Quantum Field Theory II:
Quantum Electrodynamics

Eberhard Zeidler

Quantum Field Theory II: Quantum Electrodynamics

A Bridge between Mathematicians and Physicists

Springer

Eberhard Zeidler
Max Planck Institute
for Mathematics in the Sciences
Inselstr. 22-26
04103 Leipzig
Germany

ISBN 978-3-540-85376-3 e-ISBN 978-3-540-85377-0

DOI 10.1007/978-3-540-85377-0

Library of Congress Control Number: 2006929535

Mathematics Subject Classification (2000): 35-XX, 47-XX, 49-XX, 51-XX, 55-XX, 81-XX, 82-XX

© 2009 Springer-Verlag Berlin Heidelberg

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer. Violations are liable to prosecution under the German Copyright Law.

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Cover design: WMXDesign GmbH

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com

TO FRIEDRICH HIRZEBRUCH
IN GRATITUDE

Preface

And God said, Let there be light; and there was light.
Genesis 1,3

Light is not only the basis of our biological existence, but also an essential source of our knowledge about the physical laws of nature, ranging from the seventeenth century geometrical optics up to the twentieth century theory of general relativity and quantum electrodynamics.
Folklore

Don't give us numbers: give us insight!
A contemporary natural scientist to a mathematician

The present book is the second volume of a comprehensive introduction to the mathematical and physical aspects of modern quantum field theory which comprehends the following six volumes:

Volume I: Basics in Mathematics and Physics
Volume II: Quantum Electrodynamics
Volume III: Gauge Theory
Volume IV: Quantum Mathematics
Volume V: The Physics of the Standard Model
Volume VI: Quantum Gravitation and String Theory.

It is our goal to build a bridge between mathematicians and physicists based on the challenging question about the fundamental forces in

- macrocosmos (the universe) and
- microcosmos (the world of elementary particles).

The six volumes address a broad audience of readers, including both undergraduate and graduate students, as well as experienced scientists who want to become familiar with quantum field theory, which is a fascinating topic in modern mathematics and physics.

For students of mathematics, it is shown that detailed knowledge of the physical background helps to motivate the mathematical subjects and to discover interesting interrelationships between quite different mathematical topics. For students of physics, fairly advanced mathematics are presented, which is beyond the usual curriculum in physics. The strategies and the structure of the six volumes are thoroughly discussed in the Prologue to Volume I. In particular, we will try to help the reader to understand the basic ideas behind the technicalities. In this connection, the famous ancient story of Ariadne's thread is discussed in the Preface to Volume I. In terms of this story, we want to put the beginning of Ariadne's thread in quantum field theory into the hands of the reader.

> *The present volume is devoted to the physics and mathematics of light.*

It contains the following material:

Part I: Introduction
- Chapter 1: Mathematical Principles of Modern Natural Philosophy
- Chapter 2: The Basic Strategy of Extracting Finite Information from Infinities – Ariadne's Thread in Renormalization Theory
- Chapter 3: The Power of Combinatorics
- Chapter 4: The Strategy of Equivalence Classes in Mathematics

Part II: Basic Ideas in Classical Mechanics
- Chapter 5: Geometrical Optics
- Chapter 6: The Principle of Critical Action and the Harmonic Oscillator as a Paradigm

Part III: Basic Ideas in Quantum Mechanics
- Chapter 7: Quantization of the Harmonic Oscillator – Ariadne's Thread in Quantization
- Chapter 8: Quantum Particles on the Real Line – Ariadne's Thread in Scattering Theory
- Chapter 9: A Glance at General Scattering Theory.

Part IV: Quantum Electrodynamics (QED)
- Chapter 10: Creation and Annihilation Operators
- Chapter 11: The Basic Equations in Quantum Electrodynamics
- Chapter 12: The Free Quantum Fields of Electrons, Positrons, and Photons
- Chapter 13: The Interacting Quantum Field, and the Magic Dyson Series for the S-Matrix
- Chapter 14: The Beauty of Feynman Diagrams in QED
- Chapter 15: Applications to Physical Effects

Part V: Renormalization
- Chapter 16: The Continuum Limit
- Chapter 17: Radiative Corrections of Lowest Order
- Chapter 18: A Glance at Renormalization to all Orders of Perturbation Theory
- Chapter 19: Perspectives

We try to find the right balance between the mathematical theory and its applications to interesting physical effects observed in experiments. In particular, we do not consider purely mathematical models in this volume.

