Trajectory Planning for Automatic Machines and Robots

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To Francesca and Morena

Preface

This book deals with the problems related to planning motion laws and trajectories for the actuation system of automatic machines, in particular for those based on electric drives, and robots. The problem of planning suitable trajectories is relevant not only for the proper use of these machines, in order to avoid undesired effects such as vibrations or even damages on the mechanical structure, but also in some phases of their design and in the choice and sizing of the actuators. This is particularly true now that the concept of "electronic cams" has replaced, in the design of automatic machines, the classical approach based on "mechanical cams".

The choice of a particular trajectory has direct and relevant implications on several aspects of the design and use of an automatic machine, like the dimensioning of the actuators and of the reduction gears, the vibrations and efforts generated on the machine and on the load, the tracking errors during the motion execution.

For these reasons, in order to understand and appreciate the peculiarities of the different techniques available for trajectory planning, besides the mathematical aspects of their implementation also a detailed analysis in the time and frequency domains, a comparison of their main properties under different points of view, and general considerations related to their practical use are reported.

For these reasons, we believe that the contents of this book can be of interest, besides for students of Electrical and Mechanical Engineering courses, also for engineers and technicians involved in the design and use of electric drives for automatic machines.

We would like to thank all the persons and colleagues which have contributed to this book. In particular, we would like to thank Claudio Bonivento, for the initial suggestions and motivations, and Alberto Tonielli for the discussions on electric drives and their use. The colleagues and friends Roberto Zanasi, Cesare Fantuzzi, and Alessandro De Luca have contributed not only with several constructive comments, but also with the development of some of the algorithms presented in this book.

Finally, the help of all the students that have worked on these arguments developing software and executing experimental activities, as well as the co-operations and discussions with technicians and engineers of several industries with their problems related to the design, control, and trajectory planning for automatic machines, are gratefully acknowledged.

Bologna,
June 2008

Luigi Biagiotti
Claudio Melchiorri

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1

Trajectory Planning

This book deals with the problem of *trajectory planning*, i.e. of the computation of desired motion profiles for the actuation system of automatic machines. Because of their wide use, only electric drives are considered here, and their motion is defined in the context of the real-time control of automatic machines with one or more actuators, such as packaging machines, machine-tools, assembly machines, industrial robots, and so on. In general, for the solution of this problem some specific knowledge about the machine and its actuation system is also required, such as the *kinematic model (direct and inverse)* (usually the desired movement is specified in the *operational space*, while the motion is executed in the *actuation space* and often these domains are different) and the *dynamic model of the system* (in order to plan suitable motion laws that allow to execute the desired movement with proper loads and efforts on the mechanical structure). Moreover, for the real-time execution of the planned motion, it is necessary to define proper *position/velocity control algorithms*, in order to optimize the performances of the system and to compensate for disturbances and errors during the movements, such as saturations of the actuation system. Several techniques are available for planning the desired movement, each of them with peculiar characteristics that must be well known and understood. In this book, the most significant and commonly adopted techniques for trajectory planning are illustrated and analyzed in details, taking into account the above mentioned problems.

1.1 A General Overview on Trajectory Planning

Basically, the trajectory planning problem consists in finding a relationship between two elements belonging to different domains: time and space. Accordingly, the trajectory is usually expressed as a parametric function of the time,

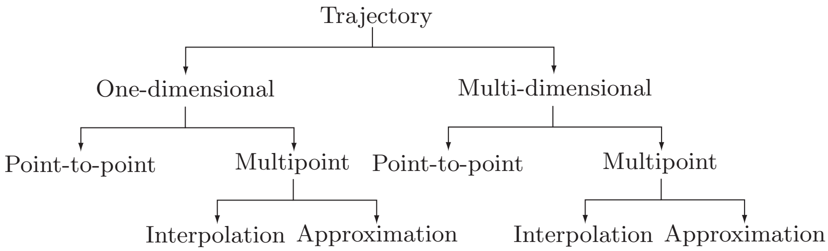


Fig. 1.1. Main trajectory categories.

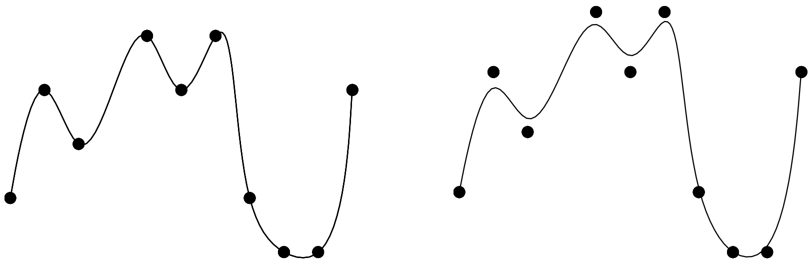
which provides at each instant the corresponding desired position. Obviously, after having defined this function, also other aspects related to its implementation must be considered, such as time discretization (automatic machines are controlled by digital control systems), saturation of the actuation system, vibrations induced on the load, and so on.

As shown in Fig. 1.1, the main distinction among the various categories of trajectories consists in the fact that they can be one- or multi-dimensional. In the first case they define a position for a one degree-of-freedom (dof) system, while in the latter case a multidimensional working space is considered. From a formal point of view, the difference between these two classes of trajectories consists in the fact that they are defined by a scalar ($q = q(t)$) or a vectorial ($\boldsymbol{p} = \boldsymbol{p}(t)$) function. However, the differences are deeper if one considers the approaches and the tools used in the two cases for their computation. Between one- and multi-dimensional trajectories, there is a class of trajectories with intermediate characteristics, namely single-axis motion laws applied to a multi-axis system, composed by several actuators arranged in a so-called master-slave configuration. In this case the motions of the single actuators, although one-dimensional, cannot be designed separately but must be properly coordinated/synchronized[1].

