Sturm-Liouville Theory

Past and Present

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(Editors)

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Preface

Charles François Sturm, through his papers published in the 1830's, is considered to be the founder of Sturm-Liouville theory. He was born in Geneva in September 1803. To commemorate the 200th anniversary of his birth, an international colloquium in recognition of Sturm's major contributions to science took place at the University of Geneva, Switzerland, following a proposal by Andreas Hinz. The colloquium was held from 15 to 19 September 2003 and attended by more than 60 participants from 16 countries. It was organized by Werner Amrein of the Department of Theoretical Physics and Jean-Claude Pont, leader of the History of Science group of the University of Geneva. The meeting was divided into two parts. In the first part, historians of science discussed the many contributions of Charles Sturm to mathematics and physics, including his pedagogical work. The second part of the colloquium was then devoted to Sturm-Liouville theory. The impact and development of this theory, from the death of Sturm to the present day, was the subject of a series of general presentations by leading experts in the field, and the colloquium concluded with a workshop covering recent research in this highly active area.

This drawing together of historical presentations with seminars on current mathematical research left participants in no doubt of the degree to which Sturm's original ideas are continuing to have an impact on the mathematics of our own times. The format of the conference provided many opportunities for exchange of ideas and collaboration and might serve as a model for other multidisciplinary meetings.

The organizers had decided not to publish proceedings of the meeting in the usual form (a complete list of scientific talks is appended, however). Instead it was planned to prepare, in conjunction with the colloquium, a volume containing a complete collection of Sturm's published articles and a volume presenting the various aspects of Sturm-Liouville theory at a rather general level, accessible to the non-specialist. Thus Jean-Claude Pont will edit a volume[1] containing the collected works of Sturm accompanied by a biographical review as well as abundant historical and technical comments provided by the contributors to the first part of the meeting.

The present volume is a collection of twelve refereed articles relating to the second part of the colloquium. It contains, in somewhat extended form, the survey lectures on Sturm-Liouville theory given by the invited speakers; these are the first

[1] *The Collected Works of Charles François Sturm*, J.-C. Pont, editor (in preparation).

six papers of the book. To complement this range of topics, the editors invited a few participants in the colloquium to provide a review or other contribution in an area related to their presentation and which should cover some important aspects of current interest. The volume ends with a comprehensive catalogue of Sturm-Liouville differential equations. At the conclusion of the Introduction is a brief description of the articles in the book, placing them in the context of the developing theory of Sturm-Liouville differential equations. We hope that these articles, besides being a tribute to Charles François Sturm, will be a useful resource for researchers, graduate students and others looking for an overview of the field.

We have refrained from presenting details of Sturm's life and his other scientific work in this volume. As regards Sturm-Liouville theory, some aspects of Sturm's original approach are presented in the contributions to the present book, and a more detailed discussion will be given in the article by Jesper Lützen and Angelo Mingarelli in the companion volume. Of course, the more recent literature concerned with this theory and its applications is strikingly vast (on the day of writing, MathSciNet yields 1835 entries having the term "Sturm-Liouville" in their title); it is therefore unavoidable that there may be certain aspects of the theory which are not sufficiently covered here.

The articles in this volume can be read essentially independently. The authors have included cross-references to other contributions. In order to respect the style and habits of the authors, the editors did not ask them to use a uniform standard for notations and conventions of terminology. For example, the reader should take note that, according to author, inner products may be anti-linear in the first or in the second argument, and deficiency indices are either single natural numbers or pairs of numbers. Moreover, there are some differences in terminology as regards spectral theory.

The colloquium would not have been possible without support from numerous individuals and organizations. Financial contributions were received from various divisions of the University of Geneva (Commission administrative du Rectorat, Faculté des Lettres, Faculté des Sciences, Histoire et Philosophie des Sciences, Section de Physique), from the History of Science Museum and the City of Geneva, the Société Académique de Genève, the Société de Physique et d'Histoire Naturelle de Genève, the Swiss Academy of Sciences and the Swiss National Science Foundation. To all these sponsors we express our sincere gratitude. We also thank the various persons who volunteered to take care of numerous organizational tasks in relation with the colloquium, in particular Francine Gennai-Nicole who undertook most of the secretarial work, Jan Lacki and Andreas Malaspinas for technical support, Danièle Chevalier, Laurent Freland, Serge Richard and Rafael Tiedra de Aldecoa for attending to the needs of the speakers and other participants. Special thanks are due to Jean-Claude Pont for his enthusiastic collaboration over a period of more than three years in the entire project, as well as to all the speakers of the meeting for their stimulating contributions.

As regards the present volume, we are grateful to our authors for all the efforts they have put into the project, as well as to our referees for generously

giving of their time. We thank Norrie Everitt, Hubert Kalf, Karl Michael Schmidt, Charles Stuart and Peter Wittwer who freely gave their scientific advice, Serge Richard who undertook the immense task of preparing manuscripts for the publishers, and Christian Clason for further technical help. We are much indebted to Thomas Hempfling from Birkhäuser Verlag for continuing support in a fruitful and rewarding partnership.

The cover of this book displays, in Liouville's handwriting, the original formulation by Sturm and Liouville, in the manuscript of their joint 1837 paper, of the regular second-order boundary value problem on a finite interval. The paper, which is discussed here by W.N. Everitt on pages 47–50, was presented to the Paris Académie des sciences on 8 May 1837 and published in Comptes rendus de l'Académie des sciences, Vol. IV (1837), 675–677, as well as in Journal de Mathématiques Pures et Appliquées, Vol. 2 (1837), 220–223. The original manuscript, with the title "Analyse d'un Mémoire sur le développement des fonctions en séries, dont les différents termes sont assujettis à satisfaire à une même équation différentielle linéaire contenant un paramètre variable", is preserved in the archives of the Académie des sciences to whom we are much indebted for kind permission to reproduce an extract.

