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Rolf Schneider · Wolfgang Weil

Stochastic and Integral Geometry

Springer

Rolf Schneider
Mathematisches Institut
Albert-Ludwigs-Universität
Eckerstr. 1
79104 Freiburg
Germany
rolf.schneider@math.uni-freiburg.de

Wolfgang Weil
Institut für Algebra und Geometrie
Universität Karlsruhe
Englerstraße 2
76128 Karlsruhe
Germany
weil@math.uka.de

Series Editors:

Joe Gani
Chris Heyde
Centre for Mathematics and its Applications
Mathematical Sciences Institute
Australian National University
Canberra, ACT 0200
Australia
gani@maths.anu.edu.au

Thomas G. Kurtz
Department of Mathematics
University of Wisconsin - Madison
480 Lincoln Drive
Madison, WI 53706-1388
USA
kurtz@math.wisc.edu

Peter Jagers
Mathematical Statistics
Chalmers University of Technology
and Göteborg (Gothenburg) University
412 96 Göteborg
Sweden
jagers@chalmers.se

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Preface

Stochastic Geometry deals with mathematical models for random geometric structures and spatial data, as they frequently arise in modern applications. As a mathematical discipline, stochastic geometry came into life in the last third of the twentieth century, but its roots and the close connections between geometric probability and integration techniques using invariant measures (though not under this name) date back much farther. The famous Buffon needle problem of 1777 was solved by what seems to be the first application of integral calculus to a probability question. A variety of problems in *Geometric Probability* was treated in the late nineteenth and early twentieth century. After the role of invariant measures had become clear, the discipline of *Integral Geometry* was initiated in the 1930s, mostly by Wilhelm Blaschke and his school. The book *Integral Geometry and Geometric Probability* by Luis Santaló (1976) summarizes the concepts and results of the preceding development. Interpretations of integral geometric results in terms of geometric probability abound in that work. At that time, David Kendall and Georges Matheron had already developed, independently, a theory of *Random Sets*, and Roger Miles had written his pioneering thesis on Poisson processes of certain geometric objects. The book *Random Sets and Integral Geometry* by Matheron (1975) presented the new field of Stochastic Geometry in its intimate relation with Integral Geometry. Applications in *Spatial Statistics* and *Stereology*, later also in *Image Analysis*, contributed to a rapid development. The classical integral geometry of Euclidean spaces is well suited to the treatment of random sets and point processes with invariance properties, like stationarity and isotropy. The necessity of studying structures which exhibit anisotropy, or even without spatial homogeneity, grew hand in hand with new developments in integral geometry, coming from *Geometric Measure Theory*. In particular, Federer's local formulas for curvature measures proved useful, and *Translative Integral Geometry* was promoted, meeting the needs of stationary structures.

Over many years, we both gave courses on Integral Geometry or Stochastic Geometry in Freiburg and Karlsruhe. This led to the joint publication of lecture notes in German, under the titles of *Integralgeometrie* (1992) and

Stochastische Geometrie (2000). It was always our plan later to amalgamate both topics in one extended monograph in English. During the time we worked on this project, the field of stochastic geometry has expanded considerably in various directions, too many to include them all in one volume. We decided to concentrate on our original idea, namely to present the basic models of stochastic geometry and their properties, the fundamental concepts and formulas of integral geometry, and the interrelations between these two fields.

In this book, therefore, we have three main aims: to give a sound mathematical foundation for the most basic and general models of stochastic geometry, namely random closed sets, particle processes, and random mosaics, to introduce the reader to the parts of integral geometry that are relevant for the applications in stochastic geometry, and, naturally, to demonstrate such applications. Since the strength of integral geometry lies in the computation of mean values and in integral transformations, this means that we develop mainly a 'first order theory' of stochastic geometry, centering around expectations. This restricted concept, with its foundational character, implies that essential and interesting parts of stochastic geometry are missing: we do not treat special point process models other than Poisson processes, nor higher order moment measures, limit theorems, spatial statistics, practical procedures, simulations; however, we comment on some of these developments in the section notes. The integral geometry here is taylored to its use in stochastic geometry; this influences the selection of topics as well as the approach, which is measure theoretic rather than differential geometric. Another restriction may be seen in the predominance of *invariance* and *independence*. The first means that we study (except in one chapter providing an outlook) only random sets and geometric point processes that are stationary (spatially homogeneous) or even stationary and isotropic, in distribution. Invariance of measures and distributions is the leitmotiv of this volume; it underlies both the stochastic geometry parts and the integral geometric parts. On the stochastic side, there is a preference for independence assumptions, as for example in the prominent role of Poisson processes, with their strong independence properties. Very often, only invariance and independence assumptions allow simple approaches and lead to beautiful results. The confinement to the fundamentals of stochastic geometry leaves us room for emphasizing the geometry; in fact, in integral as well as in stochastic geometry, we draw a richer picture than sketched above, and we include various topics of geometric appeal. For example, there is a chapter on *Geometric Probability*, since this area has seen a recent revival with many interesting problems and results.

Naturally, this book employs notions and results from other fields. We make use of some basic facts from general topology, from the theory of topological groups and homogeneous spaces of Euclidean geometry and their invariant measures, and from the geometry of convex sets; further, some more specialized results concerning geometric inequalities and additive functionals on convex bodies are needed. Anticipating that the familiarity of the readers with these topics will not be uniform, we have collected the required material in an Appendix; this should be consulted whenever necessary. This also allows us to start directly with the fundamental notion in this book, the concept of a random closed set.

We are grateful to many colleagues for their helpful comments on early drafts of our book. Special thanks go to Paul Goodey, Günter Last and Werner Nagel, for providing useful hints after reading parts of the final manuscript, and in particular to Daniel Hug, who has carefully read all of it. He prevented us from including a number of flaws and made many suggestions for improvements. We also thank the Mathematisches Forschungsinstitut Oberwolfach for giving us the opportunity to spend some time, working on our mansucript, in their wonderful 'Research in Pairs' programme.

Freiburg i. Br., Karlsruhe *Rolf Schneider*
Spring 2008 *Wolfgang Weil*

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Prolog

1.1 Introduction

Since this book is about relations between stochastic geometry and integral geometry, we begin with an imaginary experiment that demonstrates the need for and use of integral geometry for certain geometric probability questions and at the same time leads in a natural way to a basic model of stochastic geometry.

We assume that K and W are given convex bodies (nonempty compact convex sets) in d-dimensional Euclidean space $\mathbb{R}^d$. The body K serves to generate a random field of congruent copies of K, and the body W plays the role of an 'observation window'. The random field consists of countably many congruent copies of K which are laid out in space randomly and independently, overlappings being allowed. The number of bodies in the random field that hit (that is, have nonempty intersection with) the observation window W is a random variable. We ask for its distribution. This is, of course, not a meaningful question, as long as no stochastic model for the random field of convex bodies is specified. In a few steps, we shall introduce some natural assumptions, which motivate a precise model and lead to an explicit formula for the desired distribution.

