Sheet Metal Forming Processes
The concept of virtual manufacturing has been developed in order to increase the industrial performances, being one of the most efficient ways of reducing the manufacturing times and improving the quality of the products. Numerical simulation of metal forming processes, as a component of the virtual manufacturing process, has a very important contribution to the reduction of the lead time. The finite element method is currently the most widely used numerical procedure for simulating sheet metal forming processes. The accuracy of the simulation programs used in industry is influenced by the constitutive models and the forming limit curves models incorporated in their structure. From the above discussion, we can distinguish a very strong connection between virtual manufacturing as a general concept, finite element method as a numerical analysis instrument and constitutive laws, as well as forming limit curves as a specificity of the sheet metal forming processes. Consequently, the material modeling is strategic when models of reality have to be built.

The book gives a synthetic presentation of the research performed in the field of sheet metal forming simulation during more than 20 years by the members of three international teams: the Research Centre on Sheet Metal Forming—CERTETA (Technical University of Cluj-Napoca, Romania); AutoForm Company from Zürich, Switzerland and VOLVO automotive company from Sweden.

The first chapter presents an overview of different Finite Element (FE) formulations used for sheet metal forming simulation, now and in the past. The objective of this chapter is to give a general understanding of the advantages and disadvantages of the various methods in use. The first section is dedicated to some of the necessary ingredients of the fundamentals of continuum mechanics for large deformation problems. These are needed for a better understanding of the forthcoming FE-formulations.

A more extended chapter is devoted to the presentation of the phenomenological yield criteria. Due to the fact that this chapter is only a synthetic overview of the yield criteria, the reader interested in some particular formulation should also read the original paper listed in the reference section. We have tried to use the symbols adopted by the authors, especially in the mathematical relationships defining
the yield stresses and the coefficients of plastic anisotropy. This decision has been made in order to facilitate the reading of the original papers. Of course, under these circumstances, the coherency of the notations cannot be preserved. As one may see in the list of symbols, several identifiers have different meanings. The reader should take this aspect into account. This chapter gives a more detailed presentation of the yield criteria implemented in the commercial programs used for the finite element simulation (emphasizing the formulations proposed by the CERTETA team—BBC models—implemented in the AutoForm commercial code) or the yield criteria having a major impact on the research progress. To improve the springback prediction, a novel approach to model the Bauschinger effect has been developed and implemented in the commercial code AutoForm. Consequently, an extended section of this chapter has been dedicated to the modeling of the Bauschinger effect, especially in the AutoForm model.

The sheet metal formability is discussed in a separate chapter. After presenting the methods used for the formability assessment, the discussion focuses on the Forming Limit Curves (FLC). Experimental methods used for limit strains determination and the main factors influencing the FLC are presented in detail. A section is dedicated to the use of Forming Limit Diagrams in industrial practice. Theoretical predictions of the FLCs are presented in an extended section. In this context, the authors emphasize their contributions to the mathematical modeling of FLCs. A special section has been devoted to present an original implicit formulation of the Hutchinson–Neale model, developed by the authors of this chapter, used for calculating the FLCs of thin sheet metals. The commercial programs (emphasizing the FORM CERT program) and the semi-empirical models for FLC prediction are presented in the last sections of the chapter.

The aspects related to the numerical simulation of the sheet metal forming processes are discussed in the last chapter of the book. The role of simulation in process planning, part feasibility and quality, process validation and robustness are presented based on the AutoForm solutions. The performances of the material models are proved by the numerical simulation of various sheet metal forming processes: bulge and stretch forming, deep-drawing and forming of the complex parts. A section has been devoted to the robust design of sheet metal forming processes. Springback is the major quality concern in the stamping field. Consequently, two sections of this chapter are focused on the springback analysis and Computer Aided Springback Compensation (CASP).

The authors wish to express their gratitude to Dr. Waldemar Kubli, founder and CEO, Dr. Mike Selig, CTO and Markus Thomma, CMD of AutoForm Company, for their support of the book project. They have created favorable conditions for the AutoForm team in order to make this book possible. The authors also wish to thank Dr. Alan Leacock from University of Ulster (UK) for his help in proofing the English of the manuscript. Prof. Banabic wishes to express his thanks to his former PhD students Dr. L. Paraianu, Dr. P. Jurco, Dr. M. Vos, Dr. G. Cosovici and his current PhD students G. Dragos and I. Bichis for their help in preparing and editing this book.
The book will be of interest to both the research and industrial communities. It is useful for the students, doctoral fellows, researchers and engineers who are mainly interested in the material modeling and numerical simulation of sheet metal forming processes.

Cluj-Napoca, Romania

Dorel Banabic

December 2009
## Contents

1. **FE-Models of the Sheet Metal Forming Processes** .......................... 1  
   1.1 Introduction .................................................. 2  
   1.2 Fundamentals of Continuum Mechanics .......................... 3  
      1.2.1 Introduction ........................................... 3  
      1.2.2 Strain Measures ........................................ 4  
      1.2.3 Stress Measures ........................................ 8  
   1.3 Material Models .............................................. 9  
   1.4 FE-Equations for Small Deformations ........................... 11  
   1.5 FE-Equations for Finite Deformations ........................... 13  
   1.6 The ‘Flow Approach’—Eulerian FE-Formulations  
      for Rigid-Plastic Sheet Metal Analysis ....................... 16  
   1.7 The Dynamic, Explicit Method .................................. 18  
   1.8 A Historical Review of Sheet Forming Simulation ............... 21  
References .......................................................... 24  

2. **Plastic Behaviour of Sheet Metal** ...................................... 27  
   2.1 Anisotropy of Sheet Metals .................................... 30  
      2.1.1 Uniaxial Anisotropy Coefficients ...................... 30  
      2.1.2 Biaxial Anisotropy Coefficient ....................... 36  
   2.2 Yield Criteria for Isotropic Materials ........................ 39  
      2.2.1 Tresca Yield Criterion ................................ 41  
      2.2.2 Huber–Mises–Hencky Yield Criterion .................. 42  
      2.2.3 Drucker Yield Criterion ................................ 43  
      2.2.4 Hershey Yield Criterion ................................ 44  
   2.3 Classical Yield Criteria for Anisotropic Materials ............ 45  
      2.3.1 Hill’s Family Yield Criteria ......................... 45  
      2.3.2 Yield Function Based on Crystal Plasticity  
         (Hershey’s Family) ....................................... 61  
      2.3.3 Yield Criteria Expressed in Polar Coordinates .......... 74  
      2.3.4 Other Yield Criteria .................................... 75  
   2.4 Advanced Anisotropic Yield Criteria ............................ 76  
      2.4.1 Barlat Yield Criteria .................................. 77  
      2.4.2 Banabic–Balan–Comsa (BBC) Yield Criteria .......... 81
2.4.3 Cazacu–Barlat Yield Criteria ................................. 84
2.4.4 Vегter Yield Criterion ........................................ 87
2.4.5 Polynomial Yield Criteria ...................................... 88