Geneva, September 2004

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Introduction

David Pearson

Charles François Sturm was born in Geneva on 29 September 1803[1]. He received his scientific education in this city, in which science has traditionally been of such great importance. Though he was later drawn to Paris, where he settled permanently in 1825 and carried out most of his scientific work, he has left his mark also on the city of Geneva, where his name is commemorated by the Place Sturm and the Rue Charles-Sturm. On the first floor of the Museum of History of Science, in its beautiful setting with magnificent views over Lake Geneva, you can see some of the equipment with which his friend and collaborator Daniel Colladon pursued his research on the lake into the propagation of sound through water[2].

Sturm's family came to Geneva from Strasbourg a few decades before his birth. He frequently moved house, and at least two of the addresses where he spent some of his early years can still be found in Geneva's old town[3,4].

Not only did Charles Sturm leave his mark on Geneva, but his rich scientific legacy is recognized by mathematicians and scientists the world over, and continues to influence the direction of mathematical development in our own times[5]. In

[1]This corresponds to the sixth day of the month of Vendémiaire in year XII of the French revolutionary calendar then in use in the Département du Léman.

[2]Colladon was the physicist and experimentalist of this partnership, while Sturm played an important role as theoretician. Their joint work on sound propagation and compressibility of fluids was recognized in 1827 by the award of the Grand Prix of the Paris Academy of Sciences.

[3]The address 29, Place du Bourg-de-Four was home to ancestors of Charles Sturm in 1798. The present building appears on J.-M. Billon's map of Geneva, dated 1726, which is the earliest extant cadastral map of the city. The home of Charles Sturm in 1806, with his parents and first sister, was 11, Rue de l'Hôtel-de-Ville. The building now on this site was constructed in 1840. The two houses are in close proximity.

[4]For details on Sturm's life, see the biographical notice by J.-C. Pont and I. Benguigui in *The Collected Works of Charles François Sturm*, J.-C. Pont, editor (in preparation), as well as Chapter 21 of the book by P. Speziali, *Physica Genevensis, La vie et l'oeuvre de 33 physiciens genevois*, Georg, Chêne-Bourg (1997).

[5]Sturm was already judged by his contemporaries to be an outstanding theoretician. Of the numerous honours which he received during his lifetime, special mention might be made of the Grand Prix in Mathematics of the Paris Academy, in 1834, and membership of the Royal Society of London as well as the Copley Medal, in 1840. The citation for membership of the Royal Society was as follows: "Jacques Charles François Sturm, of Paris, a Gentleman eminently distinguished for his original investigations in mathematical science, is recommended by us as a proper person

bringing together leading experts in the scientific history of Sturm's work with some of the major contributors to recent and contemporary mathematical developments in related fields, the Sturm Colloquium provided a unique opportunity for the sharing of knowledge and exchange of new ideas.

Interactions of this kind between individuals from different academic backgrounds can be of great value. There is, of course, a powerful argument for mathematics to take note of its history. Mathematical results, concepts and methods do not spring from nowhere. Often new results are motivated by existing or potential applications. Some of Sturm's early work on sound propagation in fluids is a good example of this, as are his fundamental contributions to the theory of differential equations, which were partly motivated by problems of heat flow. Some of the later developments in areas that Sturm had initiated proceeded in parallel with one of the revolutions in twentieth century physics, namely quantum mechanics. New ideas in mathematics need to be considered in the light of the mathematical and cultural environment of their time.

Sturm's mathematical publications covered diverse areas of geometry, algebra, analysis, mechanics and optics. He published textbooks in analysis and mechanics, both of which were still in use as late as the twentieth century[6].

To most mathematicians today, Sturm's best-known contributions, and those which are usually considered to have had the greatest influence on mathematics since Sturm's day, have been in two main areas.

The first of Sturm's major contributions to mathematics was his remarkable solution, presented to the Paris Academy of Sciences in 1829 and later elaborated in a memoir of 1835[7], of the problem of determining the number of roots, on a given interval, of a real polynomial equation of arbitrary degree. Sturm found a complete solution of this problem, which had been open since the seventeenth century. His solution is algorithmic; a sequence of auxiliary polynomials (now called Sturm

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to be placed on the list of Foreign members of the Royal Society". The Copley Medal was in recognition of his seminal work on the roots of real polynomial equations and was the second medal awarded that year, the first having gone to the chemist J. Liebig. The citation for the Medal was: "Resolved, by ballot. – That another Copley Medal be awarded to M. C. Sturm, for his "Mémoire sur la Résolution des Equations Numériques," published in the Mémoires des Savans Etrangers for 1835". Sturm is also one of the few mathematicians commemorated in the series of plaques at the Eiffel tower in Paris.

[6]Both of these books were published posthumously, Sturm having died on 18 December 1855. The analysis text went through 15 editions, of which the last printing was as late as 1929. A reference for the first edition is: *Cours d'analyse de l'École polytechnique* (2 vols.), published by E. Prouhet (Paris, 1857–59). The text was translated into German by T. Fischer as: *Lehrbuch der Analysis* (Berlin, 1897–98). The first edition of the mechanics text was: *Cours de mécanique de l'École polytechnique* (2 vols.), published by E. Prouhet (Paris, 1861). The fifth and last edition, revised and annotated by A. de Saint-Germain, was in print at least until 1925.

[7]The full text of Sturm's resolution of this problem is to be found in: *Mémoire sur la résolution des équations numériques*, in the journal Mémoires présentés par divers savans à l'Académie Royale des Sciences de l'Institut de France, sciences mathématiques et physiques **6** (1835), 271–318 (also cited as Mémoires Savants Étrangers). See also *The Collected works of Charles François Sturm*, J.-C. Pont, editor (in preparation) for further discussion of this work.

functions), is calculated, and the number of roots on an interval is determined by the signs of the Sturm functions at the ends of the intervals. Sturms work on zeros of polynomials undoubtedly influenced his work on related problems for solutions of differential equations, which was to follow.