In this book, the design of one-dimensional trajectories is firstly considered. Then, the problem of their coordination/synchronization is addressed and, finally, the planning of motions in the three-dimensional space is taken into account.

The techniques reported in this book, both for one-dimensional and multi-dimensional trajectories, are also classified depending on the fact that the desired motion is defined by assuming initial and final points only (*point-to-point trajectories*) or by considering also a set of intermediate via-points which must properly interpolated/approximated (*multipoint trajectories*). In

[1] In the literature, the two terms “coordination” and “synchronization” are used as synonyms [1]



(a) (b)

Fig. 1.2. Interpolation (a) and approximation (b) of a set of data points.

the former case, a complex motion is obtained by joining several[2] point-to-point trajectories which are individually optimized by considering for each of them the initial and final boundary conditions on velocity, acceleration, etc., and the constraints on their maximum values. Conversely, in the case of multipoint trajectories), by specifying the intermediate points it is possible to define arbitrarily complex motions and the trajectory is found as the solution of a global optimization problem which depends on the conditions imposed on each via-point and on the overall profile. Moreover, it is possible to adopt different criteria for the definition of the motion profile on the basis of the given via-points, which are not necessarily crossed by the trajectory. In particular, two types of fitting can be distinguished:

- *Interpolation*: the curve crosses the given points for some values of the time, Fig. 1.2(a).
- *Approximation*: the curve does not pass exactly through the points, but there is an error that may be assigned by specifying a prescribed tolerance, Fig. 1.2(b).

The latter approach can be useful when, especially in multi-dimensional trajectories, a reduction of the speed/acceleration values along the curve is desirable, at the expense of a lower precision.

1.2 One-dimensional Trajectories

Nowadays the design of high speed automatic machines, whose actuation systems is mainly based on electric drives, generally involves the use of several actuators distributed in the machine and of relatively simple mechanisms, see for example Fig. 1.3, where the sketch of a packaging machine is reported. About twenty motion axes are present in a machine of this type. The so-called *electronic cams* and *electronic gears* are employed for the generation

[2] At least two (three) segments are necessary for a typical periodic motion composed by a rise and a return phase (and, in case, by a dwell phase).

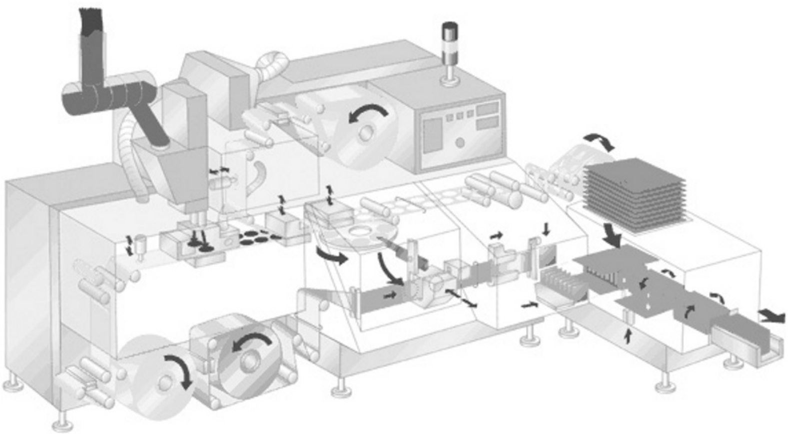


Fig. 1.3. Sketch of an automatic machine for tea packaging (courtesy IMA).

of motion where needed, in place of a single or few actuators and complex kinematic chains. In this manner, more flexible machines can be obtained, able to cope with the different production needs required from the market, [2]. In this context, the problem of trajectory planning has assumed more and more importance [3] since, once the displacement and its duration have been defined, the choice of the modality of motion from the initial to the final point has important implications with respect to the sizing of the actuators, the efforts generated on the structure, and the tracking capabilities of the specified motion (tracking error). Therefore, it is necessary to carefully consider the different types of point-to-point trajectories which could be employed with a specific system (actuation and load). As a matter of fact, for given boundary conditions (initial and final positions, velocities, accelerations, etc.) and duration, the typology of the trajectory has a strong influence on the peak values of the velocity and acceleration in the intermediate points, as well as on the spectral content of the resulting profile. For this reason, in the first part of the book the most common families of trajectories used in the industrial practice are described, providing their analytical expression. Then, these trajectories are analyzed and compared, by taking into account both the frequency aspects and the achievable performances for the overall machine.

1.3 Mechanical Cams and Electronic Cams

Mechanical cams have a very long history. Although some authors trace back their origin even to the Paleolithic age, as referred in [4], certainly Leonardo da Vinci can be considered as one of the first pioneers of the 'modern' de-

