gl}(n, \mathbb{R})$. The adjoint representation of $\mathfrak{gl}(n, \mathbb{R})$ is given by $\mathrm{ad}_A(B) = [A, B] = AB - BA$.

- The adjoint orbits of $\mathrm{SO}(3,\mathbb{R})$ are spheres centered at the origin of $\mathbb{R}^3 \simeq \mathfrak{so}(3,\mathbb{R})$ and the origin itself.
- The adjoint orbits of $\mathrm{SL}(2,\mathbb{R})$ are contained in the sets of similar matrices. By writing $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}(2,\mathbb{R})$, one sees that the adjoint orbits lie in the level sets of $\Delta = -(a^2+bc) = \text{const}$: matrices that are conjugate to each other have the same determinant. Note, however, that not all matrices in $\mathfrak{sl}(2,\mathbb{R})$ that have the same determinant are conjugate. For instance, the matrices with determinant $\Delta = 0$ constitute three different orbits: the origin and two other orbits, cones, passing through the matrices $\left(\begin{smallmatrix} 0 & \pm 1 \\ 0 & 0 \end{smallmatrix}\right)$, respectively. For $\Delta \neq 0$ the $\mathrm{SL}(2,\mathbb{R})$-orbits are either one-sheet hyperboloids or connected components of the two-sheet hyperboloids $a^2 + bc = \text{const}$, since the group $\mathrm{SL}(2,\mathbb{R})$ is connected.
- Let G be the set of orientation-preserving affine transformations of the real line. That is, $G = \{(a,b) \mid a,b \in \mathbb{R},\ a > 0\}$, and $(a,b) \in G$ acts on $x \in \mathbb{R}$ via $x \mapsto ax+b$. The Lie algebra of G is $\mathbb{R}^2$, and its adjoint orbits are the affine lines
$$\{(\alpha,\beta) \in \mathbb{R}^2 \mid \alpha = \text{const} \neq 0, \quad \beta \text{ arbitrary}\},$$
the two rays
$$\{(\alpha,\beta) \in \mathbb{R}^2, \alpha = 0, \beta < 0\} \text{ and } \{(\alpha,\beta) \in \mathbb{R}^2, \alpha = 0, \beta > 0\},$$
and the origin $\{(0,0)\}$; see Figure 2.2.
- Let M be a compact manifold. The adjoint orbits of the current group $\mathrm{GL}(n,\mathbb{C})^M$ in its Lie algebra $\mathfrak{gl}(n,\mathbb{C})^M$ are given by fixing the (smoothly dependent) Jordan normal form of the current at each point of the manifold M.
- Let M be a compact manifold. The adjoint representation of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ is given by coordinate changes of the vector field: for a $\phi \in \mathrm{Diff}(M)$ one has $\mathrm{Ad}_\phi : v \mapsto \phi_* v \circ \phi^{-1}$. The adjoint representation of $\mathrm{Vect}(M)$ on itself is given by the negative of the usual Lie bracket of vector fields: $\mathrm{ad}_v\, w = \frac{\partial v}{\partial x} w(x) - \frac{\partial w}{\partial x} v(x)$ in any local coordinate x.

Exercise 2.3 Verify the latter formula for the action of $\mathrm{Diff}(M)$ on $\mathrm{Vect}(M)$ from the definition of the group adjoint action. (Hint: express the diffeomorphisms corresponding to the vector fields $v(x)$ and $w(x)$ in the form
$$g(t): \quad x \mapsto x + tv(x) + o(t), \quad h(s): \quad x \mapsto x + sw(x) + o(s), \qquad t,s \to 0,$$
and find the first several terms of $g(t)h(s)g^{-1}(t)$.)

2.2 The Coadjoint Representation

The dual object to the adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$ is called the coadjoint representation of G on $\mathfrak{g}^*$, the dual space to $\mathfrak{g}$.

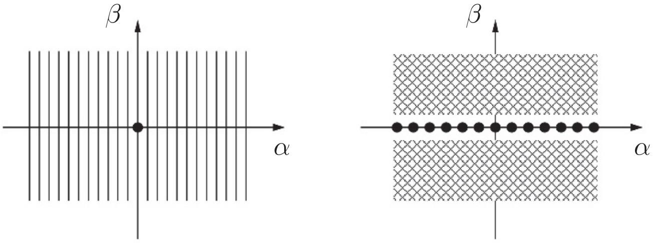

Fig. 2.2. Adjoint and coadjoint orbits of the group of affine transformations on the line.

Definition 2.4 The *coadjoint representation* Ad^* of the group G on the space $\mathfrak{g}^*$ is the dual of the adjoint representation. Let $\langle\ ,\ \rangle$ denote the pairing between $\mathfrak{g}$ and its dual $\mathfrak{g}^*$. Then the *coadjoint action of the group* G on the dual space $\mathfrak{g}^*$ is given by the operators $\mathrm{Ad}^*_g : \mathfrak{g}^* \to \mathfrak{g}^*$ for any $g \in G$ that are defined by the relation

$$\langle \mathrm{Ad}^*_g(\xi), X\rangle := \langle \xi, \mathrm{Ad}_{g^{-1}}(X)\rangle \tag{2.3}$$

for all ξ in $\mathfrak{g}^*$ and $X \in \mathfrak{g}$. The orbits of the group G under this action on $\mathfrak{g}^*$ are called the *coadjoint orbits* of G.

The differential $\mathrm{ad}^* : \mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ of the group representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ at the group identity $e \in G$ is called the *coadjoint representation of the Lie algebra* $\mathfrak{g}$. Explicitly, at a given vector $Z \in \mathfrak{g}$ it is defined by the relation

$$\langle \mathrm{ad}^*_Z(\xi), X\rangle = -\langle \xi, \mathrm{ad}_Z(X)\rangle.$$

Remark 2.5 The dual space of a Fréchet space is not necessarily again a Fréchet space. In this case, instead of considering the full dual space to an infinite-dimensional Lie algebra $\mathfrak{g}$, we will usually confine ourselves to considering only appropriate "smooth duals," the functionals from a certain G-invariant Fréchet subspace $\mathfrak{g}^*_s \subset \mathfrak{g}^*$. Natural smooth duals will be different according to the type of the infinite-dimensional groups considered, but they all have a (weak) nondegenerate pairing with the corresponding Lie algebra $\mathfrak{g}$ in the following sense: for every nonzero element $X \in \mathfrak{g}$, there exists some element $\xi \in \mathfrak{g}^*_s$ such that $\langle \xi, X\rangle \neq 0$, and the other way around. This ensures that the coadjoint action is uniquely fixed by equation (2.3). The pair $(\mathfrak{g}^*_s, \mathrm{Ad}^*|_{\mathfrak{g}^*_s})$ is called the regular (or smooth) part of the coadjoint representation of G, and, abusing notations, we will usually skip the index s.

Example 2.6

- In the first three cases of Example 2.2, there exists a G-invariant inner product on $\mathfrak{g}$ that induces an isomorphism between $\mathfrak{g}$ and $\mathfrak{g}^*$ respecting the group actions. Hence the adjoint and coadjoint representations of the groups G are isomorphic, and the coadjoint orbits coincide with the adjoint ones.