His second major mathematical contribution, or rather a whole series of contributions, was to the theory of second-order linear ordinary differential equations. In 1833 he read a paper to the Academy of Sciences on this subject, to be followed in 1836 by a long and detailed memoir in the Journal de Mathématiques Pures et Appliquées. This memoir was one of the first to appear in the journal, which had recently been founded by Joseph Liouville, who was to become a collaborator and one of Sturm's closest friends in Paris. It contained the first full treatment of the oscillation, comparison and separation theorems which were to bear Sturm's name, and was succeeded the following year by a remarkable short paper, in the same journal and in collaboration with Liouville, which established the basic principles of what was to become known as Sturm-Liouville theory[8]. The problems treated in this paper would be described today as Sturm-Liouville boundary value problems (second-order linear differential equations, with linear dependence on a parameter) on a finite interval, with separated boundary conditions. Sturm's earlier work had shown that such problems led to an infinity of possible values of the parameter. The collaboration between Sturm and Liouville took the theory some way forward by proving the expansion theorem, namely that a large class of functions could be represented by a Fourier-type expansion in terms of the family of solutions to the boundary value problem. In modern terminology, the solutions would later be known as eigenfunctions and the corresponding values of the parameter as eigenvalues.

The 1837 memoir, published jointly by Sturm and Liouville, was to become the foundation of a whole new branch of mathematics, namely the spectral theory of differential operators. Sturm-Liouville theory is central to a large part of modern analysis. The theory has been successively generalized in a number of directions, with applications to Mathematical Physics and other branches of modern science. This volume provides the reader with an account of the evolution of Sturm-Liouville theory since the pioneering work of its two founders, and presents some of the most recent research. The companion volume will treat aspects of the work of Sturm and his successors as a branch of the history of scientific ideas. We believe that the two volumes together will provide a perspective which will help to make clear the significant position of Sturm-Liouville theory in modern mathematics.

Sturm-Liouville theory, as originally conceived by its founders, may be regarded, from a modern standpoint, as a first, tentative step towards the development of a spectral theory for a class of second-order ordinary differential operators.

[8]For a more extended treatment of the early development of Sturm-Liouville theory, with detailed references, see the paper on Sturm and differential equations by J. Lützen and A. Mingarelli in the companion volume, as well as the first contribution by Everitt to this volume.

Liouville had already covered in some detail the case of a finite interval with two regular endpoints and boundary conditions at each endpoint. He regarded the resulting expansion theorem in terms of orthogonal eigenfunctions[9] as an extension of corresponding results for Fourier series, and the analysis was applicable only to cases for which, in modern terminology, the spectrum could be shown to be pure point. In fact the term "spectrum" itself, in a sense close to its current meaning, only began to emerge at the end of the nineteenth and the beginning of the twentieth century, and is usually attributed to David Hilbert.

The first decade of the twentieth century was a period of rapid and highly significant development in the concepts of spectral theory. A number of mathematicians were at that time groping towards an understanding of the idea of continuous spectrum. Among these was Hilbert himself, in Göttingen. Hilbert was concerned not with differential equations (though his work was to have a profound impact on the spectral analysis of second-order differential equations) but with what today we would describe as quadratic forms in the infinite-dimensional space l^2. Within this framework, he was able to construct the equivalent of a spectral function for the quadratic form, in terms of which both discrete and continuous spectrum could be defined. Examples of both types of spectrum could be found, and from these examples emerged the branch of mathematics known as spectral analysis. For the first time, spectral theory began to make sense even in cases where the point spectrum was empty. The time was ripe for such developments, and the theory rapidly began to incorporate advances in integration and measure theory coming from the work of Lebesgue, Borel, Stieltjes and others.

As far as Sturm-Liouville theory itself is concerned, the most significant progress during this first decade of the twentieth century was undoubtedly due to the work of the young Hermann Weyl. Weyl had been a student of Hilbert in Göttingen, graduating in 1908. (He was later, in 1930, to become professor at the same university.) His 1910 paper[10] did much to revolutionise the spectral theory of second-order linear ordinary differential equations. Weyl's spectrum is close to the modern definition via resolvent operators, and his analysis of endpoints based on limit point/limit circle criteria anticipates later ideas in functional analysis in which deficiency indices play the central role. For Weyl, continuous spectrum was not only to be tolerated, but was totally absorbed into the new theory. The expansion theorem, from 1910 onwards, was to cover contributions from both discrete and continuous parts of the spectrum. Weyl's example of continuous spectrum, corresponding to the differential equation $-d^2f(x)/dx^2 - xf(x) = \lambda f(x)$ on the

[9]Liouville's proof of the expansion theorem was not quite complete in that it depended on assumptions involving some additional regularity of eigenfunctions. Later extensions of this theory, as well as a full and original proof of completeness of eigenfunctions, can be found in the article by Bennewitz and Everitt in this volume.

[10]A full discussion of Weyl's paper and its impact on Sturm-Liouville theory is to be found in the first contribution by Everitt to this volume.

half line $[0, \infty)$, could hardly have been simpler[11]. And, perhaps most importantly, with Weyl's 1910 paper complex function theory began to move to the center stage in spectral analysis.

The year 1913 saw a further advance through the publication of a research monograph by the Hungarian mathematician Frigyes Riesz[12], in which he continued the ideas of Hilbert, with the new point of view that it was the linear operator associated with a given quadratic form, rather than the form itself, which was to be the focus of analysis. In other words, Riesz shifted attention towards the spectral theory of linear operators. In doing so he was able to arrive at the definition of spectrum in terms of the resolvent operator, to define a functional calculus for linear operators, and to explore the idea of what was to become the resolution of the identity for bounded self-adjoint operators. An important consequence of these results was that it became possible to incorporate many of Weyl's results on Sturm-Liouville problems into the developing theory of functional analysis. Thus, for example, the role of boundary conditions in determining self-adjoint extensions of differential operators could then be fully appreciated.

The modern theory of Sturm-Liouville differential equations, which grew from these beginnings, was profoundly influenced by the emergence of quantum mechanics, which also had its birth in the early years of the twentieth century. At the heart of the development of a mathematical theory to meet the demands of the new physics was John von Neumann[13].