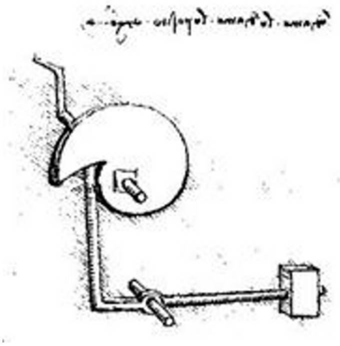
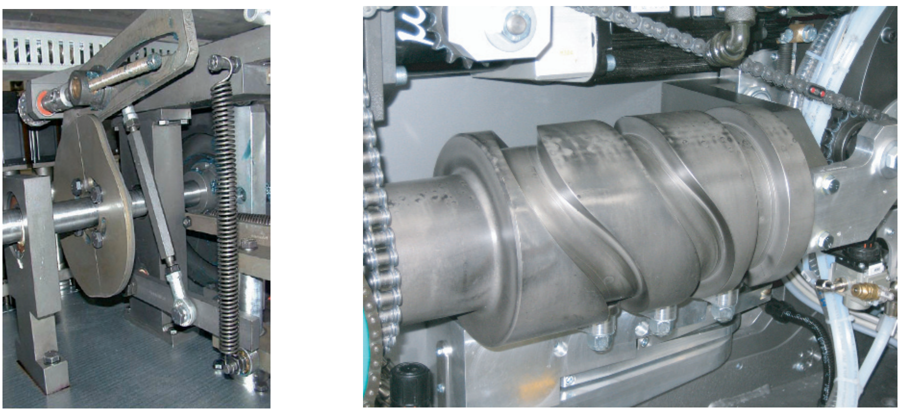


Fig. 1.4. A mechanical cam designed by Leonardo da Vinci.

sign of cam mechanisms, with his design of some machines based on these mechanisms, Fig. 1.4.

In the last decades, mechanical cams have been widely used in automatic machines for transferring, coordinating and changing the type of motion from a *master* device to one or more *slave* systems, Fig. 1.5. With reference to Fig. 1.6 the body C, the *cam*, is supposed to move at a constant rotational velocity, and therefore its angular position θ is a linear function of time. The body F, the *follower*, has an alternative motion $q(\theta)$ defined by the profile of the cam. The design of mechanical cams, especially for planar mechanisms, has been extensively and carefully investigated, and a wide literature is available on this topic, see for example [4, 5, 6, 7, 8, 9].

As already mentioned, *mechanical cams* are nowadays substituted more and more often by the so-called *electronic cams*. The goal is to obtain more



(a) (b)

Fig. 1.5. Mechanical cams, part of an automatic machine (courtesy IMA).

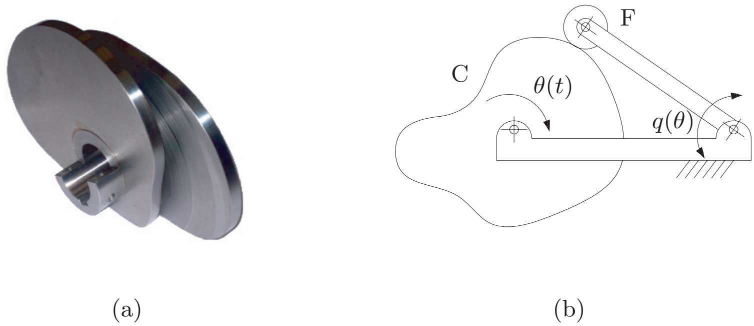


Fig. 1.6. (a) A mechanical cam; (b) working principle of a simple mechanical cam (C) with the follower (F).

flexible machines, with improved performances, easy to be re-programmed, and possibly at lower costs. With electronic cams, the motion $q(t)$ is directly obtained by means of an electric actuator, properly programmed and controlled to generate the desired motion profile. Therefore, the need of designing cams to obtain the desired movement has been progressively replaced by the necessity of planning proper trajectories for electric motors.

In multi-axis machines based on mechanical cams, the synchronization of the different axes of motion is simply achieved by connecting the slaves to a single master (the coordination is performed at the mechanical level), while in case of electronic cams the problem must be considered in the design of the motion profiles for the different actuators (the synchronization is performed at the software level, see Fig. 1.7). A common solution is to obtain the synchronization of the motors by defining a master motion, that can be either virtual (generated by software) or real (the position of an actuator of the machine), and then by using this master position as "time" (i.e. the variable $\theta(t)$ in Fig. 1.6(b)) for the other axes.

1.4 Multi-dimensional Trajectories

Properly speaking, the term *trajectory* denotes a path in the three-dimensional space. For example, the Merriam-Webster dictionary defines the trajectory as "the curve that a body describes in space", [10].

Although in the case of a machine composed by several motors each of them can be independently programmed and controlled (control in the *joint space*), many applications require a coordination among the different axes of motion with the purpose of obtaining a desired multi-dimensional trajectory in the *operational space* of the machine. This is the case of tool machines used to cut, mill, drill, grind, or polish a given workpiece, or of robots which

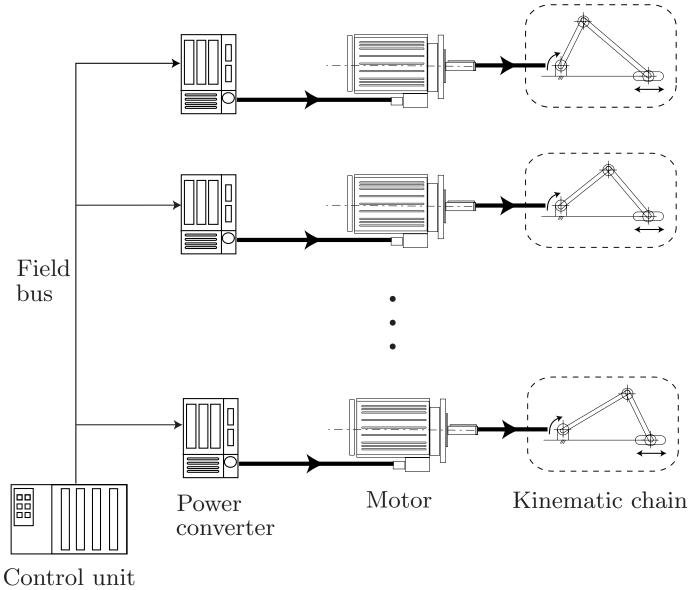


Fig. 1.7. Structure of a multi-axis system based on electronic cams.

must perform tasks in the three-dimensional space, such as spot welding, arc welding, handling, gluing, etc.

