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ERNST ZERMELO

Collected Works
Gesammelte Werke

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Zermelo in 1907

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Heinz-Dieter Ebbinghaus,
Craig G. Fraser,
Akihiro Kanamori

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Collected Works
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VOLUME I
BAND I

Set Theory,
Miscellanea

Mengenlehre,
Varia

VOLUME II
BAND II

Calculus of Variations,
Applied Mathematics,
and Physics

Variationsrechnung,
Angewandte Mathematik
und Physik

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Edited by
Herausgegeben von

Heinz-Dieter Ebbinghaus,
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Springer

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Preface to the Zermelo edition

This is a complete edition of the published works of Ernst Zermelo which moreover includes selected correspondence and unpublished manuscripts. Zermelo is generally acknowledged for his pioneering work in axiomatic set theory and for introducing the axiom of choice as a basic principle of mathematics. In contrast, his work in applied mathematics and physics, despite its originality, is hardly recognized or has even been attributed to others. This edition of Zermelo's collected works provides a picture of the entire mathematician. It appears in two volumes. The first volume comprises Zermelo's published papers in set theory and the foundation of mathematics together with isolated papers of an algebraic, analytic, or number-theoretic character. The second volume is dedicated to Zermelo's work in the calculus of variations, mathematical physics, and fluid dynamics. Both volumes are supplemented by selected notes and manuscripts, mainly from Zermelo's *Nachlass*, which throw additional light on his papers, reflect his point of view, or are unpublished continuations of published work. To the best judgment of the editors, the selected notes and manuscripts fully and faithfully represent the essential unpublished writings of Zermelo concerning mathematics. Nevertheless, a possible edition of a third volume comprising further unpublished notes and letters from the *Nachlass* has *expressis verbis* been left open.

Both volumes contain some writings by other authors which include contributions actually written by Zermelo or which react to criticism Zermelo had made. Details are given in the prefaces to the respective volumes.

In order to provide access to a wider audience, the original papers are printed face to face with English translations. As both versions use the same layout, it is easy to go from the translation to the original version and vice versa. The layout itself tries to preserve the appearance of the original papers. For details we refer to the editorial information below.

Each paper or coherent group of papers is preceded by an introductory note which comments on contents, motivation, aims, and influence of the paper(s) concerned. Written by an expert in the field, it came to its final form in discussions with the editors.

Each volume contains a full bibliography of Zermelo together with a schematic *curriculum vitae* which will enable the reader to become acquainted with the personal circumstances from which a paper arose. In addition, Volume I starts with a more detailed biographical sketch of Zermelo's life and work.

Many of these features found their inspiration in the exemplary edition of Kurt Gödel's collected works by Solomon Feferman, John W. Dawson, Jr., and others.

The edition of Zermelo's collected works has a prehistory. Already as early as 1912, at the age of 41 and faced with a serious recurrence of his tuberculosis, Zermelo conceived plans for an edition of his collected papers, but did not pursue them when his health improved. In 1949, under likewise deplorable personal circumstances, he tried again, this time approaching several publishers, among them Springer-Verlag. But the difficult situation in post-war Germany precluded such an enterprise. Immediately after Zermelo's death, in 1953, the historian of mathematics Helmuth Gericke and the philosopher Gottfried Martin, who had gotten to know Zermelo in the 1930s in Freiburg, started work on a two-volume edition, in 1956 gaining Paul Bernays as a third editor. Support was provided by the Kant-Gesellschaft. However, the plans were not realized; in 1962 work on the edition came to a definite end.

When in early 2004 new plans for an edition of Zermelo's collected works became more concrete, they found the enthusiastic support of Martin Peters of Springer-Verlag. In discussions with him it became clear very quickly that the edition should provide English translations and detailed comments. As Zermelo had been a member of the Heidelberger Akademie der Wissenschaften, the editors turned to the academy for financial support. The application found the warm backing of Hans Günter Dosch, then Sekretar of the class for mathematics and the sciences of the academy. The application was successful. Even more, besides providing generous funding, the academy offered to let the edition appear in its regular series of publications of the class for mathematics and the sciences published by Springer-Verlag.