Von Neumann joined Hilbert as assistant in Göttingen in 1926, the very year that Schrödinger first published his fundamental wave equation. The Schrödinger equation is, in fact, a partial differential equation, but, in the case of spherically symmetric potentials such as the Coulomb potential, the standard technique of separation of variables reduces the equation to a sequence of ordinary differential equations, one for each pair of angular momentum quantum numbers. In this way, under the assumption of spherical symmetry, Sturm-Liouville theory can be applied to the Schrödinger equation.

Von Neumann found in functional analysis the perfect medium for understanding the foundations of quantum mechanics. Quantum theory led in a natural way to a close correspondence (one could almost say identification, though that would not quite be true) of the physical objects of the theory with mathematical objects drawn from the theory of linear operators (usually differential operators) in Hilbert space. The state of a quantum system could be described by a normalized element (or vector, or wave function) in the Hilbert space. Corresponding to each

[11] Later it was to emerge that examples of this kind could be interpreted physically in terms of a quantum mechanical charged particle moving in a uniform electric field.

[12] F. Riesz, *Les systèmes d'équations linéaires à une infinité d'inconnues*, Gauthier-Villars, Paris (1913). See also J. Dieudonné, *History of functional analysis*, North-Holland, Amsterdam (1981). With Riesz we begin to see the development of an "abstract" operator theory, in which the special example of Sturm-Liouville differential operators was to play a central role.

[13] Von Neumann established a mathematical framework for quantum theory in his book *Mathematische Grundlagen der Quantenmechanik*, Springer, Berlin (1932). An English translation appeared as *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955).

quantum observable was a self-adjoint operator, the spectrum of which represented the range of physically realizable values of the observable. Both point spectrum and continuous spectrum were important – in the case of the hydrogen atom the energy spectrum had both discrete and continuous components, the discrete points (eigenvalues of the corresponding Schrödinger operator) agreeing closely with observed energy levels of hydrogen, and the continuous spectrum corresponding to states of positive energy.

Von Neumann quickly saw the implications for quantum mechanics of the new theory, and played a major part in developing the correspondence between physical theory and the analysis of operators and operator algebras. Physics and mathematical theory were able to develop in close parallel for many years, greatly to the advantage of both. He developed to a high art the spectral theory of self-adjoint and normal operators in abstract Hilbert space. A complete spectral analysis of self-adjoint operators in Hilbert space, generalizing the earlier results of Riesz, was just one outcome of this work, and a highly significant one for quantum theory. Similar results were independently discovered by Marshall Stone, who expounded the theory in his book published in 1932. (See the first article by Everitt.)

Of central importance for the future development of applications to mathematical physics, particularly in scattering theory which existed already in embryonic form in the work of Heisenberg, was the realization that the Lebesgue decomposition of measures into its singular and absolutely continuous (with respect to Lebesgue measure) components led to an analogous decomposition of the Hilbert space into singular and absolutely continuous subspaces for a given self-adjoint operator. Moreover, these two subspaces are mutually orthogonal. The singular subspace may itself be decomposed into two orthogonal components, namely the subspace of discontinuity, spanned by eigenvectors, and the subspace of singular continuity. Physical interpretations have been found for all of these subspaces, though in most applications only the discontinuous and absolutely continuous subspaces are non-trivial. In the case of the Hamiltonian (energy operator) for a quantum particle subject to a Coulomb force, the discontinuous subspace is the subspace of negative energy states and describes bound states of the system, whereas the absolutely continuous subspace corresponds to scattering states, which have positive energy.

The influence of the work of Charles Sturm and his close friend and collaborator Joseph Liouville may be found in the numerous modern developments of the theory which bears their names. A principal aim of this volume is to follow in detail the evolution of the theory since its early days, and to present an overview of the most important aspects of the theory as it stands today at the beginning of the twenty-first century.

We are grateful indeed to Norrie Everitt for his contributions to this volume, as author of two articles and coauthor of another. Over a long mathematical career, he has played an important role in the continuing progress of Sturm-Liouville theory.

The first of Norrie's articles in this volume deals with the development of Sturm-Liouville theory up to the year 1950, and covers in particular the work of

Weyl, Stone and Titchmarsh, of whom Norrie was himself a one-time student. (He also had the good fortune, on one occasion, to have encountered Weyl, who was visiting Titchmarsh at the time.)

Don Hinton's article is concerned with a series of results which follow from Sturm's original oscillation theorems developed in 1836 for second-order equations. Criteria are obtained for the oscillatory nature of solutions of the differential equation, and implications for the point spectrum are derived. Extensions of the theory to systems of equations and to higher-order equations are described.

Joachim Weidmann's contribution considers the impact of functional analysis on the spectral theory of Sturm-Liouville operators. Starting from ideas of resolvent convergence, it is shown how spectral behavior for singular problems may in appropriate cases be derived through limiting arguments from an analysis of regular problems. Conditions are obtained for the existence (or non-existence) of absolutely continuous spectrum in an interval.

Spectral properties of Sturm-Liouville operators are often derived, directly or indirectly, as a consequence of an established link between large distance asymptotic behavior of solutions of the associated differential equation and spectral properties of the corresponding differential operator. In the case of complex spectral parameter, the existence of solutions which are square-integrable at infinity may be described by the values of an analytic function, known as the Weyl-Titchmarsh m-function or m-coefficient, and spectral properties of Sturm-Liouville operators may be correlated with the boundary behavior of the m-function close to the real axis. The article by Daphne Gilbert explores further the link between asymptotics and spectral properties, particularly through the concept of subordinacy of solutions, an area of spectral analysis to which she has made important contributions.

A useful resource for readers of this volume, particularly those with an interest in numerical approaches to spectral analysis, will be the catalogue of Sturm-Liouville equations, compiled by Norrie Everitt with the help of colleagues. More than 50 examples are described, with details of their Weyl limit point/limit circle endpoint classification, the location of eigenvalues, other spectral information, and some background on applications. This collection of examples from an extensive literature should also provide a reference to some of the sources in which the interested reader can find further details of the theory and its applications, as well as numerical data on spectral properties.

In collaboration with Christer Bennewitz, Everitt has contributed a new version of the proof of the expansion theorem for general Sturm-Liouville operators, incorporating both continuous and discontinuous spectra.