In these applications, it is necessary to specify

1. The *geometric path* $\boldsymbol{p} = \boldsymbol{p}(u)$ to be followed, including also the orientation along the curve.
2. The modality by means of which the geometric path must be tracked, that is the *motion law* $u = u(t)$.

The curve followed by the end effector must be designed on the basis of the constraints imposed by the task (e.g. the interpolation of a given set of via-points), while the determination of the motion law descends from other constraints, such as the imposition of the conditions on the maximum velocities, accelerations, and torques that the actuation system is able to provide.

From the composition of the geometric path and of the motion law the complete trajectory is obtained

$$\tilde{\boldsymbol{p}}(t) = \boldsymbol{p}(u(t))$$

as shown in Fig. 1.8. Once the desired movement is specified, the inverse kinematics[3] of the mechanism is employed to obtain the corresponding trajectory in the actuation (joint) space, where the motion is generated and controlled.

[3] The *direct kinematics* of a mechanical device is a (nonlinear) function $\boldsymbol{q} \to \boldsymbol{p} = \boldsymbol{f}(\boldsymbol{q})$ mapping the joint positions $\boldsymbol{q} = [q_1, q_2, \ldots, q_n]^T$ (i.e. the actuators' positions) to the corresponding position/orientation $\boldsymbol{p}$ of a specific point of the

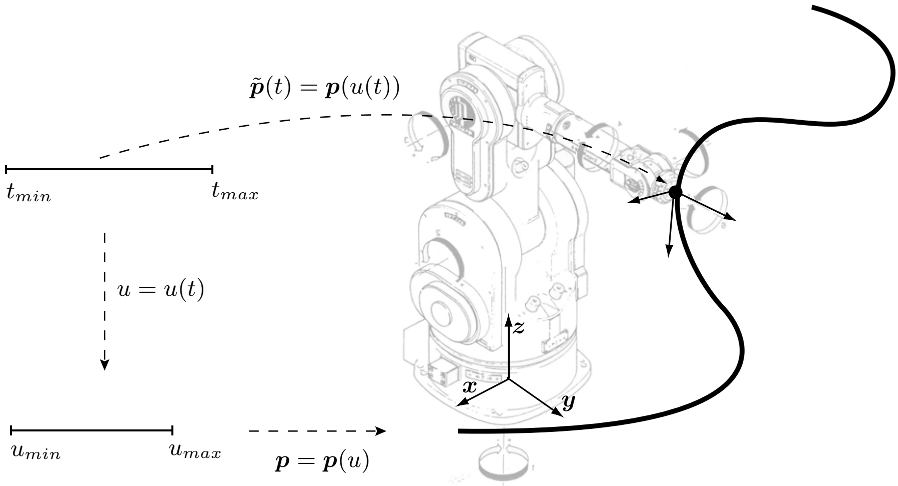


Fig. 1.8. A multi-dimensional trajectory defined in the working space of an industrial robot (courtesy COMAU).

1.5 Contents and Structure of this Book

A relevant, detailed bibliography is available for the problem of moving parts of automatic machines by means of mechanical cams, and in particular for the problem of the determination of the best cam profile in order to obtain the desired motion at the load. As already mentioned, among the numerous and good reference books, one can refer for example to [4, 5, 6, 7, 8, 9]. On the other hand, a similar bibliography concerning the solution of the same problems by means of electric actuators is not currently available. These problems, although in a rather simplified fashion, are partially faced in robotics [11, 12, 13], but limited to the illustration of simple motion profiles and planning of operational space trajectories.

In this book, the main problems related to the planning of trajectories in the joint space are discussed, with particular reference to electric actuators for automatic machines. The case of trajectories defined in the operational space is also considered, discussing the interpolation and approximation techniques for planning motions in the 3D space.

Specifically, the following topics are illustrated:

Part 1 *Basic motion profiles*

- Chapter 2. The basic functions for defining simple trajectories are illustrated: polynomial, trigonometric, exponential and based on the Fourier

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machine in the operational space. The *inverse kinematics* is the inverse function $\boldsymbol{p} \rightarrow \boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{p}) = \boldsymbol{g}(\boldsymbol{p})$.

series expansion. The main properties of these basic functions are presented and discussed.

- Chapter 3. More complex trajectories are presented, defined in order to obtain specific characteristics in terms of motion, velocity, acceleration, such as the trapezoidal or the double S.
- Chapter 4. Trajectories interpolating a set of via-points are presented. In particular, the interpolation by means of polynomial functions, the cubic splines, the B-splines, and techniques for the definition of “optimal” (i.e. minimum time) trajectories are illustrated.

Part 2 *Elaboration and analysis of trajectories*

- Chapter 5. The problems of kinematic and dynamic “scaling” of a trajectory are discussed. Comments on the synchronization of several motion axes are given.
- Chapter 6. The trajectories are analyzed and compared by taking into account the effects produced on the actuation system. For this purpose, the maximum and the root mean square values of the velocities and accelerations, consequence of the different motion profiles, are taken into account.
- Chapter 7. The trajectories are analyzed by considering their frequency properties and their influence on possible vibration phenomena in the mechanical system.

Part 3 *Trajectories in the operational space*

- Chapter 8. The problem of trajectory planning for automatic machines, and in particular for robot manipulators, is considered in the operational space. The basic tools to solve this problem are illustrated, along with some examples.
- Chapter 9. The problem of the analytical composition of the geometric path with the motion law is considered in detail. The goal is to define parametric functions of time so that given constraints on velocities, accelerations, and so on, are satisfied.