2.5 BBC 2005 Yield Criterion ........................................ 91
2.5.1 Equation of the Yield Surface ............................... 91
2.5.2 Flow Rule Associated to the Yield Surface ................. 92
2.5.3 BBC 2005 Equivalent Stress .................................... 92
2.5.4 Identification Procedure ...................................... 94
2.5.5 Particular Formulations of the BBC 2005 Yield Criterion 105

2.6 BBC 2008 Yield Criterion ........................................ 106
2.6.1 Equation of the Yield Surface ............................... 107
2.6.2 BBC 2008 Equivalent Stress .................................... 108
2.6.3 Identification Procedure ...................................... 109

2.7 Recommendations on the Choice of the Yield Criterion ... 113
2.7.1 Comparison of the Yield Criteria ............................ 113
2.7.2 Evaluating the Performances of the Yield Criteria ....... 116
2.7.3 Mechanical Parameters Used by the Identification Procedure of the Yield Criteria ........................................ 118
2.7.4 Implementation of the Yield Criteria in Numerical Simulation Programmes ..................................... 118
2.7.5 Overview of the Anisotropic Yield Criteria Developing .... 120
2.7.6 Perspectives ...................................................... 120

2.8 Modeling of the Bauschinger Effect ............................. 121
2.8.1 Reversal Loading in Sheet Metal Forming Processes .... 121
2.8.2 Experimental Observations .................................... 122
2.8.3 Physical Nature of the Bauschinger Effect ................... 124
2.8.4 Phenomenological Modelling ................................ 125

References ............................................................. 135

3 Formability of Sheet Metals ........................................ 141
3.1 Introduction ........................................................ 142
3.2 Evaluation of the Sheet Metal Formability ....................... 147
3.2.1 Methods Based on Simulating Tests ........................ 147
3.2.2 Limit Dome Height Method ................................... 151
3.3 Forming Limit Diagram ............................................ 152
3.3.1 Definition: History ............................................ 152
3.3.2 Experimental Determination of the FLD .................... 156
3.3.3 Methods of Determining the Limit Strains ................. 162
3.3.4 Factors Influencing the FLC .................................. 165
3.3.5 Use of Forming Limit Diagrams in Industrial Practice .... 175
3.4 Theoretical Predictions of the Forming Limit Curves ........ 179
3.4.1 Swift’s Model ................................................... 180
3.4.2 Hill’s Model .................................................... 182
3.4.3 Marciniak–Kuczynski (M–K) and Hutchinson–Neale (H–N) Models ........................................ 182
4 Numerical Simulation of the Sheet Metal Forming Processes

4.1 AutoForm Solutions
  4.1.1 The Role of Simulation in Process Planning
  4.1.2 Material Data in Digital Process Planning
  4.1.3 Feasibility (Part Feasibility)
  4.1.4 Manufacturability (Process Validation)
  4.1.5 Capability (Robustness)
  4.1.6 Simulation Result 'Quality'
  4.1.7 Comprehensive Digital Process Planning

4.2 Simulation of the Elementary Forming Processes
  4.2.1 Simulation of the Bulge Forming Process
  4.2.2 Simulation of Stretch Forming of Spherical Cup
  4.2.3 Simulation of Cross Die

4.3 Simulation of the Industrial Parts Forming Processes
  4.3.1 Simulation of an Outer Trunklid
  4.3.2 Simulation of a Sill Reinforcement for Volvo C30

4.4 Robust Design of Sheet Metal Forming Processes
  4.4.1 Variability of the Material Parameters
  4.4.2 AutoForm-Sigma
  4.4.3 Robust Design: Case Studies
  4.4.4 Conclusion

4.5 The Springback Analysis
  4.5.1 Introduction
  4.5.2 Example Description
  4.5.3 The Influences on the Accuracy of Springback Simulation
  4.5.4 The Optimized Numerical Parameters of Springback Simulation: Final Validation Settings
  4.5.5 The Simulation of Numisheet 2005 Benchmark #1: Decklid Inner Panel
  4.5.6 Conclusion

4.6 Computer Aided Springback Compensation
  4.6.1 Introduction
  4.6.2 The Basic Methodologies of Computer-Aided Springback Compensation
  4.6.3 The Influences of the Quality of Computer Aided Springback Compensation
4.6.4 The Recommended Work Flow of Computer-Aided Springback Compensation ................. 285
4.6.5 The Springback Compensation of Numisheet 2005 Benchmark #1 .......................... 287
4.6.6 Conclusion .................................. 293
References ........................................ 294
Index .............................................. 297
List of the Authors

Prof. Dorel Banabic
Professor at the Technical University of Cluj-Napoca
Director of the Research Centre in Sheet Metal Forming – CERTETA
27 Memorandumului, 400114 Cluj Napoca, Romania
e-mail: banabic@tcm.utcluj.ro
URL: www.certeta.utcluj.ro

Dr. Bart Carleer
AutoForm Engineering Deutschland GmbH
Emil-Figge-Str. 76-80, 44227 Dortmund, Germany
e-mail: bart.carleer@autoform.de
URL: www.autoform.com

Dr. Dan-Sorin Comsa
Reader at the Technical University of Cluj Napoca
15 C. Daicoviciu, 400020 Cluj Napoca, Romania
e-mail: dscomsa@tcm.utcluj.ro
URL: www.certeta.utcluj.ro

Eric Kam
AutoForm Engineering USA, Inc.
560 Kirts Blvd, Suite 113, Troy, Michigan 48084-4141, USA
e-mail: eric.kam@autoform.com
URL: www.autoform.com

Dr. Andriy Krasovskyy
Formerly AutoForm Development GmbH
Technoparkstrasse 1, CH-8005 Zurich, Switzerland
URL: www.autoform.com
**Prof. Kjell Mattiasson**  
Chalmers University of Technology  
SE-412 96 Goteborg, Sweden  
e-mail: mailto:kjellm@chalmers.se  
URL: www.chalmers.se

Volvo Cars Safety Centre  
Dept. 91432/PV 22, SE-405 31 Goteborg, Sweden  
e-mail: kmattias@volvocars.com  
URL: www.volvocars.com

**Dr. Matthias Sester**  
AutoForm Development GmbH  
Technoparkstrasse 1, CH-8005 Zurich, Switzerland  
e-mail: matthias.sester@autoform.ch  
URL: www.autoform.com