The article by Barry Simon presents some recent results related to Sturm's oscillation theory for second-order equations. The cases of both Schrödinger operators and Jacobi matrices (which may be regarded as a discrete analogue of Schrödinger operators) are considered. A focus of this work is the establishment of a connection between the dimension of spectral projections and the number of zeros of appropriate functions defined in terms of solutions of the Schrödinger equation. Some deep results in spectral theory follow from this analysis, and there are links with the theory of orthogonal polynomials on the unit circle.

Yoram Last has provided a review of progress over recent years in spectral theory for discrete and continuous Schrödinger operators. Of particular interest has been the progress in analysis of spectral types, with a finer decomposition of spectral measures than hitherto, and the development of new ways of characterizing absolutely continuous and singular continuous spectrum.

Rafael del Río's article is an exposition of recent results relating to the influence of boundary conditions on spectral behavior. For Schrödinger operators, a change of boundary condition will not affect the location of absolutely continuous spectrum, whereas the nature of singular spectrum may be profoundly influenced by choice of boundary conditions.

In view of the major influence that Sturm-Liouville theory has had over the years on the development of spectral theory for linear differential equations, it is not surprising that there have been many attempts to extend the ideas and methods to nonlinear equations. Chao-Nien Chen describes some recent results in the nonlinear theory, with particular emphasis on the characterization of nodal sets, an area related to Sturm's original ideas on oscillation criteria in the linear case.

Another productive area of research into Sturm-Liouville theory is the extension of the theory to partial differential equations. Sturm had himself published results on zero sets for parabolic linear partial differential equations in a paper of 1836. In their contribution to this volume, Victor Galaktionov and Petra Harwin survey recent progress in this area, including extensions to some quasilinear equations.

A continuing and flourishing branch of spectral theory, with applications in many areas, is that of inverse spectral theory. The aim of inverse theory is to derive the Sturm-Liouville equation from its spectral properties. An early example of this kind of result was the proof, due originally to Borg in 1946, that for the Schrödinger equation with potential function q over a finite interval and subject to boundary conditions at both endpoints, the spectrum for the associated Schrödinger operator for two distinct boundary conditions at one endpoint (and given fixed boundary condition at the other endpoint) is sufficient to determine q uniquely. This result has been greatly extended over recent years, for example to systems of differential equations, and some of the more recent developments are treated in the survey by Mark Malamud.

We believe that the contents of this book will confirm that Sturm-Liouville theory has, indeed, a very rich Past and a most active and influential Present. It is our hope, too, that the book will help to contribute to a continuing productive Future for this fundamental branch of mathematics and its applications.

W.O. Amrein, A.M. Hinz, D.B. Pearson
Sturm-Liouville Theory: Past and Present, 1–27

Sturm's 1836 Oscillation Results Evolution of the Theory

Don Hinton

This paper is dedicated to the memory of Charles François Sturm

Abstract. We examine how Sturm's oscillation theorems on comparison, separation, and indexing the number of zeros of eigenfunctions have evolved. It was Bôcher who first put the proofs on a rigorous basis, and major tools of analysis where introduced by Picone, Prüfer, Morse, Reid, and others. Some basic oscillation and disconjugacy results are given for the second-order case. We show how the definitions of oscillation and disconjugacy have more than one interpretation for higher-order equations and systems, but it is the definitions from the calculus of variations that provide the most fruitful concepts; they also have application to the spectral theory of differential equations. The comparison and separation theorems are given for systems, and it is shown how they apply to scalar equations to give a natural extension of Sturm's second-order case. Finally we return to the second-order case to show how the indexing of zeros of eigenfunctions changes when there is a parameter in the boundary condition or if the weight function changes sign.

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1. Introduction

In a series of papers in the 1830's, Charles Sturm and Joseph Liouville studied the qualitative properties of the differential equation

$$\frac{d}{dx}\left(K\frac{dV}{dx}\right) + GV = 0, \quad \text{for} \quad x \geq \alpha \tag{1.1}$$

where K, G, and V are real functions of the two variables x, r. Their work began research into the qualitative theory of differential equations, i.e., the deduction of properties of solutions of the differential equation directly from the equation and without benefit of knowing the solutions. However, it was half a century before significant interest in the qualitative theory took hold. In (1.1) and elsewhere, we consider only real solutions unless otherwise indicated.

In more modern notation (for spectral theory it is convenient to have the leading coefficient negative; for the oscillation results of Sections 2 and 3, we return to the convention of positive leading coefficient), (1.1) would be written as

$$-(py')' + qy = 0, \quad x \in I, \tag{1.2}$$

or as (when eigenvalue problems are studied)

$$-(py')' + qy = \lambda wy, \quad x \in I, \tag{1.3}$$

where the real functions p, q, w satisfy

$$p(x),\ w(x) > 0 \text{ on } I\ , 1/p, q, w \in L_{\mathrm{loc}}(I), \tag{1.4}$$

where $L_{\mathrm{loc}}(I)$ denotes the locally Lebesgue integrable functions on I. These are the minimal conditions the coefficients must satisfy for the initial value problem,

$$-(py')' + qy = 0, \quad x \in I, \quad y(a) = y_0, \quad y'(a) = y_1,$$

to have a unique solution. Sturm imposed no conditions on his coefficients, but was perhaps thinking of continuous coefficients. It is fair to say that thousands of papers have been written concerning the properties of solutions of (1.2), and hundreds more are published each year. Tony Zettl has called (1.2) the world's most popular differential equation. A recent check in math reviews shows 8178 entries for the word "oscillatory", 3284 entries for "disconjugacy", 1412 entries for "non-oscillatory", and even 62 for "Picone identity". The applications of (1.2) and (1.3) are ubiquitous. Their appearance in problems of heat flow and vibrations were well known since the work of Fourier. They play an important role in quantum mechanics where the problems are singular in the sense that I is an interval of infinite extent or where at a finite endpoint a coefficient fails to satisfy certain integrability conditions. Today we can find numerically with computers the solutions of (1.2) or the eigenvalues and eigenfunctions associated with (1.3). However, even with current technology, there are still problems which give computational difficulty such as computing two eigenvalues which are close together. Codes such as SLEIGN2 [9] (developed by Bailey, Everitt, and Zettl) or the NAG routines give quickly and accurately the eigenvalues and eigenfunctions of large classes of Sturm-Liouville problems. The recent text by Pryce [85] is devoted to the numerical solution of Sturm-Liouville problems.