In the first step, we consider a much simpler situation. We take a ball B_r of radius r and origin 0 that contains the observation window W, and we consider only one randomly moving copy of K, under the condition that it hits B_r. We ask for the probability that it also hits W. There is a geometrically very natural way of specifying a probability distribution of a randomly moving convex body that satisfies the side condition. A random congruent copy of K can be represented in the form $\widetilde{g}K$, where $\widetilde{g}$ is a random element of the group G_d of rigid motions. The locally compact group G_d carries an essentially unique Haar measure, that is, a locally finite Borel measure that is similarly under left and right multiplications and is not identically zero. We denote this measure, with a suitable normalization, by μ. A natural probability distribution of a random congruent copy of K hitting B_r is then obtained by restricting μ as required by the side condition, normalizing, and taking an image measure. Thus, in our situation we define a probability measure $\mathbb{Q}$ on the space $\mathcal{K}$ of convex bodies (with its usual topology) in $\mathbb{R}^d$ by

$$\mathbb{Q}(A) := \frac{\mu(\{g \in G_d : gK \cap B_r \neq \emptyset,\ gK \in A\})}{\mu(\{g \in G_d : gK \cap B_r \neq \emptyset\})}$$

for Borel sets $A \subset \mathcal{K}$. A random congruent copy of K hitting B_r is then, by definition, a random convex body with distribution $\mathbb{Q}$.

Now the probability, denoted by p, that a random congruent copy of K hitting B_r also hits W, is well defined. If we put

$$\mu(K, M) := \mu(\{g \in G_d : gK \cap M \neq \emptyset\})$$

for convex bodies K and M, this probability is given by

$$p = \frac{\mu(K, W)}{\mu(K, B_r)}. \tag{1.1}$$

The computation of $\mu(K, M)$ is a typical task of integral geometry. First, we assume that K is a ball of radius ρ. If the Haar measure μ is suitably normalized, the measure of all motions g that bring K into a hitting position with M is just the measure of all translations that bring the center of K into the parallel body

$$M + B_\rho := \{m + b : m \in M,\ b \in B_\rho\},$$

and hence is the volume of this body. By the **Steiner formula** of convex geometry, this volume is a polynomial of degree at most d in the parameter ρ. It is convenient to write it in the form

$$\lambda_d(M + B_\rho) = \sum_{i=0}^{d} \rho^{d-i} \kappa_{d-i} V_i(M), \tag{1.2}$$

where λ_d is the Lebesgue measure on $\mathbb{R}^d$ and κ_j is the volume of the j-dimensional unit ball. This defines the **intrinsic volumes** $V_0, \dots, V_d$, which are important functionals on the space of convex bodies.

The intrinsic volumes, which appear naturally in the computation of the measure $\mu(K, M)$ for the special case $K = B_\rho$, are also sufficient to handle the general case. The **principal kinematic formula** of integral geometry, specialized to convex bodies, states that

$$\mu(K, M) = \sum_{i=0}^{d} \alpha_{di} V_i(K) V_{d-i}(M), \tag{1.3}$$

with certain explicit constants α_{di}. From (1.1) and (1.3) we obtain

$$p = \frac{\sum_{i=0}^{d} \alpha_{di} V_i(K) V_{d-i}(W)}{\sum_{i=0}^{d} \alpha_{di} V_i(K) V_{d-i}(B_r)}, \tag{1.4}$$

which depends only on the intrinsic volumes of K and W (and on r).

In the second step, we consider $m \geq 2$ independent, identically distributed random convex bodies, each with distribution $\mathbb{Q}$, thus each one is a random congruent copy of K hitting B_r. For $k \in \{0, 1, \ldots, m\}$, we denote by p_k the probability that the fixed body W is hit by exactly k of the random congruent copies of K. By the independence, we obtain a binomial distribution, thus

$$p_k = \binom{m}{k} p^k (1-p)^{m-k},$$

with p given by (1.4).

In the third step, we choose m depending on the radius r and let r tend to ∞, in such a way that

$$\lim_{r\to\infty} \frac{m}{\lambda_d(B_r)} = \gamma > 0$$

with a constant γ. Since

$$\lim_{r\to\infty} \frac{\mu(K, B_r)}{\lambda_d(B_r)} = 1,$$

we obtain $\lim_{r\to\infty} mp = \gamma\mu(K, W) =: \theta$, and hence

$$\lim_{r\to\infty} p_k = \frac{\theta^k}{k!} \mathrm{e}^{-\theta} \tag{1.5}$$

with

$$\theta = \gamma \sum_{i=0}^{d} \alpha_{di} V_i(K) V_{d-i}(W). \tag{1.6}$$

We have found, not surprisingly, a Poisson distribution. Its parameter is expressed explicitly in terms of the constant γ, which can be interpreted as the number density of our random system of convex bodies, and the intrinsic volumes of K and W.

The original question and the answer given by (1.5) and (1.6) are found in a paper by Giger and Hadwiger [260]. The answer, though nice and explicit, is still not entirely satisfactory. We have computed a limit of probabilities and found a Poisson law. However, this Poisson distribution is not yet interpreted as the distribution of a well-defined random variable. What we would prefer, and what is needed for applications, is a model that allows us to consider from the beginning countably infinite systems of randomly placed convex bodies, with suitable independence properties.

This goal is readily achieved by employing suitable point processes. For the purpose of this introduction, a point process in $\mathbb{R}^d$ is a measurable map from the underlying probability space into the measurable space of locally finite subsets of $\mathbb{R}^d$. In particular, let Ξ be a Poisson point process of intensity γ in $\mathbb{R}^d$, with a translation invariant distribution. We choose a Poisson process since its built-in independence properties reflect the independence assumptions made above in the second step. With each point of Ξ, we associate a congruent copy of K, in the following way. For easier visualization, we suppose that $0 \in K$. We may assume that $\Xi = \{\xi_1, \xi_2, \ldots\}$, with a measurable numeration. Let $(\vartheta_1, \vartheta_2, \ldots)$ be an independent sequence of random rotations of $\mathbb{R}^d$, each with distribution given by the invariant probability measure on the rotation group SO_d; let this sequence be independent of Ξ. Then $\{\xi_i + \vartheta_i K,\ i = 1, 2, \ldots\}$ defines a random field X of convex bodies which are congruent copies of K. For this model one can compute that the probability, say q_k, of the event that the fixed observation window W is hit by precisely k bodies of the field X, is given by

$$q_k = \frac{\theta^k}{k!} \mathrm{e}^{-\theta}, \tag{1.7}$$

with θ according to (1.6).

This very special model can immediately be generalized. There is no particular reason for attaching to the points ξ_i of the Poisson process Ξ only rotation images $\vartheta_i K$ of a fixed convex body K. One may as well attach to $\xi_1, \xi_2, \ldots$ random convex bodies $K_1, K_2, \ldots$, chosen independently and independent from Ξ, according to some given rotation invariant probability distribution on the space $\mathcal{K}$ of convex bodies. Essentially equivalent is the assumption that X is a Poisson process in the locally compact space $\mathcal{K}$, which is **stationary** and **isotropic**, that is, whose distribution is invariant under translations and rotations. Again, let q_k denote the probability that the observation window W is hit by k bodies of the particle process X. The intrinsic volumes $V_i(K)$ appearing in (1.6), or rather $\gamma V_i(K)$, must now be replaced by suitable densities. Under a mild integrability condition on X (which is assumed in the following), it can be shown that the limit

$$\overline{V}_i(X) = \lim_{r \to \infty} \frac{1}{\lambda_d(rW)} \mathbb{E} \sum_{K \in X,\, K \subset rW} V_i(K) \tag{1.8}$$

exists for every convex body W with $\lambda_d(W) > 0$ and is finite and independent of W; here $\mathbb{E}$ denotes mathematical expectation. The number $\overline{V}_i(X)$ is called the **density of the** ith **intrinsic volume**, or the ith **specific intrinsic volume**, of the particle process X. If we now replace (1.6) by

$$\theta := \sum_{i=0}^{d} \alpha_{di} \overline{V}_i(X) V_{d-i}(W),$$

then (1.7) still holds.