> *It is our philosophy that the reader should learn quantum field theory by studying a realistic model, as given by quantum electrodynamics.*

Let us discuss the main structure of the present volume. In Chapters 1 through 4, we consider topics from classical mathematics which are closely related to modern quantum field theory. This should help the reader to understand the basic ideas behind quantum field theory to be considered in this volume and the volumes to follow. In Chapter 1 on the mathematical principles of modern natural philosophy, we discuss

- the infinitesimal strategy due to Newton and Leibniz,
- the optimality principle for processes in nature (the principle of critical action) and the calculus of variations due to Euler and Lagrange, which leads to the fundamental differential equations in classical field theory,
- the propagation of physical effects and the method of the Green's function,
- harmonic analysis and the Fourier method for computing the Green's functions,
- Laurent Schwartz's theory of generalized functions (distributions) which is related to the idea that the measurement of physical quantities by devices is based on averaging,
- global symmetry and conservation laws,
- local symmetry and the basic ideas of modern gauge field theory, and
- the Planck quantum of action and the idea of quantizing classical field theories.

Gauge field theory is behind both

- the Standard Model in elementary particle physics and
- Einstein's theory of gravitation (i.e., the theory of general relativity).

In quantum field theory, a crucial role is played by *renormalization.* In terms of physics, this is based on the following two steps:

- the regularization of divergent integrals, and
- the computation of effective physical parameters measured in experiments (e.g., the effective mass and the effective electric charge of the electron).

Renormalization is a highly technical subject. For example, the full proof on the renormalizability of the electroweak sector of the Standard Model in particle physics needs 100 pages. This can be found in:

> E. Kraus, Renormalization of the electroweak standard model to all orders, Annals of Physics **262** (1998), 155–259.

Whoever wants to understand quantum field theory has to understand the procedure of renormalization. Therefore, the different aspects of renormalization theory will be studied in all of the six volumes of this series of monographs. This ranges from

- resonance phenomena for the anharmonic oscillator (classical bifurcation theory),
- the Poincaré–Lindstedt series (including small divisors) in celestial mechanics,

- and the Kolmogorov–Arnold–Moser (KAM) theory for perturbed quasi-periodic oscillations (e.g., in celestial mechanics) based on sophisticated iterative techniques (the hard implicit function theorem)

to the following fairly advanced subjects:

- the Feynman functional integral (the Faddeev–Popov approach),
- the Wiener functional integral (the Glimm–Jaffe approach),
- the theory of higher-dimensional Abelian integrals (algebraic Feynman integrals),
- Hopf algebras and Rota–Baxter algebras in combinatorics (the modern variant of the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) approach due to Kreimer),
- the Riemann–Hilbert problem and the Birkhoff decomposition (the Connes–Kreimer approach),
- Hopf superalgebras (the Brouder–Fauser–Frabetti–Oeckl (BFFO) approach),
- characterization of physical states by cohomology and algebraic renormalization (the Becchi–Rouet–Stora–Tyutin (BRST) approach),
- the Riesz–Gelfand theory of distribution-valued meromorphic functions (construction of the Green's functions),
- wave front sets and Hörmander's multiplication of distributions (the Stueckelberg–Bogoliubov–Epstein–Glaser–Scharf approach),
- the Master Ward identity as a highly non-trivial renormalization condition and the generalized Dyson–Schwinger equation (the Dütsch–Fredenhagen approach),
- q-deformed quantum field theory (the Wess–Majid–Wachter–Schmidt approach based on the q-deformed Poincaré group, quantum groups, and the q-analysis on specific classes of q-deformed quantum spaces),
- deformation of bundles and quantization (the Weyl–Flato–Sternheimer–Fedosov–Kontsevich approach),
- microlocal analysis and renormalization on curved space-times (the Radzikowski–Brunetti–Fredenhagen–Köhler approach),
- renormalized operator products on curved space-times (the Wilson–Hollands–Wald approach to quantum field theory),
- natural transformations of functors in category theory and covariant quantum field theory on curved space-time manifolds (the Brunetti–Fredenhagen–Verch approach),

as well as

- one-parameter Lie groups and the renormalization group,
- attractors of dynamical systems in the space of physical theories (the Wilson–Polchinski–Kopper–Rivasseau approach to renormalization based on the renormalization group),
- the Master Ward Identity and the Stueckelberg–Petermann renormalization group (the Dütsch–Fredenhagen approach),
- motives in number theory and algebraic geometry, the Tannakian category, and the cosmic Galois group as a universal (motivic) renormalization group (the Connes–Marcolli approach),
- noncommutative geometry and renormalization (the Grosse–Wulkenhaar approach).

The recent work of Alain Connes, Dirk Kreimer, and Matilde Marcolli shows convincingly that renormalization is rooted in highly nontrivial mathematical structures. We also want to emphasize that the theory of many-particle systems (considered in statistical physics and quantum field theory) is deeply rooted in the theory of operator algebras. This concerns

- von Neumann algebras (the von Neumann approach),
- C^*-algebras (the Gelfand–Naimark–Segal approach),
- local nets of operator algebras (the Haag–Kastler approach) and,
- noncommutative geometry (the Connes approach).