**Mats Sigvant PhD**  
Technical Expert, Sheet Metal Forming Simulation  
Stamping CAE, Volvo Car Corporation  
Dept. 81153/26HK3, Olofstrom, Sweden  
e-mail: msigvan1@volvocars.com  
URL: www.volvocars.com

**Xiaojing Zhang PhD**  
AutoForm Engineering Deutschland GmbH  
Emil-Figge-Str. 76-80, 44227 Dortmund, Germany  
e-mail: xiaojing.zhang@autoform.de  
URL: www.autoform.com
Contributions of the Authors

Dorel Banabic  Co-ordination of the book
              Sections 2.1–2.4, 2.7; Chapter 3
Bart Carleer  Section 4.4
Dan-Sorin Comsa Sections 2.5, 2.6, 3.4.4
Eric Kam      Section 4.1
Andriy Krasovskyy Section 2.8
Kjell Mattiasson Chapter 1
Matthias Sester Sections 4.2.1, 4.2.3
Mats Sigvant  Sections 4.2.2, 4.3
Xiaojing Zhang  Sections 4.5, 4.6
Chapter 1
FE-Models of the Sheet Metal Forming Processes

List of Special Symbols

[A] matrix with constants describing the anisotropy of the material
[B] strain matrix
[B_L] linear strain matrix
[B_NL] nonlinear strain matrix
\( c \) speed of sound
\( d \) rate of deformation tensor
\( \dot{d} \) plastic rate of deformation
D elastic constitutive tensor
[D] matrix of the elastic constitutive tensor
[D_T] matrix form of the constitutive tensor
\( ds \) lengths of the vector \( dx \)
\( dS \) lengths of the vector \( dX \)
\( dv \) volume element in the current configuration
\( dV \) volume element in the reference configuration
\( E \) Young’s modulus
\( E_1, E_2 \) principal values of Lagrangian strains
\( \dot{E} \) Green’s strain tensor
\( \dot{E} \) Green strain rate tensor
\( f \) yield function
\( \{f\} \) load
\( \{f\}^{\text{ext}} \) external force vector
\( \{f\}^{\text{int}} \) internal force vector
\( F \) deformation gradient tensor
\( [K] \) linear stiffness matrix
\( [K_G] \) geometric stiffness matrix
\( [K_M] \) material stiffness matrix
\( [K_S] \) initial stress stiffness matrix
\( [K_T] \) tangent matrix
\( L \) length
\( L \) velocity gradient tensor
1.1 Introduction

In the current section an overview of different Finite Element (FE) formulations used for sheet metal forming simulation, now and in the past, will be given. The theories of FE-simulation of large deformation problems will be briefly touched upon herein, but for thorough presentations the reader is referred to the many existing text books on the subject, e.g. Belytschko et al. [1], Zienkiewicz and Taylor [2] and Crisfield [3]. The object is rather to give a general understanding of the advantages and disadvantages of the various methods in use. Review articles on sheet forming simulation and comparative studies of different FE-procedures are found in e.g. Honecker and Mattiasson [4], Oñate and Agelet de Saracibar [5], Oñate et al. [6], Mattiasson [7], Kawka et al. [8], Wenner [9], Wang et al. [10], Mattiasson [11],
Makinouchi [12] and Wenner [13]. State-of-the-art articles on the utilization of sheet forming simulation today, and outlooks against the future are given in Banabic and Tekkaya [14] and Roll and Weigand [15].

Finite Element Methods (FEM) have been developed and used for sheet forming simulations since the 1970s, when the continuum mechanics foundations for problems involving large displacements and large strains became well established. FE-procedures for sheet forming analysis can be classified into two main groups depending on if they are based on an elastic-plastic or a rigid-plastic material model. Large strain formulations may be based on either an Eulerian or a Lagrangian description of motion, leading to two basically different FE-procedures with nodal velocities and nodal incremental displacements, respectively, as primary unknowns.

1.2 Fundamentals of Continuum Mechanics

1.2.1 Introduction

In [16] and [17] an extended presentation of the fundamentals of continuum mechanics for large deformation problems and the theory of phenomenological plasticity are given. Here, some of the necessary ingredients for the forthcoming FE-formulations will be briefly presented.

When discussing the kinematics of continua it is important to make clear the meanings of the terms point and particle. The word point will be used to designate a certain location in space, while the word particle will denote a small part of a material continuum. There are basically two ways of describing the motion of such a continuum.

In the material or Lagrangian description the independent variables are the particle \( P \) and the time \( t \). The motion may then be expressed by an equation of the form

\[
x = x(P, t)
\]

where \( x \) is the position vector of the particle \( P \) at time \( t \). It is common to write this equation in terms of the position \( X \) of the particle \( P \) in the reference configuration, i.e.

\[
x = x(X, t)
\]

In the spatial or Eulerian description attention is fixed to a given region in space (a point) instead of a certain particle of a continuum. Independent variables are the present time \( t \) and the present position \( x \) of the particle that occupied the point \( X \) at time \( t=0 \). The motion is thus expressed by

\[
X = X(x, t)
\]
If Eqs. (1.2) and (1.3) represent one-to-one mappings with continuous partial derivatives, the two mappings are unique inverses of each other.

The Eulerian description is the description best suited for fluid mechanics problems, as it focuses attention on a certain region in space, which enables us to observe the flow in a point in a channel or a wind tunnel. The Lagrangian formulation has traditionally been used in solid and structural mechanics problems, in which there normally exist a natural reference configuration with known stresses and deformations. In a Finite Element context, the primary variables in the Lagrangian formulation are *displacements*, while in the Eulerian formulation they are *velocities*.

In some metal forming processes, especially bulk forming processes, the metal flow resembles that of a fluid, and the problems have consequently been solved by means of the Eulerian formulation. But even a problem like sheet metal forming has been analyzed by means of this approach as will be discussed in one of the coming sections.

In the following we will focus on the Lagrangian approach. To generalize a formulation for small deformations to large deformations adds a great deal of complexity. There are a number of different ways to formulate and solve the problem. Here we will only present a couple of possible alternatives. As mentioned in the introduction of this chapter, the reader is referred to one or more of the previously mentioned textbooks to get a more complete coverage of the subject.

The main challenge when formulating the basic equations in the Lagrangian formulation is to do it in such a way that they become independent of rigid-body rotations. This means that a rigid body rotation should not give rise to additional strains and stresses. We say that the formulation must be *objective* or *frame invariant*, which also implies that the strain and stress measures being a part of the formulation also must satisfy the objectivity requirement. There are a number of such strain and stress measures appearing in the literature, but we will here limit our discussion to only a few of them.