Together with the Poisson particle process X, we consider its union set,

$$Z := \bigcup_{K \in X} K.$$

Under the mentioned integrability assumption, this is almost surely a closed set. Thus, we obtain an example of a random closed set. Generally, a **random closed set** in $\mathbb{R}^d$ is a measurable map from the underlying probability space into the space of closed subsets of $\mathbb{R}^d$, endowed with a suitable topology and the induced Borel σ-algebra. Random closed sets are, besides particle processes, the second basic model of stochastic geometry. The random closed set obtained here is of a special type: besides being stationary and isotropic, it is the union set of a Poisson particle process. Random closed sets generated in this way are known as **Boolean models**. Due to the strong independence properties of Poisson processes, Boolean models are mathematically more tractable than general random closed sets. We give one example, after introducing specific intrinsic volumes of the random set Z.

In a certain analogy to (1.8), we want to define the ith **specific intrinsic volume** of the random closed set Z by

$$\overline{V}_i(Z) = \lim_{r\to\infty} \frac{1}{\lambda_d(rW)}\, \mathbb{E}\, V_i(Z \cap rW). \tag{1.9}$$

This is indeed possible. By the properties of the generating particle process X, the set $Z \cap rW$ is, for a convex body W, almost surely the union of finitely many convex bodies. The intrinsic volume V_i has a unique additive and measurable extension from $\mathcal{K}$ to the lattice of finite unions of convex bodies. With this extension, also denoted by V_i, the random variable $V_i(Z \cap rW)$ is well defined, and the limit (1.9) exists for every convex body W with positive volume, it is finite and independent of W. The numbers $\overline{V}_0(Z), \dots, \overline{V}_d(Z)$ are, in several respects, the simplest and most basic parameters for a quantitative description of a stationary random set. They include the specific volume $\overline{V}_d(Z)$, the specific surface area $2\overline{V}_{d-1}(Z)$, and the specific Euler characteristic $\overline{V}_0(Z)$.

A special and remarkable property of the stationary and isotropic Boolean model Z is now the fact that the specific intrinsic volumes of Z can be expressed explicitly in terms of the specific intrinsic volumes of the generating particle process X, and conversely! The latter fact is rather surprising at first sight: it says that, in principle, the specific intrinsic volumes of the particle process can be determined by observing its union set. This is astonishing, since observation of the union set does not allow us to observe individual particles. The explanation for this seeming paradox lies in the strong independence properties of Poisson processes. The first two of the mentioned relations, connecting the specific volumes and the specific surface areas of the Poisson particle process X and of its union set Z, are given by

$$\overline{V}_d(Z) = 1 - \mathrm{e}^{-\overline{V}_d(X)},$$

$$\overline{V}_{d-1}(Z) = \overline{V}_{d-1}(X)\mathrm{e}^{-\overline{V}_d(X)}.$$

The remaining relations are more complicated. Their proof is a typical application of iterated kinematic formulas of integral geometry.

Poisson processes of convex bodies and their union sets, as described, are interesting and tractable models of stochastic geometry, but are, of course, too special for many applications. Part I of our book, on foundations of stochastic geometry, begins with an introduction to general random closed sets in a topological space. The basic space, as in the treatment of point processes, is assumed to be locally compact and to have a countable base. This generality is sufficient, but it is also required for the geometric models to be introduced. Some prerequisites from general topology are collected in the Appendix. Point processes and marked point processes are the subject of Chapter 3.

Since the point processes we introduce live in quite general spaces, the 'points' can themselves be geometric objects, such as compact or convex subsets of $\mathbb{R}^d$, submanifolds or planes of a fixed dimension. This leads to the geometric models which are the subject of Chapter 4. We study particle processes and their union sets, and the geometry of processes of flats. Geometric results are treated to an extent that does not yet require special knowledge from integral geometry, but considerable use is made of results from convex geometry. The latter are made available in the Appendix.

The quantitative description of random closed sets and particle processes in $\mathbb{R}^d$ requires the definition of suitable parameters. In the spatially homogeneous case one may hope that real-valued parameters already carry useful information. Let X be a stationary particle process, Z a stationary random closed set, and φ a suitable function. In analogy to (1.8) and (1.9) above, it is a plausible attempt to define φ-**densities** by a double averaging process, stochastically and spatially, in the form

$$\overline{\varphi}(X) = \lim_{r \to \infty} \frac{1}{\lambda_d(rW)} \, \mathbb{E} \sum_{K \in X,\, K \subset rW} \varphi(K) \tag{1.10}$$

and

$$\overline{\varphi}(Z) = \lim_{r \to \infty} \frac{1}{\lambda_d(rW)} \, \mathbb{E} \, \varphi(Z \cap rW), \tag{1.11}$$

where W is, say, a convex body with positive volume. Clearly, such a procedure requires appropriate assumptions. In general, $Z \cap rW$ will have a well-defined Lebesgue measure, but not, for example, a well-defined surface area or Euler characteristic, and other appropriate functions φ are even harder to think of. In most of the quantitative investigations we shall therefore restrict ourselves to particle processes X and random closed sets Z with the properties that $K \in X$ and $Z \cap W$, for a convex body W, are almost surely **polyconvex**, that is, can be represented as finite unions of convex bodies. From the viewpoint of modeling real materials and structures, this is not a severe restriction, since such objects can be approximately represented by unions of large numbers of small convex bodies. The advantage of this restriction is that a series of geometrically meaningful functions φ becomes available. Since we want to generate sets as unions of convex bodies, the functions φ to be considered must have a simple behavior under taking unions; therefore, we demand finite additivity. More precisely, a real function φ on the space $\mathcal{K}$ of convex bodies is called **additive** or a **valuation** if

$$\varphi(K \cup M) = \varphi(K) + \varphi(M) - \varphi(K \cap M)$$

whenever $K, M, K \cup M \in \mathcal{K}$. Every continuous valuation on $\mathcal{K}$ has a unique extension to an additive function on the system of polyconvex sets. For translation invariant, additive functions φ on polyconvex sets, suitable measurability and integrability conditions are sufficient to ensure the existence of the densities $\overline{\varphi}(Z)$ according to (1.11). The densities (1.10) already exist under weaker assumptions. In isotropic situations, the relevant functions φ are well known. By a remarkable theorem of Hadwiger, every continuous, rigid motion invariant valuation on $\mathcal{K}$ is a linear combination of the intrinsic volumes. This explains the predominant role of the intrinsic volumes in large parts of this book. The required facts about additive functionals on convex bodies and their proofs can be found in the Appendix.