As a warmup, we show in Chapter 2 that the regularization of divergent expressions represents a main subject in the history of mathematics starting with Euler in the eighteenth century. In this connection, we will consider

- the regularization of divergent series, and
- the regularization of divergent integrals.

In particular, in Sect. 2.1.3, we will discuss the classical Mittag–Leffler theorem on meromorphic functions f. If the function f has merely a finite number of poles, then the method of partial fraction decomposition works well. However, as a rule, this method fails if the function f has an infinite number of poles. In this case, Mittag–Leffler showed in the late nineteenth century that one has to subtract special terms, which are called subtractions by physicists.

The subtractions force the convergence of the infinite series.

This is the prototype of the method of iteratively adding subtractions in the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) approach to renormalization theory. The corresponding iterative algorithm (called the Bogoliubov R-operation) has to be constructed carefully (because of nasty overlapping divergences). This was done by Nikolai Bogoliubov in the 1950s. An ingenious explicit solution formula for this iterative method was found by Wolfhart Zimmermann in 1969. This is the famous *Zimmermann forest formula.* In the late 1990s, it was discovered by Dirk Kreimer that the sophisticated combinatorics of the Zimmermann forest formula can be understood best in terms of a Hopf algebra generated by Feynman diagrams. By this discovery, the modern formulation of the BPHZ approach is based on both Hopf algebras and Rota–Baxter algebras.

As a warmup, in Chapter 3, we give an introduction to the modern combinatorial theory, which was founded by Gian-Carlo Rota (MIT, Cambridge, Massachusetts) in the 1960s. This includes both Hopf algebras and Rota–Baxter algebras.

Surprisingly enough, it turns out that the Zimmermann forest formula is closely related to methods developed by Lagrange in the eighteenth century when studying the solution of the Kepler equation for the motion of planets in celestial mechanics.

In modern terminology, the Lagrange inversion formula for power series expansions is based on the so-called Faà di Bruno Hopf algebra.[1] This will be studied in Sect. 3.4.3.

[1] The Italian priest and mathematician Francesco Faà di Bruno (1825–1888) was beatified in 1988.

In physics, symmetries are basic. For describing symmetries in terms of mathematics, there are two approaches based on

- groups and
- Hopf algebras.

In 1941, Heinz Hopf wanted to compute the cohomology of topological groups. Hopf discovered that the cohomology ring of topological groups is equipped with an additional structure which is called a Hopf algebra structure today. This additional algebraic structure is based on the notion of the coproduct. Roughly speaking, the concept of Hopf algebra is dual to the concept of group. Hopf algebras are intimately related to quantum groups. We will show in Chapter 3 that:

> *The product and the coproduct of a Hopf algebra model the fusion and the splitting of objects (e.g., elementary particles), respectively.*

In terms of analysis, the algebra of linear differential operators with constant coefficients can be equipped with the structure of a Hopf algebra. Here,

- the coproduct is related to the Leibniz product rule of differentiation, and
- the coinverse (also called the antipode) is related to the integration-by-parts formula (see Sect. 3.3.1).

The integration-by-parts formula is a special case of the general Stokes integral theorem, which lies at the heart of the duality between homology and cohomology in topology. This duality plays a key role for the mathematical description of processes in nature. In particular, cohomology is deeply rooted in Maxwell's theory of electrodynamics (see Sect. 4.4.7).

Incredible cancellations. When doing computations in renormalization theory, as a big surprise, physicists and mathematicians encounter incredible cancellations of a large amount of terms. This dramatically simplifies the final result. In terms of mathematics, a sophisticated combinatorics is behind these cancellations. The prototype for this is given by the Faà di Bruno Hopf algebra mentioned above.

The language of modern mathematics. We do not assume that the reader of this series of monographs is familiar with the language used in modern mathematics. In this connection, we want to help the reader. For example, many notions in advanced mathematics and hence in modern mathematical physics are based on mathematical operations applied to equivalence classes. For example, this concerns

- the construction of quantum states as equivalence classes of elements of Hilbert spaces (and the relation to projective geometry),
- the Gelfand–Naimark–Segal (GNS) construction for representing the elements of an abstract C^*-algebra as observables on a Hilbert space (the algebraic approach to quantum theory),
- the Wightman reconstruction theorem for axiomatically defined quantum fields (via the GNS-construction),

- moduli spaces of Riemann surfaces (modulo conformal equivalence) and physical states in string theory.

This leads to quotient spaces in algebra, analysis, geometry, and topology, which will be encountered again and again in this series of monographs (e.g., homology groups, cohomology groups, homotopy groups, and K-theory in topology). Chapter 4 serves as an introduction to quotient structures in mathematics and physics. The idea of the quotient ring (modulo a fixed integer) can be traced back at least to the *Disquisitiones arithmeticae* written by the young Gauss (1777–1855) in 1801.[2] In order to give the reader a feel for the usefulness of working with equivalence classes, we will consider the following examples:

- Gaussian quotient rings (modulo a prime number) and coding theory (as warmup for quantum information),
- quotient fields and Heaviside's symbolic method in electrical engineering (the Mikusiński operational calculus),
- physical fields, observers, bundles, and cocycles,
- deformation, mapping classes, and topological charges,
- loop spaces and higher homotopy groups,
- the projective and the injective limit of mathematical structures (e.g., topological spaces), and
- the rigorous approach to Leibniz's infinitesimals via ultrafilters (nonstandard analysis).