### 1.2.2 Strain Measures

We will first introduce a tensor called the *deformation gradient* tensor $\mathbf{F}$, defined by

$$
\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}; \quad F_{ij} = \frac{\partial x_i}{\partial X_j}.
$$

(1.4)

This tensor is *not* objective, but plays an important role in the derivation of the above mentioned strain and stress tensors. The tensor relates a line segment in current and reference configurations, respectively, as

$$
d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X}; \quad dx_i = F_{ij} dX_j
$$

(1.5)

To emphasize the distinction between the two sets of coordinates, we will use capital letters for indices on tensor components referred to the reference configuration and
lower case letters on those referred to the current configuration. The deformation gradient tensor is said to be a two-point tensor, since its components are referred to both the reference and the current configuration.

Let $dS$ and $ds$ denote the lengths of the vectors $dX$ and $dx$, respectively. The squares of these lengths may be written as

$$dS^2 = |dX|^2 = dX \cdot dX = dx_I dx_I \quad \text{(1.6)}$$

Using Eq. (1.5), we can rewrite $ds^2$ as

$$ds^2 = (F \cdot dX) \cdot (F \cdot dX) = dX \cdot (F^T \cdot F) \cdot dX \quad \text{(1.7)}$$

The difference $ds^2 - dS^2$ for two neighboring particles of a continuum is used as a measure of deformation. This difference can be written

$$ds^2 - dS^2 = dX \cdot (F^T \cdot F) \cdot dX - dX \cdot dX = \frac{1}{2} dX \cdot E \cdot dX \quad \text{(1.8)}$$

From Eq. (1.8) the definition of Green’s strain tensor is found to be

$$E = \frac{1}{2} (F^T \cdot F - I) ; \quad E_{IJ} = \frac{1}{2} (F_{kl} F_{kj} - \delta_{IJ}) \quad \text{(1.9)}$$

In a rigid body rotation the difference $ds^2 - dS^2$ is constant. This can only be accomplished if also the tensor $E$ is constant, which proves that the tensor is invariant under a rigid body rotation and, thus, objective.

An especially useful form of the strain tensor is obtained when it is expressed in displacement gradients. Define the displacement $u$ from the following relation:

$$x = X + u ; \quad x_I = X_I + u_I \quad \text{(1.10)}$$

Introducing this expression into the equation for $F$ in Eq. (1.4), we get

$$F = \frac{\partial x}{\partial X} = I + \frac{\partial u}{\partial X} ; \quad F_{ij} = \frac{\partial x_i}{\partial X_j} = \delta_{ij} + \frac{\partial u_i}{\partial X_j} \quad \text{(1.11)}$$

Green’s strain tensor can then be rewritten as

$$E = \frac{1}{2} \left[ \frac{\partial u}{\partial X} + \left( \frac{\partial u}{\partial X} \right)^T + \frac{\partial u}{\partial X} \cdot \left( \frac{\partial u}{\partial X} \right)^T \right] \quad \text{(1.12)}$$

$$E_{IJ} = \frac{1}{2} \left[ \frac{\partial u_I}{\partial X_J} + \frac{\partial u_J}{\partial X_I} + \frac{\partial u_K}{\partial X_J} \cdot \frac{\partial u_K}{\partial X_I} \right]$$
Now, consider two neighboring particles with instantaneous positions \( \mathbf{x} \) and \( \mathbf{x} + d\mathbf{x} \), respectively. The difference in velocity between these two points is

\[
d\mathbf{v} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}} d\mathbf{x}; \quad d\mathbf{v}_i = \frac{\partial v_i}{\partial x_j} dx_j
\] (1.13)

The gradient in the above equation is called the *velocity gradient* tensor and is, thus, defined by

\[
\mathbf{L} = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}; \quad L_{ij} = \frac{\partial v_i}{\partial x_j}
\] (1.14)

The velocity gradient can be decomposed into a symmetric and a skew-symmetric part according to

\[
\mathbf{L} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T) + \frac{1}{2} (\mathbf{L} - \mathbf{L}^T); \quad L_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) + \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)
\] (1.15)

The symmetric part \( \mathbf{d} \)

\[
\mathbf{d} = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T); \quad d_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)
\] (1.16)

is called *rate of deformation* tensor, while the skew-symmetric part \( \mathbf{W} \)

\[
\mathbf{W} = \frac{1}{2} (\mathbf{L} - \mathbf{L}^T); \quad W_{ij} = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)
\] (1.17)

is called the *spin* tensor.

The physical interpretation of these tensors is now quite obvious. Rewriting Eq. (1.13), we get

\[
d\mathbf{v} = (\mathbf{d} + \mathbf{W}) \cdot d\mathbf{x}; \quad d\mathbf{v}_i = (d_{ij} + W_{ij}) dx_j
\] (1.18)

If all the components \( d_{ij} \) are equal to zero at a certain point \( P \), the instantaneous motion in the neighbourhood of \( P \) is a rigid body motion. This is a consequence of the skew-symmetry of the tensor \( \mathbf{W} \). We then realize that, if all components \( W_{ij} \) are zero, we have a pure deformation without any rigid body rotation. The rate of deformation tensor is, thus, an objective tensor. It should also be observed that for a *small deformation problem* the rate of deformation tensor is simply the time derivative of the strain tensor, i.e. \( d_{ij} = \dot{\varepsilon}_{ij} \).

We shall now find the relationship between the rate of deformation tensor and the time derivative of Green’s strain tensor. From the definition of Green’s strain tensor in Eq. (1.8) we get
The left hand side of the above equation can also be written

\[
\frac{d}{dt} (s^2) = 2 \mathbf{d} \cdot \mathbf{dx} = 2 \mathbf{d} \cdot \mathbf{dv} = 2 \mathbf{dx} \cdot \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \cdot \mathbf{dx} = \mathbf{dx} \cdot (\mathbf{d} + \mathbf{W}) \cdot \mathbf{dx} = dx_k (d_{km} + W_{km}) dx_m
\]  

(1.20)

Due to the skew-symmetry of the spin tensor \( \mathbf{W} \), i.e. \( W_{ij} = -W_{ji} \), it is easy to show that

\[
\mathbf{dx} \cdot \mathbf{W} \cdot \mathbf{dx} = dx_k W_{km} dx_m = 0
\]  

(1.21)

Eq. (1.20) can then be rewritten as

\[
\frac{d}{dt} (s^2) = \mathbf{dx} \cdot \mathbf{d} \cdot \mathbf{dx} = dx_k d_{km} dx_m
\]  

(1.22)

Introducing the definition of the deformation gradient tensor \( \mathbf{F} \) according to Eq. (1.5), we can rewrite the above equation as

\[
\frac{d}{dt} (s^2) = (\mathbf{dX} \cdot \mathbf{F}^T) \cdot \mathbf{d} \cdot (\mathbf{F} \cdot d\mathbf{X})
\]  

(1.23)

Comparing Eqs. (1.19) and (1.23), we find the following relation between Green’s strain rate tensor and the rate of deformation tensor:

\[
\dot{\mathbf{E}} = \mathbf{F}^T \cdot \mathbf{d} \cdot \mathbf{F}; \quad \dot{E}_{IJ} = d_{km} F_{kI} F_{mJ}
\]  

(1.24)

From Eq. (1.24) we note that also the components of the tensor \( \dot{\mathbf{E}} \) vanish when the neighborhood of the particle considered moves like a rigid body. That is, the Green strain rate tensor is objective.