Our emphasis on polyconvex sets and intrinsic volumes and their generalizations also affects our introduction to integral geometry, in Part II of the book. A main task of integral geometry is to compute mean values of geometric functions with respect to invariant measures. Some fundamentals about invariant measures are collected in the Appendix. Specifically, we need the invariant measures on the groups and homogeneous spaces of Euclidean geometry, namely the translation, rotation and rigid motion group, and spaces such as spheres and linear or affine Grassmannians. Typical formulas of integral geometry will evaluate the integral, with respect to an invariant measure, of a function taken at the intersection of a fixed and a moving polyconvex set. First we consider fairly general additive functions and the motion group; then we concentrate on intrinsic volumes and their local versions, the curvature measures, and also study the case of the translation group. The picture is enriched by also treating some related topics.

Another subject of integral geometry is integral transforms involving invariant measures. As an example, consider an integral, with respect to d-fold Lebesgue measure in $\mathbb{R}^d$, of a function of d points where the function does, in fact, depend only on the hyperplane that is spanned (up to a set of measure zero) by the d points. Then it may be of advantage to transform the integral into one with respect to the invariant measure on the space of hyperplanes. Integral geometry provides geometric techniques for obtaining a variety of such transformation results, which are known as **Blaschke–Petkantschin formulas**. They are extremely useful, often allowing explicit calculations in geometric probabilities and stochastic geometry.

Part III of the book, on selected topics from stochastic geometry, combines the first two parts, but also aims at giving a broader picture. With this goal in mind, in Chapter 8 we present some geometric probability problems. This topic is not only the origin of stochastic geometry, but remains to be an attractive subject of many investigations. Our presentation touches convex hulls of random points, random projections of polytopes, questions about randomly moving convex bodies and flats, touching probabilities for convex bodies, and extremal problems for probabilities and expectations coming from intuitive geometric settings. As this chapter intends to paint a colorful picture, the presentation is not very systematic, and much information is to be found in the section notes.

Chapter 9 returns to the mainstream of the book and proceeds with a quantitative treatment of stationary random closed sets and particle processes. We begin with a study of the Boolean model. For more general random closed sets and for particle processes, we then introduce, as basic descriptive parameters, densities of additive functionals, in particular the specific intrinsic volumes. In their further investigation, stochastic geometry and integral geometry come close together. Intersection formulas lead to unbiased estimators for such parameters, and some selected estimation procedures are described.

Chapter 10 gives a detailed treatment of stationary random mosaics, another basic model of stochastic geometry. After a careful introduction, particular attention is paid to tessellations induced by stationary Poisson processes, either as Voronoi or Delaunay tessellations corresponding to Poisson point processes, or as hyperplane tessellations generated by a Poisson process in the space of hyperplanes. Zero cells and typical cells of stationary random mosaics provide interesting examples of random polytopes and are studied in some detail.

Chapter 11 is an outlook to non-stationary models. While, as emphasized in the preface, invariance of measures and distributions, at least under translations, is an essential feature in this book, we want to conclude with extending some of the results in previous chapters to non-stationary situations. Naturally, the statements become more involved, but it is perhaps surprising to see how the structure of the translative results is still recognizable and how the tools developed in the stationary case remain indispensable.

Part IV, the Appendix, collects basic material from other fields that is needed in the different chapters of the book. In Chapters 12 to 14, the reader will find, when necessary, the employed notions and results from general topology, the theory of invariant measures, and the geometry of convex bodies.

1.2 General Hints to the Literature

As explained in the preface, our presentation of stochastic geometry in this book has restricted aims only: to lay sound foundations for the standard models of stochastic geometry, and to prepare and describe the use of integral geometry. Although several further topics of geometric interest are touched, we are necessarily far from giving a complete picture of stochastic geometry. Therefore, in the following we list monographs and collections where the reader may find what is missing here. We shall, with a few exceptions, mention only literature of the last forty years, the period in which stochastic geometry, as it is understood today, has developed. We order the references in thematic groups and then chronologically.

Stochastic geometry:
1974 Harding, Kendall (eds.) [321] (collection of articles)
1975 Matheron [462]
1987 Stoyan, Kendall, Mecke [743] (second ed. 1995)
1988 Hall [317] (coverage processes)
1990 Ambartzumian [35]
1990 Mecke, Schneider, Stoyan, Weil [500] (DMV seminar, in German)
1993 Ambartzumian, Mecke, Stoyan [36] (in German)
1999 Barndorff–Nielsen, Kendall, van Lieshout (eds.) [80] (collection)
2004 Beneš, Rataj [90]
2007 Baddeley, Bárány, Schneider, Weil [50] (C.I.M.E. course)

Integral geometry:
1957 Hadwiger [307] (chapter 6, in German)
1968 Stoka [738] (in French)
1972 Sulanke, Wintgen [749] (chapter 5, in German)
1976 Santaló [662]
1982 Ambartzumian [34] (combinatorial integral geometry)
1994 K. Mecke [505] (applications to statistical physics, in German)
1994 Ren [635]
1997 Klain, Rota [416] (combinatorial aspects)
2007 Voss [772] (applied to stereology and image processing, in German)

Geometric probability:
1963 Kendall, Moran [397]
1978 Solomon [731]
1999 Mathai [456]

Random sets:
1993 Molchanov [543] (limit theorems)
1997 Goutsias, Mahler, Nguyen (eds.) [284] (collection of articles)
1997 Jeulin (ed.) [384] (collection of articles)
2005 Molchanov [548]
2006 Nguyen [583]

Point processes with geometric applications:
1986 Kallenberg [385]
1986 Matérn [454]
1988 Daley, Vere–Jones [194]
1992 König, Schmidt [423] (in German)
1993 Kingman [413]
2005 Daley, Vere–Jones [195]

2008 Daley, Vere–Jones [196]
2008 Illian, Penttinen, H. Stoyan, D. Stoyan [376]

Stereology:
1980 Weibel [778]
1998 Jensen [379]
2005 Baddeley, Jensen [53]

Spatial and geometric statistics:
1981 Ripley [644]
1988 Ripley [645]
1983 Diggle [204]
1991 Karr [389]
1992 D. Stoyan, H. Stoyan [746] (in German 1992, in English 1994)
1993 Cressie [185]
1997 Molchanov [546] (statistics of the Boolean model)
1999 Kendall, Barden, Carne, Le [396] (shape theory and shape statistics)
2000 van Lieshout [439]
2002 Ohser, Mücklich [587] (materials science)
2002 Torquato [759] (materials science)
2004 Møller, Waagepetersen [556]
2006 Baddeley, Gregori, Mateu, Stoica, D. Stoyan [51] (collection)

Random tessellations:
1994 Møller [553]
2000 Okabe, Boots, Sugihara, Chiu [591]

Several areas involving random geometric structures overlap more or less with stochastic geometry, or can be subsumed under it (the more so as stochastic geometry is not clearly defined), or they apply stochastic geometry. The following list is certainly not exhaustive.