For the foundation of nonstandard analysis, one needs the construction of ultrafilers via Zorn's lemma based on the axiom of choice in set theory.

The language of theoretical physics. Chapters 5 through 9 are devoted to the basic ideas of

- classical geometric optics,
- classical mechanics, and
- quantum mechanics.

Here, we want to help mathematicians who are not familiar with theoretical physics. In Chapter 5, we study Carathéodory's royal road to geometrical optics based on the fundamental duality between

- light rays and
- wave fronts

which can be traced back to the work of Huygens in the seventeenth century. In string theory, Kähler manifolds play a crucial role. In Chapter 5, we will show how Poincaré's non-Euclidean geometry on the upper half-plane is related to both geometrical optics and Kähler geometry.

[2] The enormous influence of Gauss' first masterpiece on the development of mathematics is described in the monograph by C. Goldstein, N. Schappacher, and J. Schwermer: The Shaping of Arithmetic after Gauss' Disquisitiones Arithmeticae, Springer, Berlin 2007.

Since all the models of quantum fields are based on the study of an infinite number of (slightly perturbed) harmonic oscillators in the setting of perturbation theory, we use the harmonic oscillator as a paradigm for both classical mechanics and quantum mechanics. In Chapter 6 on classical mechanics, we will study the following topics:

- Newtonian mechanics,
- Lagrangian mechanics (the Euler–Lagrange equation, the Jacobi accessory eigenvalue problem and Morse theory),
- Hamiltonian mechanics (the canonical dynamical system and the Hamilton–Jacobi partial differential equation), and
- Poissonian mechanics.

In particular, this concerns

- the Legendre transformation and contact geometry,
- the Hamiltonian flow and symplectic geometry,
- the *tangent bundle* of the position space (the position-velocity space also called the state space),
- the *cotangent bundle* of the position space (the position-momentum space also called the phase space),
- the Legendre transformation as a transformation from the tangent bundle to the cotangent bundle; the latter is equipped with a natural symplectic structure.

In terms of mathematics, the fundamental relation between symmetry and conservation laws in physics is related to

- the Noether theorem, and
- Poisson brackets and Lie's momentum map.

Quantum mechanics. The comprehensive Chapter 7 lies at the heart of this series of monographs. This chapter should help the reader to understand the different aspects of the passage from classical physics to quantum physics, by using the different procedures of *quantization*. We will use the paradigm of the harmonic oscillator in order to explain the basic ideas of the following approaches:

- Heisenberg's quantum mechanic (via creation and annihilation operators),
- Schrödinger's quantum mechanics (via the Schrödinger partial differential equation),
- Feynman's quantum mechanics (via the path integral),
- von Neumann's functional-analytic approach (via the spectral theory for self-adjoint operators in Hilbert spaces),
- von Neumann's density operator in statistical physics (via trace class operators),
- Weyl's symbolic calculus for pseudo-differential operators (deformation quantization),
- the Poincaré–Wirtinger calculus and Bargmann's holomorphic quantization,
- the Stone-von Neumann uniqueness theorem (for the fundamental commutation relations in quantum mechanics) and the Weyl functor[3] based on symplectic geometry,
- supersymmetric quantization.

[3] At this place, the general theory of mathematical structures (also called category theory) enters the theory of quantization (also called quantum mathematics).

Concerning the Feynman path integral as a fundamental tool in quantum physics, we will study the following:

- Brownian motion and the infinite-dimensional rigorous Wiener integral based on measure theory,
- the rigorous Feynman–Kac formula for diffusion processes,
- rigorous finite-dimensional Gaussian integrals, the computation of correlations and moments, the Wick theorem, and Feynman diagrams,
- rigorous definition of infinite-dimensional Gaussian integrals via zeta function regularization,
- the Wentzel–Kramers–Brioullin (WKB) method of stationary phase for the computation of Gaussian integrals, and the approximate computation of Feynman path integrals.

The Feynman path integral can be obtained from the Wiener integral by using formal analytic continuation from real time to imaginary time. This corresponds to the fact that the Schrödinger equation describes diffusion processes in imaginary time. Furthermore, in Chapter 7, we discuss the basic ideas of the *algebraic approach* to quantum mechanics by using C^*-algebras and von Neumann algebras. In this connection, we consider:

- applications to statistical mechanics (Boltzmann statistics, Bose–Einstein statistics, and Fermi–Dirac statistics),
- thermodynamic equilibrium states (Kubo–Martin–Schwinger (KMS) states) and the Tomita–Takesaki theory for von Neumann algebras,
- the Murray–von Neumann classification of factors in the theory of von Neumann algebras,
- projection operators and the main theorem of quantum logic (Gleason's extension theorem for C^*-algebras).