The editors wish to express deep gratitude to the Heidelberg academy for their ideal, financial support and to Springer-Verlag for their open-minded cooperation. In particular, many thanks go to Hans Günter Dosch and Martin Peters.

Freiburg, Toronto, and Boston
September 2009

Heinz-Dieter Ebbinghaus
Craig G. Fraser
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Preface to volume I

This first volume of the Zermelo edition focuses on Zermelo's work in set theory and the foundations of mathematics and is supplemented by his papers in pure mathematics. The published papers are accompanied by selected items from Zermelo's *Nachlass* and some letters. Whereas the papers span the time between Zermelo's first encounter with set theory around 1900 in Göttingen and the end of his mathematical research in the mid-1930s in Freiburg, the selected items mainly stem from the early 1930s, from a time when Zermelo's foundational views were in growing opposition to the mainstream developments in mathematical logic.

The primary criterion for the inclusion of an item was the degree of novelty and of insights into Zermelo's foundational views that it provides. Some items may be considered as partly independent drafts of (parts of) a published paper or as offering a variation of (parts of) a published paper (such as *s1931g* with respect to *1932a, 1932b*, and *1935*); some may be viewed as first steps of a continuation of a published paper, for example by elaborating on a feature only touched on there (such as *s1931f* and *s1932d* with respect to *1930a*); some items allow a deeper insight into Zermelo's foundational views (such as *s1929b* and *s1930d*).

The correspondence with Kurt Gödel together with one of Zermelo's letters to Reinhold Baer (*s1931b* to *s1931d*) sheds additional light on Zermelo's foundational views around 1930 which are marked by a strong aversion to finitary approaches to the foundation of mathematics. The last letter *s1941* to Paul Bernays is a revealing testament to Zermelo's mathematical disillusionment and physical deterioration in his final years. The Zermelo part of the correspondence with Reinhold Baer and Arnold Scholz, which could have provided further insights, is lost up to the letter *s1931c* to Baer.

The parts *Landau 1917b* and *D. König 1927b* of *Landau 1917a* and *D. König 1927a*, respectively, are included here because they were written by Zermelo or closely follow a note written by him.

Zermelo was well-educated in and had a continuing enthusiasm for literature and the classics. In the 1920s he translated large parts of Books V to IX of Homer's *Odyssey* into German blank verse. Part of his translation of Book V was published as *1930f*. It is given here together with an appendix which contains his entire translation of Book V.

The introductory notes are a crucial part of this edition. As a matter of fact, all the people we invited to comment on a paper or a group of papers, were instantly ready to join the project and share their experience and knowledge with us and the potential reader. The discussions we had toward securing the most informative and accurate presentations were framed by mutual cooperation and open-mindedness. We particularly appreciate the valuable help

provided by the collaborators in checking bibliographical items and making useful proposals concerning the English translations.

When work on the edition started, the most serious problem we seemed to be faced with was how to get accurate translations. As it turned out, the problem found the best solution imaginable. First, we could use the translations of *1904, 1908a*, and *1908b* which Stefan Bauer-Mengelberg had provided for Jean van Heijenoort's *From Frege to Gödel*. Furthermore, Warren Goldfarb and R. Gregory Taylor allowed us to use their English translations of *1909a* and *s1931g*, respectively. All the other translations were done by Enzo de Pellegrin. We express our deep gratitude for his extraordinary care and admire his feeling for both languages when handling Zermelo's style with its richness in nuances and its involved sentential structures.

There are many others who have supported us during our work. We express our gratitude to all of them and mention here, in particular, Ruth Allewelt of Springer-Verlag through whom we had smooth cooperation with the publisher; Barbara Hahn who, as the librarian of the Freiburg Mathematical Institute, efficiently provided the literature we needed; and Andrea Köhler of le-tex publishing services who directed the multitude of files and problems with patience and care. We appreciated the supportive interest which Gunther Jost from the Heidelberg Academy showed for the imponderabilities of our work. As ever, Martin Peters from Springer-Verlag was ready to offer valuable help and advice. As a final note, the second-named editor would like to express his special gratitude to the Lichtenberg-Kolleg at Göttingen. Awarded an inaugural fellowship there, he was able to carry out editorial work in a particularly supportive environment at the Gauß Sternwarte, in the city where Zermelo did his best-known work in set theory.