It is much more convenient to express the Green strain rate in velocity gradients referred to the reference coordinates. Taking the time derivative of Eq. (1.12) we get

\[
\dot{\mathbf{E}} = \frac{1}{2} \left[ \frac{\partial \mathbf{u}}{\partial \mathbf{X}} + \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \cdot \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right)^T + \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \cdot \left( \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \right)^T \right]
\]

(1.25)

Finally, it should be mentioned (without proof) that the determinant of the deformation gradient tensor expresses the relation between a volume element in the current and the reference configuration, respectively, so that

\[
dv = \det (\mathbf{F}) dV = J dV
\]  

(1.26)
where \(dv\) is the volume element in the current configuration and, \(dV\) is the same element in the reference configuration.

### 1.2.3 Stress Measures

There are a number of different stress measures appearing in the continuum mechanics literature. Here we will only consider three of them: The Cauchy stress \(\sigma\), the Kirchhoff stress \(\tau\) and the Second Piola-Kirchhoff (PK2) stress \(S\), together with rates of these tensors.

The Cauchy stress tensor \(\sigma\) is also known as the true stress tensor and measures the force per unit area in the current configuration. The traction or stress vector \(t\) at a boundary point is related to the Cauchy stress tensor through the relation

\[
t = \hat{n} \cdot \sigma; \quad t_i = \hat{n}_j \sigma_{ji}
\] (1.27)

where \(\hat{n}\) is an outward unit vector at the boundary point.

In a Lagrangian formulation equilibrium equations are often written in terms of variables referred to the reference configuration. To this purpose a variety of so-called pseudo stress measures have been defined, of which two will be discussed here. The first one is the Kirchhoff stress tensor \(\tau\), which frequently has been used in constitutive equations in finite strain plasticity. It is related to the Cauchy stress tensor by

\[
\tau = \frac{\rho_0}{\rho} \sigma = J \sigma; \quad \tau_{ij} = \frac{\rho_0}{\rho} \sigma_{ij} = J \sigma_{ij}
\] (1.28)

where \(\rho_0\) and \(\rho\) are the mass densities in the reference and current configurations, respectively, and \(J\) is the determinant of the deformation gradient tensor. It should be noted that for an incompressible material the Cauchy and Kirchhoff stress tensors are equal.

The Second Piola-Kirchhoff stress tensor \(S\) is defined to be energy conjugate to the Green strain rate tensor. The work rate can then be written

\[
\dot{W} = \int_v \sigma : d \, dv = \int_V S : \dot{E} \, dV = \int_v \frac{1}{J} S : \dot{E} \, dv
\] (1.29)

Introducing the relation between the rate of deformation tensor and Green’s strain rate according to Eq. (1.24), we get

\[
J \sigma : d = S : \dot{E} = S : (F^T \cdot d \cdot F)
\] (1.30)

After some tensor manipulations the following relation is obtained

\[
S = J F^{-1} \cdot \sigma \cdot F^{-T}; \quad S_{ij} = J F_{ik}^{-1} \sigma_{km} F_{jm}^{-1}
\] (1.31)
1.3 Material Models

It can be shown that the Cauchy stress tensor $\sigma$ is symmetric and objective. As a consequence of Eq. (1.31) it can be understood that also the PK2 tensor $S$ is symmetric and objective.

The constitutive equations for certain material types like hypoelastic and elastic-plastic ones are formulated in rate form. The rate of deformation tensor $d$ is a suitable deformation measure in such material laws. The problem is to choose a proper stress rate measure. The material time derivative of the Cauchy stress tensor, $\dot{\sigma}$, can be shown not to be objective. Hence, the material time derivative of the Cauchy stress tensor cannot serve as a proper stress rate measure. Of course this defect is valid for the Kirchhoff stress rate $\dot{\tau}$ as well.

There do, however, exist a number of different objective rates of the Cauchy stress tensor. Here we will only mention one of them: The Jaumann or co-rotational rate, $\sigma^\nabla$, defined by

$$\sigma^\nabla = \dot{\sigma} + \sigma \cdot W - W \cdot \sigma; \quad \sigma^\nabla_{ij} = \dot{\sigma}_{ij} + \sigma_{ik} W_{kj} - W_{ik} \sigma_{kj} \quad (1.32)$$

1.3 Material Models

In the following we will assume that the yield condition can be expressed in the form

$$f = \bar{\sigma} - H(\bar{\varepsilon}^p) = 0 \quad (1.33)$$

This relation implies that yielding occurs when the effective stress $\bar{\sigma}$, which is a scalar function of the state of stress, reaches a critical value $H$, which in turn is a function of the effective plastic strain $\bar{\varepsilon}^p$. The function $H(\bar{\varepsilon}^p)$ is usually obtained from a uniaxial stress-plastic strain curve.

The rate of deformation tensor $d$, defined in Eq. (1.16), is used as a strain rate measure in elastic-plastic constitutive equations. We will in the following assume that the rate of deformation tensor can be additively divided into an elastic and a plastic part:

$$d = d^e + d^p \quad (1.34)$$

The normality condition states that the plastic rate of deformation $d^p$ is outward normal to the yield surface $f = 0$. This is expressed as

$$d^p = \dot{\lambda} \frac{\partial f}{\partial \sigma} \quad (1.35)$$

where $\dot{\lambda}$ is a scalar function that depends on the current state of stress and strain. The relationship (1.35) is called an associated flow rule.

During plastic loading the stress point remains on the yield surface. This implies that $\dot{f} = 0$, which is known as the consistency condition. In the present case we find
\[ \dot{f} = \frac{\partial f}{\partial \sigma} : \sigma^\nabla = H' \dot{\varepsilon}^p = \frac{\partial f}{\partial \sigma_{km}} \sigma^\nabla_{km} - H' \dot{\varepsilon}^p = 0 \]  

(1.36)

where \( H' = dH/d\bar{\varepsilon}^p \) is the slope of the uniaxial stress-plastic strain curve, and \( \sigma^\nabla \) is an objective stress rate tensor.