1981 Adler [1] (random fields)
1982 Serra [729] (image analysis and mathematical morphology)
1996 Meesters, Roy [509] (continuum percolation)
2003 Penrose [598] (random geometric graphs)
2007 Adler, Taylor [2] (random fields)

Introductory surveys, emphasizing different aspects of stochastic geometry, were written by Baddeley [44, 45, 49], Cruz–Orive [189], Stoyan [741, 742], Weil [785], Weil and Wieacker [806].

1.3 Notation and Conventions

We collect here some basic notation, which will be used throughout the book. More detailed explanations of fundamental notions are found in the Appendix. The reader is advised to consult Chapters 12 to 14 whenever the notions and results from general topology, the theory of invariant measures, or convex geometry that we use do not appear sufficiently familiar.

Let E be a set. We denote by $\mathbf{P}(E)$ the power set, that is, the system of all subsets of E. For a subset $A \subset E$, the complement of A is denoted by A^c and the indicator function by $\mathbf{1}_A$. When one of the latter two notions is used, it will be clear from the context to which basic set E it refers. We also write $\mathbf{1}\{x \in A\}$ instead of $\mathbf{1}_A(x)$, if convenient.

Let E be a topological space. Most of the considered spaces will be locally compact or compact; by definition, this includes the Hausdorff property. Let A a subset of E. Then $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{bd} A$ are, respectively, the closure, the interior and the boundary of A. The system of closed, open, and compact subsets of E is denoted, in this order, by $\mathcal{F}$, $\mathcal{G}$, $\mathcal{C}$. If necessary to avoid ambiguities, we also write $\mathcal{F}(E)$, $\mathcal{G}(E)$, $\mathcal{C}(E)$. A prime always indicates the corresponding system of nonempty sets, thus $\mathcal{F}'$, $\mathcal{G}'$, $\mathcal{C}'$ are the systems of nonempty closed, open, compact subsets of E, respectively. The vector space of continuous real functions on E is denoted by $\mathbf{C}(E)$, and $\mathbf{C}_c(E)$ is the subspace of functions with compact support.

A measure or signed measure on a topological space E will always be defined on the σ-algebra $\mathcal{B}(E)$ of Borel sets of the space, unless a different domain is indicated. $\mathcal{B}(E)$ is the smallest σ-algebra in E containing the open sets. Also measurability, of sets or mappings, refers to Borel σ-algebras, if no other σ-algebras are mentioned explicitly. We write

$$\mu^r := \mu \otimes \ldots \otimes \mu \qquad (r \text{ factors})$$

for the r-fold product of a measure μ. The restriction of a measure μ to a measurable set A is denoted by $\mu \llcorner A$, thus $(\mu \llcorner A)(B) := \mu(B \cap A)$ for all B in the domain of μ. If X, Y are topological spaces, ρ is a measure on X and $f : X \to Y$ is a measurable map, we denote the image measure of ρ under f by $f(\rho)$.

In probabilistic considerations, the underlying probability space will generally be denoted by $(\Omega, \mathbf{A}, \mathbb{P})$. If ξ is a random variable, then $\mathbb{P}_\xi$ denotes its distribution. We employ the usual abbreviations, such as $\mathbb{P}(\xi \in A) := \mathbb{P}(\{\omega \in \Omega : \xi(\omega) \in A\})$. The expected value of a real random variable ξ is denoted by $\mathbb{E}\,\xi$.

Most of our investigations take place in Euclidean space. $\mathbb{R}^d$ is the d-dimensional real Euclidean vector space, with scalar product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. The distance of two points $x, y \in \mathbb{R}^d$ is denoted by $d(x, y) := \|x - y\|$, the distance of two nonempty sets $K, L \subset \mathbb{R}^d$ by $d(K, L) := \inf\{d(x, y) : x \in K,\, y \in L\}$, and we write $d(K, x) = d(x, K) := d(\{x\}, K)$ for the distance of the point x from the set K.

For subsets $A, B \subset \mathbb{R}^d$, the set $A + B := \{a + b : a \in A,\, b \in B\}$ is the vector sum or Minkowski sum, $\lambda A := \{\lambda a : a \in A\}$ is the dilate of A by the number $\lambda \geq 0$, and $-A := \{-a : a \in A\}$ is the image of A under reflection in the origin. $A - B$ means $A + (-B)$. This has to be distinguished from the Minkowski difference of A and B, which is defined by

$$A \ominus B := \bigcap_{b \in B} (A - b) = \{x \in \mathbb{R}^d : B + x \subset A\}$$

(note that in some of the literature this is $A \ominus -B$). We denote by $\operatorname{conv} A$ the convex hull of the set A, and by $\operatorname{pos} A$ its positive hull.

If $A \subset \mathbb{R}^d$ and if $E \subset \mathbb{R}^d$ is an affine subspace, then $A|E$ denotes the image of A under orthogonal projection to E.

The following systems of subsets will play a prominent role. $\mathcal{K}$ is the family of compact convex subsets of $\mathbb{R}^d$. The convex ring $\mathcal{R}$ consists of all finite unions of compact convex sets; its elements are sometimes called polyconvex sets. A locally polyconvex set in $\mathbb{R}^d$ is defined by the property that its intersection with any compact convex set is polyconvex. The system of these sets is denoted by $\mathcal{S}$ and is called the extended convex ring. $\mathcal{P}$ is the family of (compact, convex) polytopes. Again, $\mathcal{K}'$, $\mathcal{R}'$, $\mathcal{S}'$, $\mathcal{P}'$ denote the corresponding systems of nonempty sets. On $\mathcal{C}'$ (and thus also on $\mathcal{K}'$) the Hausdorff metric δ is defined by

$$\delta(K, L) := \max \left\{ \max_{x \in K} \min_{y \in L} d(x, y), \max_{x \in L} \min_{y \in K} d(x, y) \right\}.$$

Some particular subsets of $\mathbb{R}^d$ will occur frequently. These are the unit ball $B^d := \{x \in \mathbb{R}^d : \|x\| \leq 1\}$, the unit sphere $S^{d-1} := \{x \in \mathbb{R}^d : \|x\| = 1\}$, and the unit cube $C^d := [0, 1]^d$. The 'half-open' cube $C_0^d := [0, 1)^d$ is useful since its translates by the vectors of $\mathbb{Z}^d$ form a decomposition of $\mathbb{R}^d$; moreover, the 'upper right' boundary $\partial^+ C^d := C^d \setminus C_0^d$ is an element of the convex ring.

Hyperplanes of $\mathbb{R}^d$ are written in the form

$$H(u, \tau) := \{x \in \mathbb{R}^d : \langle x, u \rangle = \tau\}$$

with $u \in S^{d-1}$ and $\tau \in \mathbb{R}$; this representation is unique if $\tau > 0$. For $H(u, 0)$ we often write $u^\perp$.