Freiburg and Boston
November 2009

Heinz-Dieter Ebbinghaus
Akihiro Kanamori

Editorial information

Layout. The layout of the texts as well as of the translations mirrors the layout of the originals. Emphasized words, i.e., words in italics or words spaced out or consisting of small capitals, are given in italics. Original pagebreaks are indicated in the texts by "|", and the number of the new original page beginning there is given on the margin.

Editorial annotations. These are set in double square brackets "⟦" and "⟧".

Misprints and errors. Small textual errors in the originals are tacitly corrected; larger ones are corrected with the corrections commented on in editorial annotations.

Wrong words or words missing in the originals have been replaced or added in double square brackets.

Misprints in mathematical expressions in the originals are not corrected in the texts. They are, however, corrected in the translations and noted by an editorial annotation.

Special terminology. Zermelo articulates the following distinction among *Untermenge*, *Teilmenge*, and *Teil*: A set M is a *Untermenge*, *Teilmenge*, *Teil* of the set N if $M \subseteq N$, $M \subsetneq N$, $\emptyset \neq M \subsetneq N$, respectively. However, he evidently deviates from this at times, and in order to truly reflect his usage, *Untermenge*, *Teilmenge*, and *Teil* are always translated as *subset*, *partial set*, and *part*, respectively. This convention has led to changes in the Bauer-Mengelberg translations of the Zermelo papers *1904*, *1908a*, and *1908b*.

Special symbols. Mathematical symbols which are now no longer in use, such as "$\underline{\in}$" for "$\subseteq$", are kept in the texts, but replaced by their modern analogues in the translations.

References. In the texts Zermelo's references to the literature are not altered. Translations as well as introductory notes refer to the main bibliography at the end of the volume and have the form *author(s) year of appearance*, followed by an additional index a, b, c, ... if necessary. An example: *Hahn and Zermelo 1904*. If the authors are clear from the context, their names may be omitted; for example, *1904* some lines above is short for *Zermelo 1904*.

References to page numbers are kept in both the texts and the translations; they can be traced via the original pagebreaks and the original page numbers provided in the texts.

Footnotes. Whereas the translations use natural numbers in ascending order as footnote marks, the texts preserve the original marks. It may thus happen that a page of the text may contain identical footnote marks. In such cases the original page numbers on the margin allow for quick correlation of mark and footnote.

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Akademie der Wissenschaften zu Göttingen (*1901, Landau 1917b, 1934*);

Springer Verlag (*1904, 1908a, 1908b*, their English translations *1967a,b,c*, and *1909a, 1914, 1932c*);

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Cambridge University Press (*1913*);

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Instytut Matematyczny Polskiej Akademii Nauk (*1929a, 1930a, 1930b, 1935*);

Institute for Advanced Study (Princeton) (*s1931b, s1931d*);

Ulrich Felgner (*s1931c*);

Deutsche Mathematiker-Vereinigung (*1932a*);

Berlin-Brandenburgische Akademie der Wissenschaften (*1932b*);

Archiv der ETH Zürich (*s1941*).

The photographs in this volume are taken from the Zermelo photo collection held in the Abteilung für Mathematische Logik at Freiburg University. The collection will be integrated into the Zermelo *Nachlass* housed in the Universitätsarchiv Freiburg.

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Ernst Zermelo

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Set Theory,
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Ernst Zermelo: A glance at his life and work

Heinz-Dieter Ebbinghaus

Ernst Zermelo (1871–1953) is best-known for his great, pioneering work in set theory that transformed the subject. Of focal importance are his introduction of the axiom of choice in 1904 and his axiomatization of set theory in 1908. The axiom of choice led to a methodological enrichment of mathematics, and the axiomatization was the starting point of post-Cantorian set theory. Zermelo also made significant contributions later, particularly in his 1930 introduction of the cumulative hierarchy.

Less known is that Zermelo did significant work in applied mathematics. His dissertation, for example, promoted the Weierstraßian direction in the calculus of variations; in 1912 he wrote the first paper in what is now called the theory of games; and in 1928 he created the pivotal method in the theory of rating systems. His scientific interests included also purely technical questions such as cruise controls for engines. Zermelo also made translations of poetry, which appear here in these collected works.