The effective plastic strain rate \( \dot{\varepsilon}^p \) is defined by the rate of plastic work equation

\[ \dot{W}^p = \bar{\sigma} \dot{\varepsilon}^p = \sigma : d^p \]  

(1.37)

Combining Eqs. (1.35) and (1.37), we note that

\[ \dot{\lambda} = \dot{\varepsilon}^p \]  

(1.38)

Typical for an elastic-plastic law is that there exists a relation between rates (or increments) of stress and strain. This relation can be written

\[ \sigma^\nabla = D : d^e = D : (d - d^p) \]  

(1.39)

where \( D \) is the elastic constitutive tensor.

For simplicity, we will in the following gather the Cartesian components of the tensors \( \sigma \) and \( d^p \) in two vectors (column matrices) \( \{ \sigma \} \) and \( \{ d^p \} \). For the case of plane stress these vectors will have the following appearances

\[ \{ \sigma \} = \begin{bmatrix} \sigma_x & \sigma_y & \tau_{xy} \end{bmatrix}^T \]

\[ \{ d^p \} = \begin{bmatrix} d_{xx}^p & d_{xy}^p & 2d_{xy}^p \end{bmatrix}^T \]  

(1.40)

For a quadratic yield condition, e.g. the von Mises or the Hill’48 condition, the effective stress can in matrix form be expressed as

\[ \bar{\sigma} = (\{ \sigma \}^T [A] \{ \sigma \})^{1/2} \]  

(1.41)

where \( [A] \) is a matrix with constants describing the anisotropy of the material.

The gradient to the yield surface can then be expressed as

\[ \left\{ \frac{\partial f}{\partial \sigma} \right\} = \frac{1}{\bar{\sigma}} [A] \{ \sigma \} \]  

(1.42)

and according to the normality condition the components of the plastic rate of deformation tensor are given by

\[ \{ d^p \} = \frac{\dot{\varepsilon}^p}{\bar{\sigma}} [A] \{ \sigma \} \]  

(1.43)
To give an example of the appearance of the matrix \([A]\), let us consider a transversally anisotropic material obeying the Hill’48 yield condition. Eq. (1.43) can then be expressed in matrix form as

\[
\begin{bmatrix}
\frac{d\varepsilon_x}{dP} \\
\frac{d\varepsilon_y}{dP} \\
2\frac{\tau_{xy}}{dP}
\end{bmatrix} = \frac{\dot{\varepsilon}_p}{\bar{\sigma}} \begin{bmatrix}
1 & -\frac{R}{1+R} & 0 \\
-\frac{R}{1+R} & 1 & 0 \\
0 & 0 & 2\frac{1+2R}{1+R}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\] (1.44)

Inverting Eq. (1.43), we get

\[
\{\sigma\} = \frac{\bar{\sigma}}{\dot{\varepsilon}_p} [A]^{-1} \{dP\}
\] (1.45)

Note that this equation expresses total stress in terms of rate of plastic strain. Note also that it is only for quadratic yield conditions that the normality condition can be inverted to this form.

If the total strain rates in Eq. (1.45) are replaced by plastic strain rates, i.e. the elastic part of the rate of deformation tensor is ignored, this equation will form the basis of the rigid-plastic theory. A couple of the earlier FE formulations for sheet forming simulation were based on this form of the constitutive equations. In matrix form this equation takes the form

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{\bar{\sigma}}{\dot{\varepsilon}_p} \begin{bmatrix}
1 + R & R & 0 \\
R & 1 + R & 0 \\
0 & 0 & \frac{1}{2}
\end{bmatrix} \begin{bmatrix}
\frac{d\varepsilon_x}{dP} \\
\frac{d\varepsilon_y}{dP} \\
2\frac{\tau_{xy}}{dP}
\end{bmatrix}
\] (1.46)

Using the work Eq. (1.37), we can easily derive the following expression for the effective strain rate

\[
\dot{\varepsilon}_p = \left(\{d\}^T [A]^{-1} \{d\}\right)^{1/2}
\] (1.47)

1.4 FE-Equations for Small Deformations

The dynamic equilibrium conditions of a body are readily expressed by means of the principle of virtual velocities. In terms of variables referred to the current configuration the principle is stated

\[
\int_V \sigma : \delta \mathbf{d} dV + \int_V \rho \dot{\mathbf{u}} : \delta \dot{\mathbf{u}} dV = \int_S \mathbf{t} : \delta \dot{\mathbf{u}} dS + \int_V \mathbf{f} : \delta \dot{\mathbf{u}} dV
\] (1.48)
where $\ddot{u}$ is the acceleration, $\dot{u}$ is the velocity, $t$ is the surface traction, and $f$ is the body load. Integration is performed over current volume $V$ and surface area $S$.

Assume now for a moment that we are dealing with a small deformation problem. FE-approximations can be introduced as

\[
\{ u(x) \} = [ N(x) ] \{ \tilde{u} \} \\
\{ \dot{u}(x) \} = [ N(x) ] \{ \dot{\tilde{u}} \} \\
\{ \delta \dot{u}(x) \} = [ N(x) ] \{ \delta \dot{\tilde{u}} \} 
\]

(1.49)

where $\{ \tilde{u} \}$ is a vector with nodal displacements and $[ N(x) ]$ is a matrix with base functions.

Introducing the above FE-approximations into the equation for the strain rate, Eq. (1.16), we obtain the expressions for the strain rate and virtual strain rate, respectively, according to

\[
\{ \dot{\varepsilon} \} = \{ d \} = [ B ] \{ \dot{\tilde{u}} \} \\
\{ \delta \dot{\varepsilon} \} = \{ \delta d \} = [ B ] \{ \delta \dot{\tilde{u}} \} 
\]

(1.50)

where the matrix $[B]$ is known as the strain matrix.