The following measures are used. Lebesgue measure on $\mathbb{R}^d$ is denoted by λ or, if there is danger of ambiguity, by λ_d. For $k \in \{0, \ldots, d-1\}$, λ_k is the k-dimensional Lebesgue measure on a k-dimensional affine subspace of $\mathbb{R}^d$. If E is this subspace, the Lebesgue measure on E is also denoted by λ_E. If F is a compact convex set with affine hull E, then

$$\lambda_F := \lambda_E \llcorner F.$$

The spherical Lebesgue measure on a k-dimensional great subsphere of S^{d-1} is denoted by σ_k, and we write σ instead of σ_{d-1} if this does not cause ambiguities. Occasionally, the k-dimensional Hausdorff measure is used, which is denoted by $\mathcal{H}^k$. For the Lebesgue measure of a compact set C, we often use the notation $V_d(C)$ and call it the volume of C. The intrinsic volumes $V_0(M), \ldots, V_{d-1}(M)$ of a compact convex set M are defined by the Steiner formula (1.2); they are discussed in more detail in Section 14.3.

A frequently occurring constant is the volume of the unit ball,

$$\kappa_d := \lambda_d(B^d) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1+\frac{d}{2}\right)}.$$

The surface area of the unit sphere S^{d-1} is given by

$$\omega_d := \sigma_{d-1}(S^{d-1}) = d\kappa_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}.$$

The standard groups operating on $\mathbb{R}^d$ are the translation group, which is the additive group of $\mathbb{R}^d$ and is denoted by T_d if a distinction is appropriate, the group SO_d of proper (orientation-preserving) rotations, and the group G_d of rigid motions, or orientation-preserving isometries. These groups carry their standard topologies.

The translation by the vector $x \in \mathbb{R}^d$ is denoted by t_x, thus $t_x y := y + x$ for $y \in \mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, we have $A + x := t_x A = \{a + x : a \in A\}$. If μ is a measure on $\mathbb{R}^d$, then the image measure $t_x(\mu)$ is also denoted by $t_x\mu = \mu + x$, thus $(\mu + x)(A) = \mu(t_x^{-1}A) = \mu(A - x)$ for $A \in \mathcal{B}(\mathbb{R}^d)$. Similarly, $(\vartheta\mu)(A) := \mu(\vartheta^{-1}A)$ for $\vartheta \in SO_d$.

For $k \in \{0, \dots, d\}$, the Grassmannian of k-dimensional linear subspaces of $\mathbb{R}^d$ is denoted by $G(d, k)$, and the affine Grassmannian of k-dimensional affine subspaces by $A(d, k)$; both are equipped with their standard topologies.

We denote by $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ the extended system of real numbers, and by $\mathbb{R}^+$ the set of positive real numbers.

Part I

Foundations of Stochastic Geometry

2

Random Closed Sets

A random set in a space E is defined, in agreement with the usual approach of axiomatic probability, as a set-valued random variable, that is, as a measurable map from some abstract probability space into a system of subsets of E, endowed with a suitable σ-algebra. It has turned out to be particularly tractable to assume that E is a locally compact space with a countable base and to consider the system $\mathcal{F}$ of its closed subsets, equipped with the topology of closed convergence and the induced σ-algebra of Borel sets. This approach is described in Section 2.1.

The distribution of a random closed set is completely determined by certain hitting probabilities, in particular, by its capacity functional. This gives, for every compact set $C \subset E$, the probability that the random set has nonempty intersection with C. The capacity functional can be seen in a certain analogy to the distribution function of a real random variable. Like distribution functions, the possible capacity functionals can be completely characterized. This characterization is provided by the Theorem of Choquet, for which we give a proof in Section 2.2. Some applications of this theorem are treated in Section 2.3. Special features of random closed sets in Euclidean spaces are the subject of Section 2.4.

2.1 Random Closed Sets in Locally Compact Spaces

The basic space in this chapter is a locally compact topological space E with a countable base. We denote by $\mathcal{F}$, $\mathcal{G}$, $\mathcal{C}$ the system of the closed, open, and compact subsets of E, respectively. The empty set is always included; we write $\mathcal{F}' := \mathcal{F} \setminus \{\emptyset\}$, and similarly $\mathcal{G}'$ and $\mathcal{C}'$ are defined. If necessary to avoid ambiguities, we write $\mathcal{F}(E)$, $\mathcal{G}(E)$, $\mathcal{C}(E)$ for $\mathcal{F}$, $\mathcal{G}$, $\mathcal{C}$.

Since random sets will be investigated in terms of their hitting probabilities with given sets, the following notation is fundamental. For $A \subset E$ we write

$$\mathcal{F}^A := \{F \in \mathcal{F} : F \cap A = \emptyset\},$$

$$\mathcal{F}_A := \{F \in \mathcal{F} : F \cap A \neq \emptyset\},$$

and we set

$$\mathcal{F}^A_{A_1,\ldots,A_k} := \mathcal{F}^A \cap \mathcal{F}_{A_1} \cap \ldots \cap \mathcal{F}_{A_k}$$

($:= \mathcal{F}^A$ for $k = 0$), if $k \in \mathbb{N}_0$ and $A_1, \ldots, A_k \subset E$.

Definition 2.1.1. *The* **topology of closed convergence** *on $\mathcal{F}$ is the topology generated by the set system*

$$\{\mathcal{F}^C : C \in \mathcal{C}\} \cup \{\mathcal{F}_G : G \in \mathcal{G}\}.$$

The topology of closed convergence is also known as the ‘Fell topology’. It is an example of a ‘hit-and-miss topology’.

In the following, $\mathcal{F}$ will always be equipped with the topology of closed convergence. Basic properties of this topology are proved in Chapter 12, which the reader is advised to consult when necessary. The space $\mathcal{F}$ is compact and has a countable base (Theorem 12.2.1), and the subspace $\mathcal{F}'$ is locally compact.

Lemma 2.1.1. *The σ-algebra $\mathcal{B}(\mathcal{F})$ of Borel sets of $\mathcal{F}$ is generated by either of the systems*

$$\{\mathcal{F}^C : C \in \mathcal{C}\} \qquad \textit{and} \qquad \{\mathcal{F}_G : G \in \mathcal{G}\}.$$

Proof. As shown in the proof of Theorem 12.2.1, the topology of $\mathcal{F}$ is generated by a countable subsystem of $\mathcal{A} := \{\mathcal{F}^C : C \in \mathcal{C}\} \cup \{\mathcal{F}_G : G \in \mathcal{G}\}$. Therefore, $\mathcal{A}$ generates $\mathcal{B}(\mathcal{F})$.

Let $G \in \mathcal{G}$. According to Theorem 12.1.1, there is a sequence $(C_i)_{i\in\mathbb{N}}$ of compact sets with $\bigcup_{i\in\mathbb{N}} C_i = G$, hence

$$\mathcal{F}_G = \bigcup_{i\in\mathbb{N}} \mathcal{F}_{C_i} = \bigcup_{i\in\mathbb{N}} (\mathcal{F}^{C_i})^c.$$

This shows that the system $\{\mathcal{F}^C : C \in \mathcal{C}\}$ is sufficient to generate the σ-algebra $\mathcal{B}(\mathcal{F})$.

Let $C \in \mathcal{C}$. According to Theorem 12.1.1, there is a sequence $(G_i)_{i\in\mathbb{N}}$ of open neighborhoods of C such that every open set G with $C \subset G$ contains a suitable set G_i. This yields

$$\mathcal{F}^C = \bigcup_{i\in\mathbb{N}} \mathcal{F}^{G_i} = \bigcup_{i\in\mathbb{N}} (\mathcal{F}_{G_i})^c,$$

hence also the system $\{\mathcal{F}_G : G \in \mathcal{G}\}$ is sufficient to generate $\mathcal{B}(\mathcal{F})$. □

Remark. Similarly, also each of the systems $\{\mathcal{F}_C : C \in \mathcal{C}\}$ and $\{\mathcal{F}^G : G \in \mathcal{G}\}$ generates $\mathcal{B}(\mathcal{F})$.