Four appendices close the book, with details about some aspects related to the computational issues for one-dimensional trajectories, namely efficient polynomial evaluation, matrix inversion and so on (Appendix A), the B-spline, Nurbs and Bézier definitions and properties (Appendix B), the tools for the definition of the orientation in three-dimensional space (Appendix C), and the spectral analysis of analog and digital signals (Appendix D).

1.6 Notation

In this book, the following notation is adopted.
One-dimensional trajectories:

$q(t)$	: position profile
t	: independent variable, that can be either the "time" (as normally assumed in the book) or the angular position θ of the master in a system based on electronic cams
$q^{(1)}(t)$, $\dot{q}(t)$	: time-derivative of the position (velocity profile)
$q^{(2)}(t)$, $\ddot{q}(t)$	: time-derivative of the velocity (acceleration profile)
$q^{(3)}(t)$, $\dddot{q}(t)$	: time-derivative of the acceleration (jerk profile)
$q^{(4)}(t)$	: time-derivative of the jerk (snap, jounce or ping profile)
$s(t)$	: spline function
$q_k(t)$	: k-th position segment ($k = 0, \ldots, n-1$) in multi-segment trajectories
$\tilde{q}(t')$	: reparameterization of $q(t)$ (scaling in time), $\tilde{q}(t') = q(t)$ with $t = \sigma(t')$
t_0, t_1	: initial and final time instants in point-to-point motions
T	: total duration of a point-to-point trajectory ($T = t_1 - t_0$)
q_0, q_1	: initial and final via-points in point-to-point motions
h	: total displacement ($h = q_1 - q_0$)
q_k	: k-th via-points ($k = 0, \ldots, n$) in multipoint trajectories
t_k	: k-th time instant ($k = 0, \ldots, n$) in multipoint trajectories
T_k	: duration of the k-th segment ($T_k = t_{k+1} - t_k$) in multi-segment trajectories
$\mathtt{v}_0, \mathtt{v}_1$	: initial and final velocity in point-to-point motions
$\mathtt{a}_0, \mathtt{a}_1$	: initial and final acceleration in point-to-point motions
$\mathtt{j}_0, \mathtt{j}_1$	: initial and final jerk in point-to-point motions
$\mathtt{v}_0, \mathtt{v}_n$	: initial and final velocity in multipoint motions
$\mathtt{a}_0, \mathtt{a}_n$	: initial and final acceleration in multipoint motions
$\mathtt{j}_0, \mathtt{j}_n$	: initial and final jerk in multipoints motions
$\mathtt{v}_{max}$	: maximum speed value
$\mathtt{a}_{max}$	: maximum acceleration value
$\mathtt{j}_{max}$	: maximum jerk value

Multi-dimensional trajectories:

$\boldsymbol{p}(u)$	: geometric path
p_x, p_y, p_z	: $x-$, $y-$, $z-$ components of the curve $\boldsymbol{p}$
u	: independent variable for parametric functions describing a geometric path
$u(t)$	: function of time defining the motion law
$\boldsymbol{p}^{(1)}(u)$	: derivative of the position (tangent vector) with respect to u
$\boldsymbol{p}^{(2)}(u)$	: derivative of the tangent vector (curvature vector) with respect to u
$\boldsymbol{p}^{(i)}(u)$	: i-th time-derivative of the geometric path $\boldsymbol{p}(u)$
$\boldsymbol{p}_k(u)$	: k-th curve segment ($k = 0, \ldots, n-1$) in multi-segment trajectories
$\boldsymbol{s}(u)$	: B-spline function
$\boldsymbol{n}(u)$	: Nurbs function
$\boldsymbol{b}(u)$	: Bézier function
$\tilde{\boldsymbol{p}}(t)$	: position trajectory obtained by composing the geometric path with the motion law, $\tilde{\boldsymbol{p}}(t) = \boldsymbol{p}(u) \circ u(t)$
$\tilde{p}_x$, $\tilde{p}_y$, $\tilde{p}_z$	: $x-$, $y-$, $z-$ components of the trajectory $\tilde{\boldsymbol{p}}$ as a function of the time t
$\tilde{\boldsymbol{p}}^{(i)}(t)$	: i-th derivative of the trajectory ($i = 1$ velocity, $i = 2$ acceleration, etc.)
$\tilde{p}_x^{(i)}$, $\tilde{p}_y^{(i)}$, $\tilde{p}_z^{(i)}$	: $x-$, $y-$, $z-$ components of $\tilde{\boldsymbol{p}}^{(i)}$
$\hat{\boldsymbol{p}}(\hat{u})$	: parameterization of the function $\boldsymbol{p}(u)$, $\hat{\boldsymbol{p}}(\hat{u}) = \boldsymbol{p}(u) \circ u(\hat{u})$
$\boldsymbol{q}_k$	: k-th via-points ($k = 0, \ldots, n$) in multipoints trajectories
$\boldsymbol{R}_k$	: rotation matrix defining the orientation at the k-th via-point
$\boldsymbol{t}_k$	: tangent vector at the generic k-th via-point
$\bar{u}_k$	: k-th "time instant" ($k = 0, \ldots, n$) in multipoints trajectories
$\boldsymbol{t}_0$, $\boldsymbol{t}_n$	: tangent vectors at the initial and final points in multipoints motions
$\boldsymbol{n}_0$, $\boldsymbol{n}_n$	: curvature vectors at the initial and final points in multipoints motions
G^h	: class of functions with geometric continuity up to the order h