The modern theory of operator algebras culminates in Alain Connes's noncommutative geometry, which represents the appropriate mathematical structure for a deeper understanding of the Standard Model in elementary particle physics. This will be investigated in Volume IV on quantum mathematics. For the interested reader, we refer to the following fundamental monograph:

> A. Connes and M. Marcolli, Noncommutative Geometry, Quantum Fields, and Motives, American Mathematical Society 2008.
> Internet: `http://www.math.fsu.edu/∼ marcolli/bookjune4.pdf`

Chapters 8 and 9 serve as an introduction to scattering theory, which plays a crucial role in elementary particle physics. As a paradigm for general scattering theory, we consider the scattering of a quantum particle on the real line. We consider

- the energy levels of bound states,
- the energy levels of scattering states, and distributions as generalized eigenfunctions of the Schrödinger equation,
- the transition matrix,
- the unitary S-matrix and transition probabilities for scattering processes,
- the relation between the singularities of the S-matrix in the complex energetic plane and the energy levels of stable bound states,

- unstable particles (resonances) and the second sheet of the energetic Riemann surface (the Breit–Wigner formula for the energy levels and the mean lifetime of resonances),
- stationary scattering theory, the Green's function of the Helmholtz equation, and the Lippmann–Schwinger integral equation,
- instationary scattering theory, wave operators, the absolutely continuous spectrum of the Hamiltonian, and the S-matrix in functional analysis.

Here, we do not assume that the reader is familiar with

- von Neumann's functional-analytic spectral theory for self-adjoint operators in Hilbert spaces,
- the Gelfand–Kostyuchenko theory of generalized eigenfunctions for self-adjoint operators,
- the Møller–Kato theory of wave operators in scattering theory, and
- the Weyl–Kodaira theory for singular differential operators.

For the convenience of the reader, the necessary material will be summarized at the proper place when it is needed in Volumes II and III.

Quantum electrodynamics. In the present volume, it is our main goal to illustrate the beauty of quantum electrodynamics by proceeding pragmatically.

> *We do not start with an abstract approach, but with the computation of important physical effects which are observed in experiments, including radiative corrections in lowest order of renormalization theory.*

This should help the reader in getting a feel for the essential questions. More sophisticated approaches are postponed to later volumes of this series of monographs. In the introductory Chapter 10, we study creation and annihilation operators for electrons, positrons, and photons. In Chapter 11, we formulate the classical field equations of quantum electrodynamics on the interaction between electrons and photons, by coupling the Maxwell equations of the electromagnetic field to the Dirac equation of the electron wave function. This equation depends on the gauge fixing of the four-potential for the electromagnetic field. However, it turns out that physical effects measured in experiments are independent of the choice of the gauge fixing. The point is that:

> *The classical field equations of quantum electrodynamics have to be quantized.*

In this connection, we have to distinguish between

- the single free quantum fields for electrons, positrons, and photons, and
- the total interacting quantum field for electrons, positrons, and photons.

In Chapter 12, we construct free quantum fields by using the method of Fourier quantization based on creation and annihilation operators.

In Chapter 13, we study the interacting quantum field of electrons, positrons, and photons by using

- the magic Dyson formula for the S-matrix (scattering matrix), and
- the main Wick theorem for the S-matrix, which implies the Feynman diagrams.

This is Dyson's classical approach to understanding the Feynman diagrams.[4] Originally, Feynman invented his exciting diagram technique on the basis of ingenious physical intuition. In Dyson's mathematical setting, the Feynman diagrams are nothing other than graphical representations of well-defined analytic expressions, which are effectively produced by the main Wick theorem. Feynman's use of propagators and Dyson's magic formula for the S-matrix are closely related to Lagrange's variation-of-parameter formula in celestial mechanics. Many mathematicians complain about the following situation:

> *In the physics textbooks, one reads the Feynman rules for Feynman diagrams, but it is not clear where the Feynman rules come from.*

In the present textbook, we will thoroughly study the mathematical and physical origin of both the Feynman diagrams and the Feynman rules. We will also consider applications to interesting physical effects.

In Chapter 15, we investigate the following physical effects in lowest order of perturbation theory:

- the cross section for Compton scattering between photons and electrons (improvement of the Thomson formula in classical electrodynamics),
- the cross section for the scattering of electrons in an external electromagnetic field,
- the intensity of spectral lines for bound states in an external electromagnetic field, and
- the Cherenkov radiation.