Two factors are crucial when considering Zermelo's life. First, he had a confrontational personality: he was polemical in his approach to mathematics, especially against those who disagreed with him, and this appeared in his writing and partly stimulated it. Second, he was very ill for a significant part of his middle life, and this slowed down his mathematical activities and ended his academic career when he was in his mid-forties.

Zermelo's confrontational attitude comes together with his striving for truth and objectivity, and the determination with which he stood up for his convictions. Well-educated in and open-minded about philosophy, the classics, and literature, he had the ability of engaging others in a stimulating way.

Zermelo's life and scientific work can naturally be divided into four periods: The time in Berlin, in Göttingen, in Zurich, and, finally, in Freiburg im Breisgau.

The time in Berlin comprises the years from his birth to 1897, leading scientifically to his first disciplines of specialization, the calculus of variations and mathematical physics.

The following years in Göttingen until 1910 are marked by his pioneering work in set theory under the influence and with the encouragement of David Hilbert. Scientifically this was the most fruitful period of Zermelo's life. Personally, however, it was overshadowed by the outbreak of tuberculosis of the lungs which together with a nervous constitution nearly blocked his chances of getting a permanent university position.

In spring 1910 Zermelo obtained a full professorship at the University of Zurich. The years in Switzerland began with the prospect of a fruitful academic life. But soon they were spoilt by a recurrence of his illness which kept him from any systematic scientific work and enforced his retirement already in 1916.

The time in Freiburg from 1921 until his death in 1953 was sustained by a new academic position—an honorary professorship for mathematics—and was a time of scientific work both in applied mathematics and in the foundations of mathematics. As the latter was directed against the finitary approaches of Kurt Gödel and Thoralf Skolem which were to shape mathematical logic in the 1930s, it was doomed to fail. Moreover, in 1935 Zermelo lost his professorship because of his anti-fascist disposition. The promising new start had come to an end, which left him in a state of resignation.

What follows is an overview of Zermelo's life, one that incorporates his scientific work and development. A detailed analysis of his achievements is reserved for the introductory notes preceding his papers. This overview draws on our full biography (*Ebbinghaus 2007b*) where extensive supporting documentation is provided.

1 Berlin

1.1 Youth

Ernst Friedrich Ferdinand Zermelo was born in Berlin on 27 July 1871 as the second child of the high school teacher ("Gymnasialprofessor") Dr. Theodor Zermelo (1834–1889) and his wife Maria Auguste née Zieger (1847–1878).

Zermelo had five sisters, the one-year older Anna and the younger sisters Elisabeth, Margarete, Lena, and Marie. The letters which he wrote to them throughout his life attest to a close relationship among the siblings, and care and concern on Zermelo's side.

Zermelo's mother Maria Auguste was the only child of the surgeon Dr. Ottomar Hugo Zieger and his wife Auguste née Meißner. Like her parents she suffered from poor health. Exhausted from the strain of multiple pregnancies, she died shortly before Zermelo's seventh birthday. After her death a maid took care of the household.

Zermelo's father Theodor was the son of Ferdinand Zermelo, a bookbinder from Tilsit (today Sovjetsk, Russia) on the river Memel (Neman) and his wife Bertha née Haberland. He studied mainly history and geography in Königsberg (now Kaliningrad, Russia) and in Berlin. Having attained his doctorate at the Friedrich Wilhelm University (now Humboldt University) in Berlin in 1856 with a dissertation in history, he completed his studies with the exams *pro facultate docendi,* i.e., state exams for teaching at secondary schools, which allowed him to teach history, geography, Greek, Latin, and French; he later completed an additional diploma in mathematics. In its evaluation, the examination board found him to have "an adequate knowledge of the elements of geometry and algebra for teaching mathematics in the lower grades." The board further observed that he had a solid grasp of the basic notions of logic and general grammar and showed an assured competence in their application.

In sum, Ernst Zermelo grew up in a family with an academic background, but not one with a significant orientation towards mathematics and the sciences.