$\mathbb{N}$	: set of natural numbers
$\mathbb{R}$	: set of real numbers
$\mathbb{C}$	: set of complex numbers
m	: scalar number
$\|m\|$	: absolute value
$\boldsymbol{m}$	: vector
$\|\boldsymbol{m}\|$	: vector norm
$\boldsymbol{m}^T$	: transpose of the vector $\boldsymbol{m}$
$\boldsymbol{M}$	: matrix
$\|\boldsymbol{M}\|$	: matrix norm
$\|\boldsymbol{M}\|_F$	: Frobenius norm of matrix $\boldsymbol{M}$
$\mathrm{tr}(\boldsymbol{M})$	: trace of matrix $\boldsymbol{M}$
$\mathrm{diag}\{m_1, \ldots, m_{n-1}\}$	: diagonal matrix
ω	: angular frequency
T_s	: sampling time
C^h	: class of functions continuous up to the h-th derivative
$\mathrm{floor}(\cdot)$	: integer part function
$\mathrm{sign}(\cdot)$	: sign function
$\mathrm{sat}(\cdot)$	: saturation function
$m!$	: factorial operator

Sometimes, these symbols have different meanings. Where not explicitly indicated, the new meaning is clear from the context.
For the sake of simplicity, the numerical values used in this book are considered dimensionless. In this manner, the mathematical expressions can be applied without changes to several practical cases, with different physical dimensions. In particular, positions may refer to meters, degrees, radians, ...; velocities may then refer to meters/second, degrees/second, ...; and so on.

Finally, it is worth noticing that, without loss of generality, the algorithms for one-dimensional trajectories assume that $q_1 > q_0$, and therefore the desired displacement $h = q_1 - q_0$ is always positive. If this is not the case, the basic motion profiles are unchanged, while the motions based on composition of elementary trajectories (described in Ch. 3) require the adoption of the procedure reported in Sec. 3.4.2.

Part I

Basic Motion Profiles

2

Analytic Expressions of Elementary Trajectories

The basic trajectories are illustrated, classified into three main categories: polynomial, trigonometric, and exponential. Trajectories obtained on the basis of Fourier series expansion are also explained. More complex trajectories, able to satisfy desired constraints on velocity, acceleration and jerk, can be obtained by means of a suitable composition of these elementary functions. The case of a single actuator, or axis of motion, is specifically considered. The discussion is general, and it is therefore valid to define both a trajectory in the joint space and a motion law in the operational space, see Chapter 8 and Chapter 9.

2.1 Polynomial Trajectories

In the most simple case, a motion is defined by assigning the initial and final time instant t_0 and t_1, and conditions on position, velocity and acceleration at t_0 and t_1. From a mathematical point of view, the problem is then to find a function

$$q = q(t), \qquad\qquad t \in [t_0,\ t_1]$$

such that the given conditions are satisfied. This problem can be easily solved by considering a polynomial function

$$q(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$$

where the $n+1$ coefficients a_i are determined so that the initial and final constraints are satisfied. The degree n of the polynomial depends on the number of conditions to be satisfied and on the desired "smoothness" of the resulting motion. Since the number of boundary conditions is usually even, the degree n of the polynomial function is odd, i.e. three, five, seven, and so on.

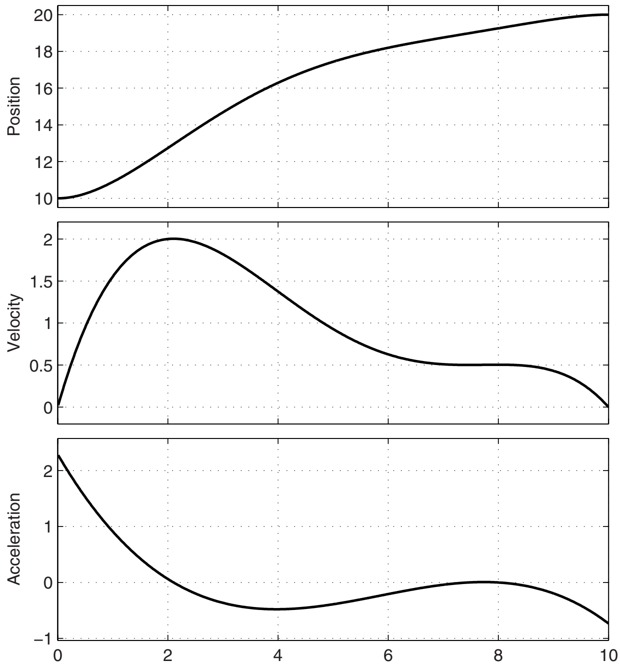

Fig. 2.1. Position, velocity and acceleration profiles of a polynomial trajectory computed by assigning boundary and intermediate conditions (Example 2.1).

In general, besides initial and final conditions on the trajectory, other conditions could be specified concerning its time derivatives (velocity, acceleration, jerk, ...) at generic instants $t_j \in [t_0,\ t_1]$. In other words, one could be interested in determining a polynomial function $q(t)$ whose k-th time-derivative assumes a specific value $q^{(k)}(t_j)$ at a given instant t_j. Mathematically, these conditions can be specified as

$$k!\ a_k + (k+1)!\ a_{k+1}\ t_j + \ldots + \frac{n!}{(n-k)!}\ a_n\ t_j^{n-k} = q^{(k)}(t_j)$$

or, in matrix form, as

$$\boldsymbol{M}\ \boldsymbol{a} = \boldsymbol{b}$$

where $\boldsymbol{M}$ is a known $(n+1) \times (n+1)$ matrix, $\boldsymbol{b}$ collects the given $(n+1)$ conditions to be satisfied, and $\boldsymbol{a} = [a_0,\ a_1,\ \ldots,\ a_n]^T$ is the vector of the unknown parameters to be computed. In principle this equation can be solved simply as

$$\boldsymbol{a} = \boldsymbol{M}^{-1}\ \boldsymbol{b}$$

although, for large values of n, this procedure may lead to numerical problems. These considerations are analyzed in more details in Chapter 4.