The following consequence is important. If the map $\varphi : T \to \mathcal{F}$ from some topological space T to $\mathcal{F}$ is upper or lower semicontinuous (see Section 12.2), then it is Borel measurable. In fact, if φ is upper semicontinuous, then $\varphi^{-1}(\mathcal{F}^C)$ is open, and hence a Borel set, for every compact set $C \in \mathcal{C}$. Since $\{\mathcal{F}^C : C \in \mathcal{C}\}$ is a generating system of $\mathcal{B}(\mathcal{F})$, the measurability of φ follows. For lower semicontinuous maps, the proof is analogous.

Lemma 2.1.2. *$\mathcal{C}$ is a Borel set in $\mathcal{F}$.*

Proof. By Theorem 12.1.1, there is a sequence $(C_i)_{i\in\mathbb{N}}$ of compact sets with $C_i \subset \operatorname{int} C_{i+1}$ for $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} C_i = E$. This yields

$$\mathcal{C} = \bigcup_{i\in\mathbb{N}} \mathcal{F}^{C_i^c},$$

where each $\mathcal{F}^{C_i^c}$ is closed, hence $\mathcal{C}$ is a Borel set in $\mathcal{F}$. $\square$

Now we introduce random closed sets.

Definition 2.1.2. *A* **random closed set** *in E is an $\mathcal{F}$-valued random variable, that is, an $(\mathbf{A}, \mathcal{B}(\mathcal{F}))$-measurable map $Z : \Omega \to \mathcal{F}$ from some probability space $(\Omega, \mathbf{A}, \mathbb{P})$ into $\mathcal{F}$. The* **distribution** *of Z is the image measure $\mathbb{P}_Z := Z(\mathbb{P})$ of $\mathbb{P}$ under Z.*

In the following, 'random closed set' always means 'random closed set in E'.

As usual in probability theory, the essential feature of a random variable is its distribution and what can be derived from it. Two random closed sets Z and Z', which may be defined on different probability spaces, are called **stochastically equivalent** if they have the same distribution. This is also written as $Z \overset{\mathcal{D}}{=} Z'$ (**equality in distribution**). Even though every random closed set Z has a canonical representation Z' with $Z' \overset{\mathcal{D}}{=} Z$, via the identical map on $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathbb{P}_Z)$, it is still more convenient to use the general representation of Definition 2.1.2, with an abstract probability space.

For $\mathbb{P}_Z(A)$, where $A \in \mathcal{B}(\mathcal{F})$, we also use the notation $\mathbb{P}(Z \in A)$, as an abbreviation for $\mathbb{P}(\{\omega \in \Omega : Z(\omega) \in A\})$, etc. If $\mathbb{P}(Z \in A) = 1$, we say that '$Z \in A$ almost surely' (a.s.).

If, in the following, several (finitely or countably many) random closed sets are treated simultaneously, we always assume that they are defined on the same probability space $(\Omega, \mathbf{A}, \mathbb{P})$. If $Z_1, \dots, Z_k$ are random closed sets, their **joint distribution** is the probability measure $\mathbb{P}_{Z_1,\dots,Z_k}$ on $\mathcal{F}^k$ defined by

$$\mathbb{P}_{Z_1,\dots,Z_k}(A_1 \times \dots \times A_k) = \mathbb{P}(Z_1 \in A_1, \dots, Z_k \in A_k)$$

for $A_1, \dots, A_k \in \mathcal{B}(\mathcal{F})$. Analogously, the joint distribution $\mathbb{P}_{Z_1,Z_2,\dots}$ of a sequence $Z_1, Z_2, \dots$ of random closed sets is defined. It is a probability measure on $\mathcal{F}^{\mathbb{N}}$. As usual, the random closed sets $Z_1, \dots, Z_k$, respectively $Z_1, Z_2, \dots,$

are called (**stochastically**) **independent** if their joint distribution is the product of their individual distributions, that is, if

$$\mathbb{P}_{Z_1,\dots,Z_k} = \mathbb{P}_{Z_1} \otimes \ldots \otimes \mathbb{P}_{Z_k},$$

respectively

$$\mathbb{P}_{Z_1,Z_2,\dots} = \bigotimes_{i\in\mathbb{N}} \mathbb{P}_{Z_i}.$$

From given random closed sets Z and Z', one can obtain new ones by means of set-theoretic or topological operations. If $\varphi : \mathcal{F} \to \mathcal{F}$ and $\psi : \mathcal{F}\times\mathcal{F} \to \mathcal{F}$ are measurable maps, then also the compositions $\varphi \circ Z$ and $\psi \circ (Z, Z')$ are measurable. Therefore, the continuity or semicontinuity results of Theorems 12.2.3, 12.2.6, 13.1.1 yield the following.

Theorem 2.1.1. *If Z and Z' are random closed sets, then also $Z \cup Z'$, $Z \cap Z'$, $\operatorname{bd} Z$ and $\operatorname{cl} Z^c$ are random closed sets. If the topological group G operates continuously on E, then for $g \in G$ also gZ is a random closed set.*

We mention some simple examples of random closed sets. Trivially, if $F \in \mathcal{F}$, the constant map $\omega \mapsto F$ from Ω into $\mathcal{F}$ is a random closed set. Therefore, Theorem 2.1.1 implies that for a random closed set Z also the intersection $Z \cap F$ with a fixed set $F \in \mathcal{F}$ is a random closed set (and similarly $Z \cup F$). If $\xi_1, \xi_2, \dots$ is a sequence of random variables with values in E, then the countable set $Z = \{\xi_1, \xi_2, \dots\}$ is a random closed set if the set $\{\xi_1(\omega), \xi_2(\omega), \dots\}$ has no accumulation points, for almost all ω. If, as in this case, $Z \cap C$ is almost surely finite for every compact set $C \in \mathcal{C}$, we say that the random closed set Z is **locally finite**.

Now we introduce, for random closed sets, a functional which can be considered as an analog to the distribution function of a real random variable. We first recall this latter notion.

For a random variable ξ with values in $(-\infty, \infty]$, the distribution function $\varphi = \varphi_\xi$ is defined by

$$\varphi_\xi(t) := \mathbb{P}(\xi \le t) = \mathbb{P}(\{\xi\} \cap (-\infty, t] \ne \emptyset), \quad t \in [-\infty, \infty).$$

It has the following properties:

(a) $0 \le \varphi \le 1$, $\varphi(-\infty) = 0$,
(b) φ is continuous from the right, that is, $t_i \downarrow t$ implies $\varphi(t_i) \to \varphi(t)$,
(c) φ is increasing, that is, $\varphi(t_0 + t_1) - \varphi(t_0) \ge 0$ for all $t_1 \ge 0$ and all $t_0 \in [-\infty, \infty)$.

The distribution function φ_ξ determines the distribution $\mathbb{P}_\xi$ uniquely. For any function φ satisfying (a), (b), (c), there exists a random variable with distribution function φ.