For the computation of terms corresponding to higher order of perturbation theory, renormalization theory is needed. In Chapter 17, we discuss the physics behind the following radiative corrections in lowest possible order of renormalization theory:

- the screening of the Coulomb potential by vacuum polarization (the Uehling potential),
- the anomalous magnetic moment of the electron (the Schwinger formula),
- the anomalous magnetic moment of the muon, and
- the Lamb shift in the spectrum of the hydrogen atom.

Unfortunately, the explicit computations (in the framework of dimensional regularization in renormalization theory) are lengthy. We will postpone these detailed computations to Volume III.

In Chapter 18, we discuss the main result telling us that quantum electrodynamics can be renormalized to all orders of perturbation theory. The final result consists of getting finite expressions in each order of perturbation theory (e.g., cross sections for scattering processes), which depend on the two fundamental free parameters

[4] Dyson's discovery of this approach is described by himself in his book, F. Dyson, *Disturbing the Universe,* Harper & Row, New York, 1979 (see page 27 of Volume I for this fascinating story).

- m_{eff} (effective mass of the electron) and
- $-e_{\text{eff}}$ (effective electric charge of the electron).

Observe the crucial fact that:

> *The free parameters m_{eff} and e_{eff} cannot be determined theoretically by quantum electrodynamics.*

They have to be determined by physical experiments. In the SI system, one obtains the following values:

$$m_{\text{eff}} = 0.511 \text{ MeV}/c^2, \qquad e_{\text{eff}} = 1.602 \cdot 10^{-19} \text{ As}.$$

A reader who wants to become familiar with quantum electrodynamics as quickly as possible should start reading with Chapter 10.

The incredible effectiveness of perturbation theory in physics. Surprisingly enough, low-order radiative corrections are sufficient for getting a fantastic coincidence between theory and experimental data. For example, the anomalous magnetic moment of the electron measured in experiments is predicted very precisely by fourth-order radiative corrections (up to 9 digits). However, the necessary amount of computations is enormous. One has to evaluate high-dimensional integrals which correspond to 891 Feynman diagrams; this needs years of supercomputer time.

A warning to the reader. In summer 1976, Arthur Wightman (Princeton University) organized a famous conference in Erice (Sicily/Italy) on renormalization theory. In the introduction to the Proceedings of this conference, he writes:[5]

> Renormalization theory is a notoriously complicated and technical subject... I want to tell stories with a moral for the earnest student: Renormalization theory has a history of egregious errors by distinguished savants (see page 967). It has a justified reputation for perversity; a method that works up to 13th order in the perturbation theory fails in the 14th order. Arguments that sound plausible often dissolve into mush when examined closely. The worst that can happen often happens. The prudent student would do well to distinguish sharply between what has been proved and what has been plausible, and in general he should watch out!

In 1999 Gerardus 't Hooft and Martinus Veltman were awarded the Nobel prize in physics for their contributions to the renormalization of the theory of electroweak interaction and for the computation of radiative corrections in the Standard Model of particle physics.

Perspectives. More advanced approaches to renormalization theory will be systematically studied in the following volumes of this series of monographs. In particular, this concerns the new approaches to perturbative quantum field theory due to Connes and Kreimer (Hopf algebras), and Brunetti, Dütsch,

[5] A. Wightman, Orientation. In: Renormalization Theory, pp. 1–20. Edited by G. Velo and A. Wightman, Reidel, Dordrecht, 1976 (reprinted with permission).

and Fredenhagen (microlocal analysis and the Master Ward Identity). In order to give the reader an overview on the large variety of different approaches to renormalization theory, we summarize important references in Section 19.11, and in Chapter 19 we sketch some basic ideas.

The propagation of light in the universe, namely,

- the deflection of light at the sun, and
- the red shift of spectral lines as a consequence of the expansion of the universe (the Hubble effect)

will be investigated in Volume III in terms of Einstein's theory of general relativity.

The basic idea of our approach to quantum electrodynamics. As a rule, mathematicians have trouble with reading some textbooks on quantum field theory written by physicists. The point is that:

> *In mathematics, one never does computations with quantities which do not exist.*[6]

In order to respect this basic principle in mathematics, we will use the *lattice approach.* That is, roughly speaking, we will proceed in the following two steps.

Step 1: The discretized physical system. We put the physical system in a cubic box of finite side length L and volume $\mathcal{V} = L^3$. The boundary conditions are given by periodicity.

- We observe the physical system in a finite time interval $[-\frac{T}{2}, \frac{T}{2}]$.
- We choose a maximal energy $E_{\max}$.
- In the 3-dimensional momentum space, we introduce a finite lattice of spacing Δp and maximal momentum $P_{\max}$. In this setting, the Fourier integral transform is replaced by a discrete Fourier transform via finite Fourier series expansions.
- We define Dyson's S-matrix for this situation.
- The main Wick theorem allows us to compute the S-matrix elements (i.e., the transition amplitudes) in an elegant manner, by eliminating the creation and annihilation operators, and replacing them by propagators (i.e., correlation functions for free fields).
- The point is that the propagators are discrete algebraic Feynman integrals, which are indeed well-defined finite sums.
- The transition amplitudes can be graphically represented by Feynman diagrams.
- The Feynman rules allow us to translate the Feynman diagrams into well-defined finite sums.
- From the transition amplitudes, we obtain the transition probabilities which yield the cross sections for scattering processes. Note that cross sections can be measured in particle accelerators.