Example 2.1 Fig. 2.1 shows the position, velocity and acceleration profiles of a polynomial trajectory computed by assigning the following conditions:

$$q_0 = 10, \qquad q_1 = 20, \qquad t_0 = 0, \qquad t_1 = 10,$$
$$\mathtt{v}_0 = 0, \qquad \mathtt{v}_1 = 0, \qquad \mathtt{v}(t=2) = 2, \qquad \mathtt{a}(t=8) = 0.$$

There are four boundary conditions (position and velocity at t_0 and t_1) and two intermediate conditions (velocity at $t = 2$ and acceleration at $t = 8$). Note that with six conditions it is necessary to adopt a polynomial at least of degree five. In this case, the coefficients a_i result

$$a_0 = 10.0000, \qquad a_1 = 0.0000, \qquad a_2 = 1.1462,$$
$$a_3 = -0.2806, \qquad a_4 = 0.0267, \qquad a_5 = -0.0009.$$

□

2.1.1 Linear trajectory (constant velocity)

The most simple trajectory to determine a motion from an initial point q_0 to a final point q_1, is defined as

$$q(t) = a_0 + a_1(t - t_0).$$

Once the initial and final instants t_0, t_1, and positions q_0 and q_1 are specified, the parameters a_0, a_1 can be computed by solving the system

$$\begin{cases} q(t_0) = q_0 = a_0 \\ q(t_1) = q_1 = a_0 + a_1(t_1 - t_0) \end{cases} \qquad \Longrightarrow \qquad \begin{bmatrix} 1 & 0 \\ 1 & T \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}$$

where $T = t_1 - t_0$ is the time duration. Therefore

$$\begin{cases} a_0 = q_0 \\ a_1 = \dfrac{q_1 - q_0}{t_1 - t_0} = \dfrac{h}{T} \end{cases}$$

where $h = q_1 - q_0$ is the displacement. The velocity is constant over the interval $[t_0, t_1]$ and its value is

$$\dot{q}(t) = \frac{h}{T} \qquad (= a_1).$$

Obviously, the acceleration is null in the interior of the trajectory and has an impulsive behavior at the extremities.

Example 2.2 Fig. 2.2 reports the position, velocity and acceleration of the linear trajectory with the conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$. Note that at $t = t_0,\ t_1$, the velocity is discontinuous and therefore the acceleration is infinite in these points. For this reason the trajectory in this form is not adopted in the industrial practice. □

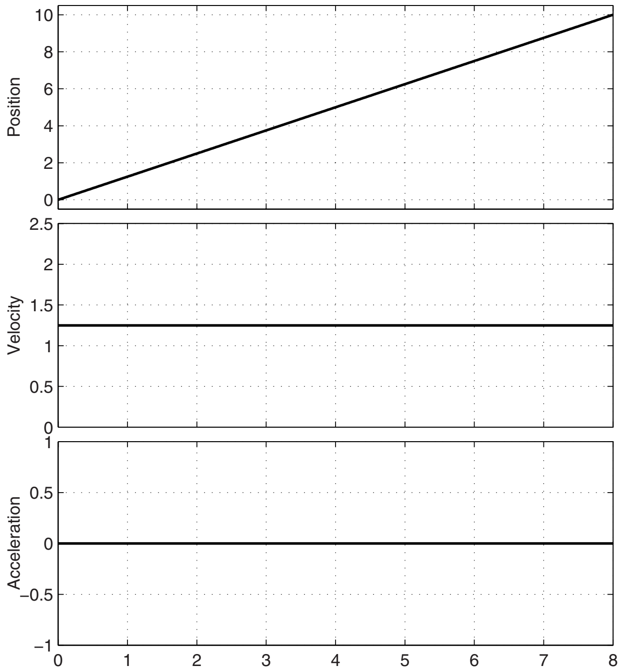

Fig. 2.2. Position, velocity and acceleration of a constant velocity trajectory, with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.

2.1.2 Parabolic trajectory (constant acceleration)

This trajectory, also known as *gravitational trajectory* or *with constant acceleration*, is characterized by an acceleration with a constant absolute value and opposite sign in the acceleration/deceleration periods. Analytically, it is the composition of two second degree polynomials, one from t_0 to t_f (the flex point) and the second from t_f to t_1, see Fig. 2.3.
Let us consider now the case of a trajectory symmetric with respect to its middle point, defined by $t_f = \frac{t_0+t_1}{2}$ and $q(t_f) = q_f = \frac{q_0+q_1}{2}$. Note that in this case $T_a = (t_f - t_0) = T/2,\ \ (q_f - q_0) = h/2$.
In the first phase, the "acceleration" phase, the trajectory is defined by

$$q_a(t) = a_0 + a_1\,(t - t_0) + a_2\,(t - t_0)^2, \qquad t \in [t_0,\ t_f].$$

The parameters a_0, a_1 and a_2 can be computed by imposing the conditions of the trajectory through the points q_0, q_f and the condition on the initial velocity $\mathtt{v}_0$

$$\begin{cases} q_a(t_0) = q_0 = a_0 \\ q_a(t_f) = q_f = a_0 + a_1\,(t_f - t_0) + a_2\,(t_f - t_0)^2 \\ \dot{q}_a(t_0) = \mathtt{v}_0 = a_1. \end{cases}$$