A tool with analogous properties exists in the theory of random closed sets.

Definition 2.1.3. *The* **capacity functional** T_Z *of the random closed set* Z *is defined by*

$$T_Z(C) := \mathbb{P}_Z(\mathcal{F}_C) = \mathbb{P}(Z \cap C \neq \emptyset) \qquad \text{for } C \in \mathcal{C}.$$

The following theorem shows that the capacity functional has properties corresponding to the properties (a), (b), (c) of a distribution function. We denote by $A_i \downarrow A$ the monotone convergence of sets A_i to A; this means that $A_{i+1} \subset A_i$ for $i \in \mathbb{N}$ and $\bigcap_{i\in\mathbb{N}} A_i = A$. Similarly, $A_i \uparrow A$ means that $A_{i+1} \supset A_i$ for $i \in \mathbb{N}$ and $\bigcup_{i\in\mathbb{N}} A_i = A$. If a function $T : \mathcal{C} \to \mathbb{R}$ is given, we define

$$S_0(C) := 1 - T(C) \qquad \text{for } C \in \mathcal{C}$$

and then, by recurrence,

$$S_k(C_0; C_1, \dots, C_k) := S_{k-1}(C_0; C_1, \dots, C_{k-1}) - S_{k-1}(C_0 \cup C_k; C_1, \dots, C_{k-1})$$

for $C_0, C_1, \dots, C_k \in \mathcal{C}$ and $k \in \mathbb{N}$. It should be kept in mind that S_k depends on T, although the notation does not reveal this.

Theorem 2.1.2. *The capacity functional* $T = T_Z$ *of a random closed set* Z *has the following properties:*

(a) $0 \leq T \leq 1$, $T(\emptyset) = 0$,
(b) *if* $C_i, C \in \mathcal{C}$ *and* $C_i \downarrow C$, *then* $T(C_i) \to T(C)$,
(c) $S_k(C_0; C_1, \dots, C_k) \geq 0$ *for* $C_0, C_1, \dots, C_k \in \mathcal{C}$ *and* $k \in \mathbb{N}_0$.

Proof. Assertion (a) follows immediately from the definition.

(b) If $C_i \downarrow C$, then the sequence $(\mathcal{F}_{C_i})_{i\in\mathbb{N}}$ is decreasing, and $\mathcal{F}_C \subset \bigcap_{i\in\mathbb{N}} \mathcal{F}_{C_i}$. We show that $\mathcal{F}_{C_i} \downarrow \mathcal{F}_C$. Let $F \in \bigcap_{i\in\mathbb{N}} \mathcal{F}_{C_i}$, then $F \cap C_i \neq \emptyset$ for all $i \in \mathbb{N}$. From $\bigcap_{i\in\mathbb{N}} C_i = C$ and the intersection property of compact sets it follows that $F \cap C = \bigcap_{i\in\mathbb{N}}(F \cap C_i) \neq \emptyset$. Hence, $F \in \mathcal{F}_C$ and thus $\bigcap_{i\in\mathbb{N}} \mathcal{F}_{C_i} = \mathcal{F}_C$. Assertion (b) now follows from the fact that the probability measure $\mathbb{P}_Z$ is continuous from above.

(c) Clearly, $S_0 \geq 0$. Using the relation

$$\mathcal{F}^{C_0}_{C_1,\dots,C_k} = \mathcal{F}^{C_0}_{C_1,\dots,C_{k-1}} \setminus \mathcal{F}^{C_0 \cup C_k}_{C_1,\dots,C_{k-1}}, \tag{2.1}$$

one shows by induction with respect to k that

$$S_k(C_0; C_1, \dots, C_k) = \mathbb{P}_Z(\mathcal{F}^{C_0}_{C_1,\dots,C_k}), \qquad k \in \mathbb{N}. \tag{2.2}$$

The assertion follows. □

A real function T on $\mathcal{C}$ satisfying (a) and (b) of Theorem 2.1.2 is called a **Choquet capacity**. (The reason for this terminology comes from the fact that T can be extended to a set function on the power set $\mathbf{P}(E)$ of E which has the properties of a capacity; see Choquet [174].) A Choquet capacity satisfying (c) is called **alternating of infinite order**. The distribution of a random closed set is uniquely determined by its capacity functional.

Theorem 2.1.3. *If Z, Z' are random closed sets with $T_Z = T_{Z'}$, then $Z \stackrel{\mathcal{D}}{=} Z'$.*

Proof. The equality $T_Z = T_{Z'}$ means that $\mathbb{P}_Z(\mathcal{F}^C) = 1 - \mathbb{P}_Z(\mathcal{F}_C) = 1 - \mathbb{P}_{Z'}(\mathcal{F}_C) = \mathbb{P}_{Z'}(\mathcal{F}^C)$. Since the system $\{\mathcal{F}^C : C \in \mathcal{C}\}$ is $\cap$-stable and by Lemma 2.1.1 generates the σ-algebra $\mathcal{B}(\mathcal{F})$, the assertion follows from a well-known uniqueness theorem of measure theory. □

Notes for Section 2.1

1. Random sets were systematically developed by Matheron [459, 460] and D.G. Kendall [395]. Important fundamental ideas can already be found in Choquet's [173] theory of capacities. The introduction given in this chapter is essentially based on Matheron's seminal book [462].

2. General introductions to the theory of random closed sets are found in the monographs by Molchanov [548] and by Nguyen [583]. As the reader is advised to consult these volumes, the section notes in this chapter will be very brief.

3. Several different aspects of the theory of random sets are described in the surveys [542, 547] of Molchanov. The volumes edited by Jeulin [384] and by Goutsias, Mahler and Nguyen [284] contain various contributions to theory and applications of random sets.

2.2 Characterization of Capacity Functionals

The capacity functional $T = T_Z$ of a random closed set Z has the properties listed in Theorem 2.1.2. These properties of a function T on the system $\mathcal{C}$ of compact sets are also sufficient for T to be the capacity functional of a random closed set. This result is known as Choquet's Theorem.

Theorem 2.2.1 (Theorem of Choquet). *Let $T : \mathcal{C} \to \mathbb{R}$ be a function with the following properties:*

(a) $0 \le T \le 1$, $T(\emptyset) = 0$,
(b) *if* $C_i, C \in \mathcal{C}$ *and* $C_i \downarrow C$, *then* $T(C_i) \to T(C)$,
(c) $S_k(C_0; C_1, \dots, C_k) \ge 0$ *for* $C_0, C_1, \dots, C_k \in \mathcal{C}$ *and* $k \in \mathbb{N}_0$.

Then there exists a uniquely determined probability measure $\mathbb{P}$ on $\mathcal{F}$ with

$$T(C) = \mathbb{P}(\mathcal{F}_C)$$

for all $C \in \mathcal{C}$.

Consequently, the function T is the capacity functional of a random closed set Z. For example, one can take for Z the identical map on the probability space $(\mathcal{F}, \mathcal{B}(\mathcal{F}), \mathbb{P})$.

The stated uniqueness is clear from Theorem 2.1.3. For the existence, we shall present a proof due to Matheron [462], with a simplification taken from