For (1.1), Sturm imposed a condition ($h(r)$ is a given function),

$$\frac{K(\alpha, r)}{V(\alpha, r)} \frac{\partial V(\alpha, r)}{\partial x} = h(r), \tag{1.5}$$

and obtained the following central result [94] (after noting that when the values of $V(\alpha, r)$, $\partial V(\alpha, r)/\partial x$ are given, the solution $V(x, r)$ is uniquely determined). We have also used Lützen's translation [74].

Theorem A. *If V is a nontrivial solution of* (1.1) *and* (1.5), *and if for all $x \in [\alpha, \beta]$,*

1. *$K > 0$ for all r and K is a decreasing function of r,*
2. *G is an increasing function of r,*
3. *$h(r)$ is a decreasing function of r,*

then $\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)$ is a decreasing function of r for all $x \in [\alpha, \beta]$.

Here decreasing or increasing means strictly. If $V(\alpha, r) = 0$, then $h(r)$ decreasing means $\partial V/\partial x \cdot \partial V/\partial r < 0$ at $x = \alpha$. Sturm's method of proof of Theorem A was to differentiate (1.1) with respect to r, multiply this by V, and then subtract this from $\partial V/\partial r$ times (1.1). After an integration by parts over $[\alpha, x]$, the resulting equation obtained is

$$\left(-V^2 \frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)\right)(x) = \left(-V^2(\alpha, r)\frac{dh}{dr}\right) + \int_\alpha^x \left[\frac{\partial G}{\partial r}V^2 - \frac{\partial K}{\partial r}\left(\frac{\partial V}{\partial r}\right)^2\right], \quad (1.6)$$

where we have used

$$-V^2 \frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right) = K\frac{\partial V}{\partial x}\frac{\partial V}{\partial r} - V\frac{\partial}{\partial r}\left(K\frac{\partial V}{\partial x}\right). \quad (1.7)$$

If we solve this equation for the term $\frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)(x)$, then we get

$$\frac{\partial}{\partial r}\left(\frac{K}{V}\frac{\partial V}{\partial x}\right)(x, r) < 0, \quad (1.8)$$

which completes the proof.

An examination of the above proof shows that the same conclusion can be reached with less restrictive hypotheses. With $K > 0$, an examination of the right-hand side of (1.6) shows that it is positive, and hence (1.8) holds under any one of the following three conditions.

$$\frac{\partial G}{\partial r} > 0, \ \frac{\partial K}{\partial r} \leq 0, \frac{dh}{dr} \leq 0, \quad (1.9)$$

$$\frac{\partial G}{\partial r} \geq 0, \ \frac{\partial K}{\partial r} \leq 0, \frac{dh}{dr} < 0, \quad (1.10)$$

$$\frac{\partial G}{\partial r} \geq 0, \ \frac{\partial K}{\partial r} < 0, \frac{dh}{dr} \leq 0, \ V \text{ is not constant.} \quad (1.11)$$

Theorem A has immediate consequences. The first is that if $x(r)$ denotes a solution of $V(x, r) = 0$, then by implicit differentiation, we get from (1.7) and (1.8) that

$$\frac{dr}{dx} = -\frac{\partial V}{\partial x}/\frac{\partial V}{\partial r} < 0. \quad (1.12)$$

Note that this implies under the conditions of Theorem A, that the roots $x(r)$ of $V(x, r)$ are decreasing with respect to r. With $K > 0$ the same conclusion may be reached by replacing the hypothesis of Theorem A with (1.9), (1.10), or (1.11).

By considering two equations, $(K_iV_i')' + G_iV_i = 0$, $i = 1, 2$, with $G_2(x) \geq G_1(x)$, $K_2(x) \leq K_1(x)$ and embedding the functions h_1, h_2, G_1, G_2 and K_1, K_2 into a continuous family, e.g., one can define

$$\hat{G}(r, x) = rG_2(x) + (1 - r)G_1(x),\ 0 \leq r \leq 1,$$

and similarly for K, Sturm was able to prove comparison theorems. In particular he proved

Theorem B (Sturm's Comparison Theorem). *For $i = 1, 2$ let V_i be a nontrivial solution of $(K_iV_i')' + G_iV_i = 0$. Suppose further that with $h_i = (K_iV_i'/V_i)(\alpha)$,*

$$h_2 < h_1, \quad G_2(x) \geq G_1(x), \quad K_2(x) \leq K_1(x), \quad x \in [\alpha, \beta].$$

Then if α, β are two consecutive zeros of V_1, the open interval (α, β) will contain at least one zero of V_2.

In case $V_i(\alpha) = 0$, the proper interpretation of infinity must be made.

This version of comparison corresponds to using the hypothesis (1.10). Other versions may be proved by using either (1.9) or (1.11). Perhaps the most widely stated version of Sturm's comparison theorem (not the version he proved) may be stated as follows.

Theorem B*. *For $i = 1, 2$ let V_i be a nontrivial solution of $(K_iV_i')' + G_iV_i = 0$ on $\alpha \leq x \leq \beta$. Suppose further that the coefficients are continuous and for $x \in [\alpha, \beta]$,*

$$G_2(x) \geq G_1(x),\ \text{with } G_2(x_0) > G_1(x_0) \text{ for some } x_0, \quad K_2(x) \leq K_1(x).$$

Then if α, β are two consecutive zeros of V_1, the open interval (α, β) will contain at least one zero of V_2.