Zermelo had a kind heart, but he was also prone to sharp and polemical reactions and did not refrain from trenchant irony when he was convinced of his opinions. Whereas the first trait might be attributable to the influence of his mother, his determination to speak his mind may be due to the example of his father. In 1875, Theodor Zermelo had published a widely acknowledged paper, *1875*, on the historian and philologist August Ludwig von Schlözer (1735–1809) of Göttingen, the most influential anti-absolutist German publicist of the early age of Enlightenment. It was his aim to raise awareness of those values expressed by Schlözer's plea for free speech, sincerity, tolerance, and humanity on which, in his opinion, the newly founded German Reich should be built. There are indications that the views of his father as reflected in this treatise and the example of von Schlözer had their effect on Zermelo.

Zermelo shared with his father an interest in poetry. As a young teacher, Theodor Zermelo had compiled a large selection of his own translations of poems from England, the United States, France, and Italy. Zermelo was later to translate parts of the Homeric epics, and he enjoyed commenting on daily events in the form of poems. At the age of thirteen, he made "a metric translation from the first book of Virgil's Aeneid" as a birthday present for his father.

In 1880 Zermelo entered the Luisenstädtisches Gymnasium in Berlin, which he finished in March 1889, only one month after the death of his father. His final school report certifies good results in religious education, German language, Latin, Greek, history, geography, mathematics, and physics. The comment in mathematics says that Zermelo followed the instructions with good understanding and that he had reliable knowledge and the ability to use it for skilfully solving problems. In physics it is testified that he was well-acquainted with a number of phenomena and laws. Under the heading "Behaviour and diligence" it is remarked that he followed the lessons with reflection, but that he occasionally showed a certain passivity as a result of physical fatigue, an indication that Zermelo already suffered from poor health during his school days.

1.2 University

In the summer semester 1889 Zermelo matriculated at the Friedrich Wilhelm University in Berlin. During his course of studies he took one semester each at Halle (winter semester 1890/91) and Freiburg (summer semester 1891). After his final examinations he moved to Göttingen (winter semester 1897/98).

When his father died in 1889, he and his five sisters became orphans. As his father's estate was needed to provide for the younger siblings, Zermelo applied for grants and succeeded in obtaining scholarships from several foundations which were affiliated with the Friedrich Wilhelm University and established to support gifted students.

In Berlin Zermelo enrolled for philosophy and took courses in mathematics, above all with Johannes Knoblauch, a student of Karl Weierstraß, and with Lazarus Fuchs. Furthermore, he attended courses on experimental physics and heard a course on experimental psychology by Hermann Ebbinghaus.

In Halle he enrolled for mathematics and physics, attending Georg Cantor's courses on elliptical functions and on number theory, Albert Wangerin's courses on differential equations and on spherical astronomy, and a course given by Edmund Husserl on the philosophy of mathematics at the time when Husserl's *Philosophie der Arithmetik* (*1891*) was about to be published. He also took part in a course on logic given by Benno Erdmann, one of the leading philosophical logicians of the time.

In Freiburg he studied mathematical physics with Emil Warburg, analytical geometry and the method of least squares with Jakob Lüroth, experimental psychology with Hugo Münsterberg, and history of philosophy with Alois Riehl. Furthermore he attended a seminar on Heinrich von Kleist.

Having returned to Berlin, he attended several courses by Max Planck on theoretical physics, among them a course on the theory of heat in the winter semester 1893/94. He also attended a course on the principle of the conservation of energy by Wilhelm Wien (summer semester 1893). In mathematics he took part in courses on differential equations by Fuchs, algebraic geometry by Georg Ferdinand Frobenius, and non-Euclidean geometry by Knoblauch. The calculus of variations, one of the central topics of Zermelo's later work, was taught by Hermann Amandus Schwarz in the summer semester 1892. In philosophy he attended "Philosophische Übungen" by the neo-Kantian Friedrich Paulsen and Wilhelm Dilthey's course on the history of philosophy. He also took a course on psychology, again by Hermann Ebbinghaus (winter semester 1893/94). In October 1894 he received his doctorate, and from December 1894 to September 1897 he served as an assistant to Max Planck at the Berlin Institute for Theoretical Physics.