The discretized equilibrium equations will now take the form

\[
\begin{bmatrix} f^{int} \end{bmatrix} + [ M ] \{ \ddot{\tilde{u}} \} = \{ f^{ext} \} 
\]

(1.51)

where $[M]$ is the consistent mass matrix, defined by

\[
[M] = \int_{V} \rho [ N ]^T [ N ] dV
\]

(1.52)

The external force vector is given by

\[
\{ f^{ext} \} = \int_{V} [ N ]^T \{ b \} dV + \int_{S} [ N ]^T \{ t \} dS
\]

(1.53)

and, finally, the internal force or stress force vector is defined by

\[
\{ f^{int} \} = \int_{V} [ B ]^T \{ \sigma \} dV
\]

(1.54)

In the case of linear elasticity the constitutive relation can be written in matrix form as

\[
\{ \sigma \} = [ D ] \{ \varepsilon \}
\]

(1.55)
where \([D]\) is the matrix form of the elastic constitutive tensor. The internal force vector can then be rewritten as

\[
\begin{bmatrix} f^{\text{int}} \end{bmatrix} = \left( \int_V \begin{bmatrix} B \end{bmatrix}^T \begin{bmatrix} D \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \ dV \right) \begin{bmatrix} \hat{u} \end{bmatrix} = \begin{bmatrix} K \end{bmatrix} \begin{bmatrix} \hat{u} \end{bmatrix} \tag{1.56}
\]

where \([K]\) is the linear stiffness matrix.

### 1.5 FE-Equations for Finite Deformations

The virtual velocity equation (1.48) is a correct representation of the dynamic equilibrium of a body, even if the deformations are of finite magnitudes, on the assumption that all quantities are measured in the current configuration. In a Lagrangian formulation it is, however, much more convenient to express equilibrium in variables referred to the reference configuration. The transformation of Eq. (1.48) can be shown to give

\[
\int_{V_0} S : \delta \dot{E} \ dV_0 + \int_{V_0} \rho_0 \ddot{u} : \delta \dot{u} \ dV_0 = \int_{S_0} \mathbf{t}^0 : \delta \dot{u} \ dS_0 + \int_{V_0} \mathbf{f}^0 : \delta \dot{u} \ dV_0 \tag{1.57}
\]

In Eq. (1.57) an index ‘0’ has been assigned to variables measured in the reference configuration. The variables \(\mathbf{t}^0\) and \(\mathbf{f}^0\) are pseudo forces per unit area and volume, respectively, in the reference configuration. They are defined by

\[
\mathbf{t}^0 \ dS_0 = \mathbf{t} \ dS; \quad \mathbf{f}^0 \ dV_0 = \mathbf{f} \ dV \tag{1.58}
\]

In accordance with the small deformation formulation in Eq. (1.50), the expressions for the strain rate and virtual strain rate in matrix notation now become

\[
\begin{align*}
\{ \dot{E} \} &= \begin{bmatrix} \hat{B} \end{bmatrix} \begin{bmatrix} \hat{u} \end{bmatrix} \\
\{ \delta \dot{E} \} &= \begin{bmatrix} \hat{B} \end{bmatrix} \begin{bmatrix} \delta \hat{u} \end{bmatrix} \tag{1.59}
\end{align*}
\]

The strain matrix \(\begin{bmatrix} \hat{B} \end{bmatrix}\) can formally be divided in two parts according to

\[
\begin{bmatrix} \hat{B} \end{bmatrix} = \begin{bmatrix} B_L \end{bmatrix} + \begin{bmatrix} B_{NL} \end{bmatrix} \tag{1.60}
\]

Here \([B_L]\) is the ordinary linear strain matrix, equivalent with the matrix \([B]\) in the small deformation theory, while \([B_{NL}]\) is a nonlinear matrix, which arises as a consequence of the displacement dependent terms in the expression for the Green strain rate \(\dot{E}\) according to Eq. (1.25).
In accordance with the small deformation theory the discretized dynamic equilibrium equations can be expressed as

\[
\begin{aligned}
\begin{bmatrix} f^{\text{int}} \end{bmatrix} + [ M ] \begin{bmatrix} \ddot{u} \end{bmatrix} &= \begin{bmatrix} f^{\text{ext}} \end{bmatrix} \\
\end{aligned}
\tag{1.61}
\]

with the consistent mass matrix

\[
[M] = \int_{V_0} \rho_0 [N]^T [N] \, dV_0 
\tag{1.62}
\]

the external force vector

\[
\begin{bmatrix} f^{\text{ext}} \end{bmatrix} = \int_{V_0} [N]^T \{ b_0 \} \, dV_0 + \int_{S_0} [N]^T \{ t_0 \} \, dS_0 
\tag{1.63}
\]

and the internal force vector

\[
\begin{bmatrix} f^{\text{int}} \end{bmatrix} = \int_{V_0} \left[ \hat{B} \right]^T \{ S \} \, dV_0 
\tag{1.64}
\]

We will henceforth assume that the problem is quasi-static, i.e. we neglect the inertia term in the equilibrium equations in Eq. (1.61). The simplified equations can then be written

\[
\begin{bmatrix} \Psi \end{bmatrix} = \int_{V_0} \left[ \hat{B} \right]^T \{ S \} \, dV_0 - \begin{bmatrix} f^{\text{ext}} \end{bmatrix} = 0 
\tag{1.65}
\]

where \{ \Psi \} is a residual vector, whose elements should be zero when the equilibrium equations are satisfied.

In order to solve the resulting set of nonlinear equations, the Newton-Raphson iterative solution procedure, or related techniques, is commonly used. This requires a linearization of Eq. (1.65) around the last obtained solution. Taking the time derivative of the equilibrium equation in Eq. (1.65), we get

\[
\begin{bmatrix} \dot{\Psi} \end{bmatrix} = \int_{V_0} \left( \left[ \hat{\hat{B}} \right]^T \{ \dot{S} \} + \left[ \hat{B} \right]^T \{ \dot{S} \} \right) \, dV_0 - \begin{bmatrix} \dot{f}^{\text{ext}} \end{bmatrix} = 0 
\tag{1.66}
\]

Assuming that the constitutive equations can be transformed to a form such as

\[
\begin{bmatrix} \dot{S} \end{bmatrix} = [D_T] \begin{bmatrix} \dot{E} \end{bmatrix} 
\tag{1.67}
\]

where \([D_T]\) is a matrix form of the constitutive tensor. It can then be shown that Eq. (1.66) can be rewritten as
1.5 FE-Equations for Finite Deformations

\[
\{ \dot{\Psi} \} = [K_T] \left\{ \dot{\hat{u}} \right\} - \{ \dot{f}^{\text{ext}} \} = 0 \tag{1.68}
\]

where the tangent matrix \([K_T]\) formally can be divided into three matrices:

\[
[K_T] = [K_M] + [K_G] + [K_S] \tag{1.69}
\]

The first of these matrices, \([K_M]\), is called the material stiffness matrix and is defined by

\[
[K_M] = \int_{V_0} \{B_L\}^T \{D_T\} \{B_L\} dV_0 \tag{1.70}
\]

This matrix is recognized as the ordinary small deformation tangent stiffness matrix. The second one, \([K_G]\), is known as the geometric stiffness matrix, and is a consequence of the nonlinear terms in the strain-displacement relation. Finally, the third matrix, \([K_S]\), is called the initial stress stiffness matrix, and is a function of the stress state in the current configuration.