Sturm's methods also yielded (in modern terminology):

Theorem C (Sturm's Separation Theorem). *If V_1, V_2 are two linearly independent solutions of $(KV')' + GV = 0$ and a,b are two consecutive zeros of V_1, then V_2 has a zero on the open interval (a, b).*

The final result of Sturm that we wish to quote concerns the zeros of eigenfunctions and is proved in his second memoir [95]. Here he considered the eigenvalue problem,

$$(k(x)V'(x))' + [\lambda g(x) - l(x)]V(x) = 0, \quad \alpha \leq x \leq \beta, \tag{1.13}$$

with separated boundary conditions,

$$k(\alpha)V'(\alpha) - hV(\alpha) = 0, \quad k(\beta)V'(\beta) + HV(\beta) = 0. \tag{1.14}$$

Further the functions k, g, and l are assumed positive. Some properties he established are:

Theorem D. *There are infinitely many real simple eigenvalues $\lambda_1, \lambda_2, \ldots$ of (1.13) and (1.14), and if $V_1, V_2, \ldots$ are the corresponding eigenfunctions, then for $n = 1, 2, \ldots$,*

1. *V_n has exactly $n - 1$ zeros in the open interval (α, β),*
2. *between two consecutive zeros of V_{n+1} there is exactly one zero of V_n.*

Theorem D relates to the spectral theory of the operator associated with (1.13) and (1.14). For (1.2) considered on an infinite interval $I = [a, \infty)$, an eigenvalue problem, in order to define a self-adjoint operator, may only require one boundary condition at a (limit point case at infinity), or it may require two boundary conditions involving both a and infinity (limit circle case at infinity). This dichotomy was discovered by Weyl. In the limit point case with $w \equiv 1$, a self-adjoint operator is defined in the Hilbert space $L^2(a, \infty)$ of Lebesgue square integrable functions by

$$L_\alpha[y] = -(py')' + qy, \quad y \in \mathcal{D},$$

where

$$\mathcal{D} = \{y \in L^2(a,\infty) : y,\, py' \in AC_{loc}, L_\alpha[y] \in L^2(a,\infty), \\ y(a)\sin\alpha - (py')(a)\cos\alpha = 0\}, \quad (1.15)$$

and AC_{loc} denotes the locally absolutely continuous functions.

Unlike the case (1.13) and (1.14) for the compact interval, the spectrum for the infinite interval may contain essential spectrum, i.e., numbers λ such that $L_\alpha - \lambda I$ has a range that is not closed, and Theorem D does not apply. However in the case of a purely discrete spectrum bounded below, a version of Theorem D carries over to the operator L_α above in the relation of the index of the eigenvalue to the number of zeros of the eigenfunction in (a, ∞) [22]. In general, one can say that the number of points in the spectrum of L_α below a real number λ_0 is infinite if and only if the equation $-(py')' + qy = \lambda_0 y$ is oscillatory, i.e., the solutions have infinitely many zeros on $[a, \infty)$. This same result carries over to self-adjoint equations of arbitrary order if the definition of oscillation in Section 4 is used [80, 99]. This basic connection has been used extensively in spectral theory. Note that if $-(py')' + qy = \lambda_0 y$ is non-oscillatory for every λ_0, then the spectrum of L_α consists only of a sequence of eigenvalues tending to infinity. Theorem D and its generalizations have also important numerical consequences. When an eigenvalue is computed, it allows one to be sure which eigenvalue it is, i.e., just count the zeros of the eigenfunction. It also allows the calculation of an eigenvalue without first calculating the eigenvalues that precede it. This feature is built into some eigenvalue codes.

A number of monographs deal almost exclusively with the oscillation theory of linear differential equations and systems. The books of Coppel [24] and Reid [88] emphasize linear Hamiltonian systems, but also contain substantial material on the second-order case. Coppel contains perhaps the most concise treatment of Hamiltonian systems; Reid is the most comprehensive development of Sturm theory. The book of Elias [29] is based on the oscillation and boundary value problem theory for two term ordinary differential equations, while Greguš [38] deals entirely with third-order equations. The text by Kreith [62] includes abstract oscillation theory as well as oscillation theory for partial differential equations. Finally the classic book by Swanson [96] has special chapters on second, third, fourth-order ordinary differential equations as well as results for partial differential equations. The reader is also referred to the survey papers of Barrett [10] and Willett [100]. The books by Atkinson [8], Glazman [37], Hartman [44], Ince [53], Kratz [61], Müller-Pfeiffer [80], and Reid [86] contain many results on oscillation theory.

As noted, the literature on the Sturm theory is voluminous. There are extensive results on difference equations, delay and functional differential equations, and partial differential equations. The Sturm theory for difference equations is similar to that of ordinary differential equations, but contains many new twists. The book by Ahlbrandt and Peterson [6] details this theory (see also the text by B. Simon in the present volume). Oscillation results for delay and functional equations as well as further work on difference equations can be found in the books by Agarwal, Grace, and O'Regan [1, 2], I. Gyori and G. Ladas [39], and L. Erbe, Q. Kong, and B. Zhang [31]. We confine ourselves to the case of ordinary differential equations and at that we are only able to pursue a few themes.

The comparison and oscillation theorems of Sturm have remained a topic of considerable interest. While the extensions and generalizations have much intrinsic interest, we believe their continued relevance is due in no small part to their intimate connection with problems of physical origin. Particularly the connections with the minimization problems of the calculus of variations and optimal control as well as the spectral theory of differential operators are important. We will discuss some of these connections below. We will trace some of the developments that have occurred with respect to the comparison and separation theorems as well as other developments related to Theorem D. The tools introduced by Picone, Prüfer, and the variational methods will be discussed and their applications to second-order equations as well as to higher-order equations and systems. Sample results will be stated and a few short and elegant proofs will be given. The problem of extending Sturm's results to systems was only considered about one hundred years after Sturm; the work of Morse was fundamental in this development. It is interesting that it was variational theory which gave the most natural and fruitful generalization of the definitions of oscillation. In a very loose way, we show that the theme of largeness of the coefficient q in $(py')' + qy = 0$ leads to oscillation in not only the second-order, but also higher-order equations, while $q \leq 0$, or $|q|$ small leads to disconjugacy.