One obtains

$$a_0 = q_0, \qquad a_1 = \mathtt{v}_0, \qquad a_2 = \frac{2}{T^2}(h - \mathtt{v}_0 T).$$

Therefore, for $t \in [t_0,\ t_f]$, the trajectory is defined as

$$\begin{cases} q_a(t) = q_0 + \mathtt{v}_0(t - t_0) + \dfrac{2}{T^2}(h - \mathtt{v}_0 T)(t - t_0)^2 \\ \dot{q}_a(t) = \mathtt{v}_0 + \dfrac{4}{T^2}(h - \mathtt{v}_0 T)(t - t_0) \\ \ddot{q}_a(t) = \dfrac{4}{T^2}(h - \mathtt{v}_0 T) \qquad \text{(constant).} \end{cases}$$

The velocity at the flex point is

$$\mathtt{v}_{max} = \dot{q}_a(t_f) = 2\frac{h}{T} - \mathtt{v}_0.$$

Note that, if $\mathtt{v}_0 = 0$, the resulting maximum velocity has doubled with respect to the case of the constant velocity trajectory. The jerk is always null except at the flex point, when the acceleration changes its sign and it assumes an infinite value.

In the second part, between the flex and the final point, the trajectory is described by

$$q_b(t) = a_3 + a_4\,(t - t_f) + a_5\,(t - t_f)^2 \qquad t \in [t_f,\ t_1].$$

If the final value of the velocity $\mathtt{v}_1$ is assigned, at $t = t_1$, the parameters a_3, a_4, a_5 can be computed by means of the following equations

$$\begin{cases} q_b(t_f) = q_f = a_3 \\ q_b(t_1) = q_1 = a_3 + a_4\,(t_1 - t_f) + a_5\,(t_1 - t_f)^2 \\ \dot{q}_b(t_1) = \mathtt{v}_1 = a_4 + 2a_5\,(t_1 - t_f) \end{cases}$$

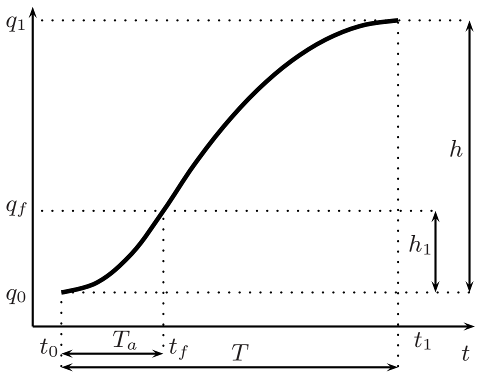

Fig. 2.3. Trajectory with constant acceleration.

from which

$$a_3 = q_f = \frac{q_0 + q_1}{2}, \qquad a_4 = 2\frac{h}{T} - \mathtt{v}_1, \qquad a_5 = \frac{2}{T^2}(\mathtt{v}_1 T - h).$$

The expression of the trajectory for $t \in [t_f,\ t_1]$ is

$$\begin{cases} q_b(t) = q_f + (2\frac{h}{T} - \mathtt{v}_1)(t - t_f) + \frac{2}{T^2}(\mathtt{v}_1 T - h)(t - t_f)^2 \\ \dot{q}_b(t) = 2\frac{h}{T} - \mathtt{v}_1 + \frac{4}{T^2}(\mathtt{v}_1 T - h)(t - t_f) \\ \ddot{q}_b(t) = \frac{4}{T^2}(\mathtt{v}_1 T - h). \end{cases}$$

Note that, if $\mathtt{v}_0 \neq \mathtt{v}_1$, the velocity profile of this trajectory is discontinuous at $t = t_f$.

Example 2.3 Fig. 2.4 reports the position, velocity and acceleration for this trajectory. The conditions $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10,\ \mathtt{v}_0 = \mathtt{v}_1 = 0$ have been assigned. □

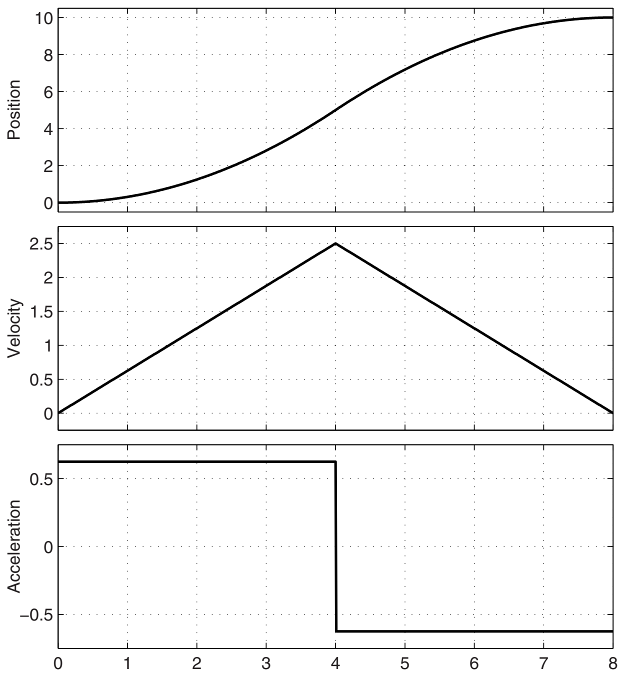

Fig. 2.4. Position, velocity and acceleration of a trajectory with constant acceleration, with $t_0 = 0,\ t_1 = 8,\ q_0 = 0,\ q_1 = 10$.