Various FE-approaches, based on a Lagrangian description, can be constructed depending on the choice of reference configuration. The most well-known Lagrangian formulations are the Total (TL) and Updated Lagrangian (UL) formulations, respectively. In the TL-formulation the initial, stress free configuration is taken as reference configuration, while in the UL-formulation the last calculated configuration is taken as reference state. These two formulations are described below.

The TL-formulation follows basically the one described above. The reference coordinates should here be interpreted as the initial ones, and the displacements are the total ones. Integrations are, furthermore, performed over initial volume and surface area, respectively. The Green strain tensor components at time \(t+\Delta t\) can either be calculated from the total displacements at time \(t+\Delta t\), or by adding the strain increment during the time increment \(\Delta t\) to the strains at time \(t\). This is justified by the fact that all tensors, even incremental ones, are referred to the same reference configuration.

One of the primary objects of the simulations is to determine the strain distribution in the blank. Principal logarithmic strains in the plane of the sheet are given by

\[
\varepsilon_1 = \ln \Lambda_1; \quad \varepsilon_2 = \ln \Lambda_2 \tag{1.71}
\]

where \(\Lambda_1\) and \(\Lambda_2\) are principal stretch values. Stretch is a measure of extensional strain of a differential line element, defined by \(\Lambda=dS/dS\). The principal stretch values can be shown to be related to the in-plane principal values of Lagrangian strain, \(E_1\) and \(E_2\), by
\[ \Lambda_1 = \sqrt{2E_1 + 1}; \quad \Lambda_2 = \sqrt{2E_2 + 1} \] (1.72)

where

\[ E_{1,2} = \frac{1}{2} (E_x + E_y) \pm \sqrt{\frac{1}{4} (E_x - E_y) + E_{xy}^2}; \] (1.73)

The UL FE-formulation follows largely the same pattern as outlined for the TL-approach above. The main differences are as follow. All coordinates entering the formulation should be the current ones. Since the displacements are measured from the reference (current) configuration, all terms involving displacements in the general formulation will vanish. This implies that the matrices \([B_{NL}]\) and \([K_G]\) of the general formulation do not enter the UL-formulation. Furthermore, integrals are carried out over current volume and area.

The calculation of total strain is much more complicated in the UL-formulation than in the TL-formulation, since the Lagrangian strain increments in each time step is referred to different configurations. This implies that they cannot be added to total strains without complicated transformations.

For three-dimensional shell and membrane elements it is usually necessary to use a local coordinate system for each element, which is redefined (updated) in each step. This implies that a new transformation matrix has to be established in each step, and that a number of transformations of displacement and load vectors between local and global systems have to be performed.

### 1.6 The ‘Flow Approach’—Eulerian FE-Formulations for Rigid-Plastic Sheet Metal Analysis

The rigid-plastic constitutive relations in Sect. 1.3 have the form of the constitutive relations for a non-Newtonian viscous fluid. In steady-state metal forming problems, such as extrusion and rolling, the velocities at a given point in space remain constant in time. The material behavior in this type of problems is similar to that of a fluid, and an Eulerian FE-approach is a natural choice. The FE mesh in such problems is fixed (Eulerian).

The Eulerian formulation has, however, also been used for the solution of rigid-plastic, transient problems, such as stretch forming and deep-drawing of metal sheets, although the material behavior in such problems bears small resemblance with a fluid flow. In such transient problems the control volume is identified with the sheet geometry in each deforming step. The element mesh has, thus, to be ‘updated’ in each step (Lagrangian).

The use of the flow approach in sheet metal forming problems has been advocated particularly by Prof. O.C. Zienkiewicz and co-workers in Swansea, and by Prof. E. Oñate in Barcelona [5, 6, 18–21]. A review of the flow approach in application to various steady-state and transient forming problems is given in Zienkiewicz [22].
In Sect. 1.3 the constitutive relations of a rigid-plastic model were derived by neglecting the elastic part of an elastic-plastic model (Eqs. 1.45, 1.46, 1.47). Some writers have derived the constitutive relations for a rigid-plastic material starting from the general form of a viscoplastic material as suggested by Perzyna [23]. If the time dependent effects in the viscoplastic model are neglected, the equations of the rigid-plastic model are recovered. It is interesting to note the analogy between the equations of rigid-plastic or viscoplastic flow, and those of small strain, linear elasticity. It is easily shown that completely analogous stress-strain relations are obtained, if the velocity and rate of deformation variables of the flow equations are interchanged by displacement and strain of the linear elasticity equations. Take for instance Hooke’s generalized law in the case of plane stress and replace Poisson’s ratio $\nu$ by $R/(1+R)$. Hooke’s law then takes the form

\begin{equation}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
2\varepsilon_{xy}
\end{bmatrix} = \frac{1}{E} \begin{bmatrix}
1 & -\frac{R}{1+R} & 0 \\
-\frac{R}{1+R} & 1 & 0 \\
0 & 0 & 2\frac{1+2R}{1+R}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\end{equation}

(1.74)

The analogy between these equations and the corresponding plastic flow equations in Eq. (1.44) is immediately seen. It is noted that the modulus of elasticity $E$ in the elasticity relations plays the role of the ‘viscosity’ $\bar{\sigma}/\dot{\varepsilon}$ in the flow relations.

The above discussed analogy makes it possible to use a standard FE-program for linear elastic analysis in large strain viscoplastic analysis with only minor modifications of the program. Basically, the same FE-equations outlined in Sect. 1.4 for linear elasticity are also applicable in the current flow approach, but with nodal displacements replaced by nodal velocities, and the constitutive relationship modified as described above. In the solution process a steady-state flow situation is assumed at every deformation level. Due to the nonlinear ‘viscosity’, $\bar{\sigma}/\dot{\varepsilon}$, an iterative solution scheme has to be employed at every step to ensure equilibrium.

The FE-equations are thus established in a standard fashion. We assume that a local Cartesian coordinate system is defined for each element and is updated in each step. Components referred to these local axes are in the following marked by a super-scribed star. Briefly, the major steps of the discretization process are the following:

**Velocity assumptions:**

\[ \{ \dot{u}^* \} = [N^*] \{ \dot{\tilde{u}}^* \} \quad (1.75) \]

**Rate of deformation-velocity relations:**

\[
d^*_{\alpha\beta} = \frac{1}{2} (\dot{u}^*_{\alpha\beta} + \dot{u}^*_{\beta\alpha})
\]

\[ \{ d^* \} = [N^*] \{ \dot{\tilde{u}}^* \} \quad (1.76) \]