Step 2: The delicate continuum limit. Explicitly, we have to study the following limits:

- $L \to \infty$ (the volume L^3 of the cubic box becomes infinite),

[6] For example, this concerns infinite renormalization constants or ill-defined infinite-dimensional functional/path integrals.

- $T \to \infty$ (the time interval becomes infinite),
- $P_{\max} \to \infty$ (i.e., $E_{\max} \to \infty$) (high-energy limit),
- $\Delta p \to 0$ (low-energy limit).

In order to force the convergence of the discrete algebraic Feynman integrals to well-defined expressions, we modify the classical Lagrangian density by setting

$$m_{\text{eff}} := m_e + \delta m, \qquad e_{\text{eff}} := e + \delta e.$$

That is, we replace the so-called bare electron mass m_e and the so-called bare electron charge $-e$ in the Lagrangian density by

$$m_e = m_{\text{eff}} - \delta m, \qquad -e = -e_{\text{eff}} + \delta e,$$

respectively. This way, the classical Lagrangian density

$$\mathcal{L}(\psi, \partial\psi, A, \partial A; m_e, e)$$

is modified by the function

$$\mathcal{L}_{\text{modified}}(\psi, \partial\psi, A, \partial A; m_{\text{eff}}, e_{\text{eff}}; \delta m, \delta e).$$

The terms multiplied by $\delta m, \delta e$ are called counterterms of the classical Lagrangian density $\mathcal{L}$. Note that in this lattice approach, δe and δm are real parameters which depend on the shape of the lattice, that is, they depend on the maximal energy $E_{\max}$. Now consider the high-energy limit

$$E_{\max} \to +\infty.$$

Roughly speaking, we have to show that $\delta m(E_{\max})$ and $\delta e(E_{\max})$ can be chosen in such a way that the finite continuum limit exists for the S-matrix elements (i.e., the transition elements). This is the procedure of *renormalization.*

Observe the following peculiarity. By the Stone–von Neumann uniqueness theorem, a *finite* number of creation and annihilation operators is uniquely determined by the commutation relations (up to unitary equivalence). This is not true anymore for an *infinite* number of creation and annihilation operators, as was shown by Lars Gårding and Arthur Wightman in 1954. However, our approach avoids the latter ambiguity, since we only work with a finite number of creation and annihilation operators before passing to the continuum limit (of the vacuum expectation values). We also would like to emphasize that our approach differs only slightly from the usual approach used by physicists. In particular, we use a notation for discrete Fourier integrals such that the formal passage to the language used by physicists is possible at each stage of our procedure.

> *For the physical quantities which can be measured in experiments, our final formulas coincide with the formulas used by physicists.*

Moreover, in each step of our procedure it is easy to pass formally to the expressions used by physicists, since the Feynman diagrams are the same.

> *This way, we hope to help mathematicians in getting a better understanding for the ingenious and beautiful approach invented by physicists.*

From the physical point of view, the modification of the classical Lagrangian density reflects the fact that:

> *Quantum effects have to be added to the classical theory.*

Intuitively, this means that the quantum fluctuations of the ground state of the quantum field of electrons, positrons, and photons influence physical effects observed in experiments. For example, this concerns the anomalous magnetic moment of the electron and the spectrum of the hydrogen atom (Lamb shift).

Convention. If we do not expressively state the opposite, we will use the SI system of physical units (international system of units) which can be found in the Appendix to Volume I. In particular, note that in Chapters 10–19 on quantum electrodynamics, we will use the energetic system with $c = 1$ (velocity of light in a vacuum), $\hbar = h/2\pi = 1$ (Planck's quantum of action), $k = 1$ (Boltzmann constant), $\varepsilon_0 = \mu_0 = 1$ (see page 790).

The Poincaré Seminar. The best way of getting information about recent developments in modern physics is to look at the books which report the lectures given at the Poincaré Seminar in Paris. Starting in 2002, this seminar has been organized by *l'Institut Henri Poincaré* in Paris (see page 1050). Bertrand Duplantier and Vincent Rivasseau write in the Foreword to *Quantum Spaces*, Birkhäuser, Basel, 2007:

> This book is the seventh in a series of lectures of the *Séminaire Poincaré*, which is directed towards a large audience of physicists and mathematicians.
> The goal of this seminar is to provide up-to-date information about general topics of great interest in physics. Both the theoretical and experimental aspects are covered, with some historical background. Inspired by the *Séminaire Bourbaki* in mathematics in its organization, hence nicknamed "Séminaire Bourbaphy," the Poincaré Seminar is held twice a year at the *Institut Henri Poincaré* in Paris, with contributions prepared in advance. Particular care is devoted to the pedagogical nature of the presentation so as to fulfill the goal of being readable by a large audience of scientists.