Looking back on Zermelo's university studies, one may emphasize that he acquired a solid and broad knowledge in both mathematics and physics, his main subjects. His advanced studies directed him to his early specialities of research, to mathematical physics and to the calculus of variations, the topic of his dissertation. He attended courses by Georg Cantor, but no course on set theory, at least until his change to Göttingen where Arthur Schoenflies gave such a course in the summer semester 1898. Furthermore, he had no instruction in mathematical logic. Given his teachers in philosophy, Riehl, Paulsen, Dilthey, and Erdmann, it can be assumed that he had a broad, overall knowledge of philosophical theories. In Halle he even became acquainted with Husserl's phenomenological philosophy of mathematics *in* *statu nascendi.* His interest in experimental psychology as taught by Hermann Ebbinghaus and Hugo Münsterberg is quite evident.

In February 1897 Zermelo passed his exams *pro facultate docendi.* In philosophy he wrote an essay entitled "What is the significance of the principle of the conservation of energy for the question of the relation between body and mind?" ("Welche Bedeutung hat das Prinzip der Erhaltung der Energie für die Frage nach dem Verhältnis von Leib und Seele?"). According to the reports he was well-informed about the history of philosophy and systematic subjects. His exams in religious education was evaluated as excellent. As to his exams in mathematics the report says that although he was not always aware of the methods in each domain, it was nevertheless evident that he had acquired an excellent mathematical education. The physics part was rated excellent. The exams in geography showed that he was excellent with respect to theoretical explanations, but that he was not that affected by "studying reality." Finally it was certified that he had the knowledge for teaching mineralogy in the high range and chemistry in the middle.

1.3 Scientific work in Berlin

1.3.1 Ph. D. thesis

Zermelo's Ph. D. thesis *Untersuchungen zur Variations-Rechnung* (*Investigations in the calculus of variations*), *1894*, was suggested and guided by Hermann Amandus Schwarz. Schwarz had studied in Berlin with Weierstraß, becoming his most eminent student. In 1892 he had been appointed his teacher's successor. Zermelo became his first Ph. D. student, and work on the thesis may have been inspired by Schwarz's lectures in the summer semester 1892.

Schwarz proposed generalizing methods and results in the calculus of variations, which Weierstraß had obtained for derivations of first order, to higher derivations. The problem concerns variational problems of type

$$J = \int_{t_1}^{t_2} F\big(x, x', \dots x^{(n)}; y, y', \dots y^{(n)}\big)\, dt \,,$$

and asks for curves

$$x = \varphi(t) \,, \quad y = \psi(t)$$

leading to an extremal value of J.

Weierstraß had treated his solution for $n = 1$ in his courses, particularly in a course given in the summer semester 1879 which had been written up under the auspices of the Berlin Mathematical Society. Zermelo studied these lecture notes in the summer of 1892, when he attended Schwarz's course on the calculus of variations. Less than two years later he succeeded in solving the problem Schwarz had set. On 23 March 1894, then 22 years old, Zermelo applied to begin the Ph. D. process. The oral examination took place on 6 October 1894.

In his report of 5 July 1884 Schwarz describes the subject as very difficult; he is convinced that Zermelo provided the very best solution and predicts a lasting influence of the methods Zermelo had developed and of the results he had obtained:

> According to my judgement the author succeeds in generalizing the main investigations of Mr. Weierstraß [...] in the correct manner. In my opinion he thus obtained a valuable completion of our present knowledge in this part of the calculus of variations. Unless I am very much mistaken, all future researchers in this difficult area will have to take up the results of this work and the way they are deduced.

He marked the thesis with the highest degree possible, *diligentia et acuminis specimen egregium.* Co-referee Fuchs shared his evaluation.

Various voices confirm the significance of Zermelo's dissertation for the development of the Weierstraßian direction in the calculus of variations. Examples may be found in Adolf Kneser's monograph on the calculus of variations (*1900*) and in Constantin Carathéodory's similarly influential monograph *1935.* Oskar Bolza (1857–1942) also gives due respect to Zermelo: his epochal monograph *1909* contains numerous quotations of Zermelo's work. Moreover, the quality of the dissertation played a major role when Zermelo was considered for university positions after his *Habilitation.*

1.3.2 Statistical mechanics

The oral examination of the Ph. D. process featured the defence of three theses that could be proposed by the candidate. Zermelo had made the following choice:

I. In the calculus of variations one has to attach importance to an exact definition of maximum or minimum more than has been done up to now.
II. It is not justified to burden physics with the task of reducing all phenomena in nature to the mechanics of atoms.
III. Measurement can be understood as the everywhere applicable means to distinguish and to compare continuously changing qualities.