Two recent survey volumes. The following two volumes try to reflect the state of the art by summarizing the most important approaches used in modern quantum field theory:

- B. Fauser, J. Tolksdorf, and E. Zeidler (Eds.), Quantum Gravity: Mathematical Models and Experimental Bounds, Birkhäuser, Basel, 2006.
- B. Fauser, J. Tolksdorf, and E. Zeidler (Eds.), Quantum Field Theory – Competitive Methods, Birkhäuser, Basel, 2008.

Acknowledgements. I am very grateful to my collaborators Bertfried Fauser and Jürgen Tolksdorf for organizing two workshops in 2005 and 2007 on recent developments in quantum field theory, and for both doing an excellent editorial job and telling me new research strategies and new results in the algebraic and geometric approach to quantum field theory. In particular, I would like to thank Bertfried Fauser for drawing my attention to the relation between quantum field theory and the following topics: Gian-Carlo Rota's work on combinatorics, the ubiquitous Hopf algebras, Rota–Baxter algebras, quantum Clifford algebras, the Rota–Stein cliffordization process, Littlewood's work on the representation theory of the symmetric group, motivic number theory, topos theory, and category theory in algebraic geometry. For illuminating discussions about promising new approaches to (perturbative) quantum field theory in both flat and curved space-time, I would like to thank Klaus Fredenhagen and the late Julius Wess, as well as Detlev Buchholz, Christian Brouder, Romeo Brunetti, Michael Dütsch, Kurusch Ebrahimi-Fard, Joel Feldman, Felix Finster, Christian Fleischhack, Alessandra Frabetti, Stefan Hollands, Harald Grosse, Jerzy Kijowski, Dirk Kreimer, Elisabeth Kraus, Alexander Lange, Matilde Marcolli, Mario Paschke, Klaus Rehren, Gerd Rudolph, Manfred Salmhofer, Alexander Schmidt, Klaus Sibold, Rainer Verch, Hartmut Wachter, and Raimar Wulkenhaar.

I would like to thank Thomas Hahn from the Max Planck Institute for Physics, Werner Heisenberg, in Munich for informing me on the state of the art in automated multi-loop computations in perturbation theory. Such sophisticated computer programs are used for preparing the experiments at the LHC (Large Hadron Collider) of CERN (European Organization for Nuclear Research at Geneva, Switzerland) (see Sect. 18.4).

On the occasion of Professor Friedrich Hirzebruch's 80th birthday on October 17 in 2007, I would like to dedicate this volume to him in gratitude. His scientific work deeply influenced the development of mathematics in the second half of the twentieth century. Nowadays physicists frequently use Hirzebruch's results in order to study the topological structure of physical fields. In 1982, Friedrich Hirzebruch founded the Max Planck Institute for Mathematics in Bonn (Germany). Mathematicians from all over the world enjoy doing research in the relaxed and highly stimulating atmosphere of this institute. In 1996, the Max Planck Institute for Mathematics in the Sciences was founded in Leipzig. Friedrich Hirzebruch was the chairman of the Founders' Committee. The staff and the visitors of our institute are very grateful to Professor Hirzebruch for his efforts made as chairman.

For helping me to save a lot of time, I am very grateful to my secretary Regine Lübke (invaluable support), Katarzyna Baier (answering patiently almost infinitely many bibliographical questions), the library team (steadily support), Kerstin Fölting (graphics and tables), Micaela Krieger-Hauwede (answering patiently my LaTeX questions), Katrin Scholz (internet searching), and Thomas Heid (computer expert). I also would like to thank the staff of the Springer publishing house in Heidelberg, Ruth Allewelt, Joachim Heinze, and Martin Peters, for the harmonious collaboration. Many years ago, my Czech colleague from Prague, the late Svatopluk Fučik, wrote the following sentence in the preface to his book on nonlinear partial differential equations: "Finally, I would like to thank myself for typing the manuscript." I resist the temptation of repeating this sentence with respect to my LaTeX file. But I would like to thank Donald Knuth for enjoying the mathematical community with the beautiful gift of the TeX tool.

In the Proverbs 31, 10 of the Bible, one reads:

> Who can find a virtuous woman? For her price is far above rubies. The heart of her husband does safely trust in her, so that he shall have no need of spoil. She will do him good and not evil all the days of her life.

I am very grateful to my beloved wife, Christine, who has been taking care of me for 40 years.

Leipzig, Summer 2008 *Eberhard Zeidler*

Contents