The first thesis reflects the special care with which Zermelo attends to the definition of maxima and minima in his dissertation in the case of higher derivations. The latter theses, in particular the second one, are clearly aimed against early atomism in physics and the mechanical explanations coming with it. They may have been adopted by Zermelo when attending Planck's lecture on thermodynamics in the winter semester 1893/94 and gained in significance within the next two years when he was an assistant to Planck.

Around 1895 the atomistic point of view, while widely accepted in chemistry, was still under debate in physics, Planck being among the sceptics. For an "atomist", the principles governing heat theory are reducible to the mechanical behaviour of the particles that constitute the system under consideration. However, as a rule, the number of atoms definitely precludes the possibility of calculating exactly how each of them will behave. To overcome this dilemma, atomists used statistical methods to describe the expected behaviour.

In Boltzmann's presentation the probability $W(s)$ of a system A to be in state s—or, up to a constant, its entropy—is measured by the number (with respect to some suitable measure) of the configurations of A that macroscopically represent s. According to this interpretation the second law now says that physical systems tend toward states of maximal probability or maximal entropy, thereby imposing a direction on the physical processes concerned.

Zermelo published his concerns about statistical mechanics in a note (*1896a*) that took direct aim at Boltzmann, and this provoked a controversy which took place in two rounds in the *Annalen der Physik und Chemie* in 1896/97. Personal invective aside, Zermelo's note led to a re-consideration of foundational questions in physics at a time when probability was emerging and competing with the paradigm of causality. Zermelo observed that the function describing the spatial coordinates of the particles of a bounded system together with their velocities falls under Henri Poincaré's recurrence theorem, which claims that the system in question will again and again approach its initial state provided this state does not belong to a set of exceptional states of measure zero. Thus, as a rule, the entropy of a bounded system cannot steadily increase. His argument became known as *Zermelo's recurrence objection*.[1]

There is evidence that Zermelo was backed by Planck. Boltzmann, aware of the stakes, felt himself compelled to give an immediate answer. In his reply, *1896*, he accepted the application of the recurrence theorem, but defended his theory by arguing that the second law would in itself be subject to probability. The increase of entropy was not absolutely certain, but a decrease or a return to the neighbourhood of the initial state was so improbable and the expected recurrence time so long that one was fully justified in excluding it.

Zermelo answered with further counter-arguments (*1896b*), among them a variant of Josef Loschmidt's reversibility objection, arguing that the notion of probability did not refer to time and, hence, could not be used to impose a direction on physical processes as is the case with the second law. With regard to this point, Boltzmann's defence *1897* brought up the possibility that the universe may be a collection of single worlds, partly in a state of growing entropy and partly in a state of decreasing entropy, thus providing a way to escape the reversibility objection.

Through the next decade, Zermelo maintained interest in statistical mechanics. In his *Habilitation* address, *1900*, given in 1899 in Göttingen, he offered

[1] Three years earlier Poincaré had already formulated the recurrence objection as one of the major difficulties when bringing together experimental results and mechanism (*1893*, 537).

a solution to a problem which he had already mentioned at the end of his rejoinder *1896b*, the problem that on the one hand, Boltzmann applied his statistical view to each state of a system developing in time, whereas, on the other hand, these states were not independent from each other but were intimately linked by the laws of mechanics. Six years later Zermelo published a German translation (*Gibbs 1905*) of Josiah Willard Gibbs' monograph on statistical mechanics (*Gibbs 1902*) which "played an important role in making Gibbs' work known in Germany" (*Uffink 2004*, 1.2). In the preface he wrote in appreciation of Gibbs' book as being the first attempt to develop strictly and on a secure mathematical basis the statistical and the probability theoretical considerations in mechanics. However, despite the growing acceptance of Boltzmann's ideas—even by Planck—Zermelo's reservations concerning the range of statistical mechanics did not get resolved, as described in detail in his review *1906* of Gibb's book. Today Boltzmann's ideas are generally accepted. However, some of Zermelo's questions such as that for a physical explanation why, say, our universe might have been in a state of low entropy when it came to exist, or the question about how Boltzmann's *Einzelwelten* fit the cosmological picture, are still subject to actual research.

2 Göttingen

2.1 Introduction

Already in the summer of 1896, while still an assistant to Planck, Zermelo applied for a position as assistant at the Deutsche Seewarte in Hamburg, the central institution for maritime meteorology of the German Reich. His application was supported by Schwarz and Planck. Apparently, Zermelo was going to give up on an academic career, henceforth dedicating his work to a mathematical treatment of practical metereological problems. For unknown reasons, however, he ultimately decided to pursue the aim of obtaining an academic position. The mathematical treatment of metereological depressions he had started seemed of sufficient worth to be extended to a *Habilitation* thesis, a post-doctoral thesis necessary for obtaining a professorship.

On 19 July 1897 Zermelo turned to Felix Klein in Göttingen for advice, and presumably Klein's answer encouraged him to go to Göttingen. He enrolled there for mathematics on 4 November 1897. Given his special field of competence, Zermelo had made a good choice, as Göttingen was a centre of applied mathematics. It was to become, moreover, the leading centre for research in the foundations of mathematics, a development that began after the turn of the century, the driving force to be David Hilbert. On the initiative of Klein he had been brought from Königsberg University two years before Zermelo arrived.

In 1897 Hilbert was still mainly working in algebraic number theory. He subsequently changed his main field of research, concentrating on the foundations of geometry and the axiomatic method. Zermelo became deeply involved from the first hour on. He grew into the role of Hilbert's most important collaborator in foundational studies in this important early period, slowly moving from applied mathematics and mathematical physics to foundational studies, set theory, and logic. Decades later he would speak of Hilbert as his "first and sole teacher in science" to whom he owes "more than to anybody else with regard to ⟦his⟧ scientific development."

2.2 Applied mathematics in Göttingen

During his first semester in Göttingen Zermelo heard Hilbert's course on irrational numbers and attended the mathematical seminar on differential equations in mechanics directed by Hilbert and Klein. He took exercises in physics with Eduard Riecke and heard thermodynamics with Oskar Emil Meyer. During the next semester he took part in a course on set theory given by Arthur Schoenflies and in Klein's mathematical seminar.

For the time being and in accord to his plans, applied mathematics and physics became Zermelo's main subjects. Hilbert's *Nachlass* contains a manuscript (*Zermelo s1899b*) in which Zermelo studied the movement of an unstretchable material thread in a potential field. In his first Göttingen publication (*1899a*) Zermelo succeeded in generalizing a result of Adolph Mayer (*1899*) on the uniqueness of the solutions of a special differential equation; this equation represents the accelerations of the points of a frictionless system in terms of their coordinates, these restricted by inequalities, and their velocities.

In late 1898, Zermelo completed his *Habilitation* thesis which he had begun already with Planck in Berlin, completing the *Habilitation* process on 4 March 1899 by giving the *Habilitation* address *1900* mentioned above.

The thesis aims at creating a systematic theory of incompressible and frictionless fluids streaming in the sphere. The starting motivation is to get "information about many a process in the development of atmospheric cyclones and ocean currents as far as they concern the whole of the earth and as far as the vertical component of the current can be neglected against its horizontal one." The first two chapters deal with the general theory, while the third chapter deals with the case of finitely many vortices and the last chapter with the special case of three.

Only the first two chapters of the *Habilitation* thesis appeared in print (*1902a*). The *Nachlass* contains the handwritten version of Chapters 3 and 4 (*s1902b, s1902c*). There are no documents that might explain why the last two chapters remained unpublished and whether this was due to a deliberate decision.

The review of the published part in the *Jahrbuch der Fortschritte der Mathematik* (Vol. 33, 0781.01) essentially quotes from the introduction. In later reports on Zermelo's achievements having to do with university position possibilities, his *Habilitation* thesis did not play a major role, whereas his