2. Extensions and more rigor

Sturm's proofs of course do not meet the standards of modern rigor. They meet the standards of his time, and are in fact correct in method and can without too much trouble be made rigorous. The first efforts to do this are due to Bôcher in a series of papers in the Bulletin of the AMS [17] and are also contained in his book [18]. Bôcher [17] remarks that "the work of Sturm may, however, be made perfectly rigorous without serious trouble and with no real modification of method". The conditions placed on the coefficients were to make them piecewise continuous. Bôcher used Riccati equation techniques in some of his proofs; we note that Sturm mentions the Riccati equation, but does not employ it in his proofs. Riccati equation techniques in variational theory go back at least to Legendre who in 1786 gave a flawed proof of his necessary condition for a minimizer of an integral functional. A correct proof of Legendre's condition using Riccati equations can be found in Bolza's 1904 lecture notes [19]. Bolza attributes this proof to Weierstrass.

Bôcher was also motivated by the oscillation theorem of Klein [58] which is a multiparameter version of Sturm's existence proof for eigenvalues. Bôcher [17] noted that Klein "had given rough geometrical proofs which however made no pretence at rigor". The general form of Klein's problem may be stated as follows, see Ince [53, p. 248]. Suppose in (1.2), q is of the form

$$q(x) = -l(x) + [\lambda_0 + \lambda_1 x + \cdots + \lambda_n x^n]g(x),$$

where p, l, g are continuous with $p(x), g(x) > 0$. Further let there be $n+1$ intervals $[a_0, b_0], \ldots, [a_n, b_n]$ with $a_0 < b_0 < a_1 < \cdots < a_n < b_n$. Suppose m_s, $s = 0, \ldots, n$ are given nonnegative integers and on each interval $[a_s, b_s]$, separated boundary conditions of the form (1.14) are given. Then there exist a set of simultaneous characteristic numbers $\lambda_0, \ldots, \lambda_n$ and corresponding functions $y_0, \ldots, y_n$ such that on each $[a_s, b_s]$, y_s has m_s zeros in (a_s, b_s) and satisfies the boundary conditions for $[a_s, b_s]$. Klein was interested in the two parameter Lamé equation

$$y'' + \frac{1}{2}\left[\frac{1}{x-e_1} + \frac{1}{x-e_2} + \frac{1}{x-e_3}\right]y' - \frac{Ax+B}{4(x-e_1)(x-e_2)(x-e_3)}y = 0$$

because of its application to physics. The text by Halvorsen and Mingarelli [40] deals with the oscillation theory of the two parameter case.

The proofs of Sturm's theorems depend on existence-uniqueness results for (1.2), and Norrie Everitt has brought to our attention that it was Dixon [25] who first proved that these are valid under only the assumption that the coefficients $1/p$, q are Lebesgue integrable functions. The details of Dixon's work may be found in N. Everitt's text in the present volume. Later Carathéodory generalized the concept of a solution of a system of differential equations to only require the equation hold almost everywhere. When (1.2) is written in system form, the Dixon and Carathéodory conditions are the same. Richardson [89, 90] extended the results of counting zeros of eigenfunctions further by allowing the weight $g(x)$ in (1.13) to not be of constant sign and called this the non-definite case. We will return to his case in Section 5. Part (1) of Theorem D, which is for the separated boundary conditions (1.14), was extended by Birkhoff [16] to the case of arbitrary self-adjoint boundary conditions.

To simplify our discussion, we will henceforth assume that all coefficients and matrix components are real and piecewise continuous unless otherwise stated.

Thinking of examples like $y'' + ky = 0, k > 0$, whose solutions are sines and cosines or the Euler equation $y'' + kx^{-2}y = 0$ which has oscillatory solutions if and only if $k > 1/4$, it is natural to pose the problem:

$$\text{When are all solutions of } (py')' + qy = 0 \text{ oscillatory on } I? \qquad (2.1)$$

We use the term *oscillatory* (*non-oscillatory*) here in the sense of infinitely (finitely) many zeros for all nontrivial solutions. Because of the Sturm separation theorem, if one nontrivial solution has infinitely many zeros, then all do, but this property fails for nonlinear equations. A second problem, not quite so obvious, but which arose naturally from the calculus of variations, is

$$\text{When is the equation } (py')' + qy = 0 \text{ disconjugate on } I? \tag{2.2}$$

The term *disconjugate* is used here to mean that no nontrivial solution has more than one zero on I. If a nontrivial solution of $(py')' + qy = 0$ has a zero at a, then the first zero of y to the right of a is called the first right *conjugate point* of a; if there are no zeros to the right of a, then we say the equation is right disconjugate. Successive zeros are isolated and hence yield a counting of conjugate points. If y satisfies $y'(a) = 0$, then the first zero of y to the right of a is called the first right *focal point* of a. If y has no zeros to the right of a, then $(py')' + qy = 0$ is called right *disfocal*. Similar definitions are made to the left. The simplest criterion for both right disconjugate and disfocal is for $q(x) \leq 0$, for then an easy argument shows y is monotone if $y(a) \geq 0$, $y'(a) \geq 0$. On a compact or open interval I disconjugacy is equivalent to there being a solution of $(py')' + qy = 0$ with no zeros on I [24, p.5]. For a half-open interval $(py')' + qy = 0$ can be disconjugate without there being a solution with no zeros as is shown by the equation $y'' + y = 0$ on $[0, \pi)$ which is disconjugate, but every solution has a zero in $[0, \pi)$.

A major advance was made by Picone [83] in his 1909 thesis. He discovered the identity

$$\left[\frac{u}{v}\left(vpu' - uPv'\right)\right]' = u(pu')' - \frac{u^2}{v}(Pv')' + (p - P)\,u'^2 + P\left(u' - \frac{u}{v}v'\right)^2 \tag{2.3}$$

which holds when u, v, pu', and Pv' are differentiable and $v(x) \neq 0$. In case u, v are solutions of the differential equations

$$(pu')' + qu = 0, \qquad (Pv')' + Qv = 0,$$

(2.3) reduces to

$$\left[\frac{u}{v}\left(vpu' - uPv'\right)\right]' = (Q - q)\,u^2 + (p - P)\,u'^2 + P\left(u' - \frac{u}{v}v'\right)^2. \tag{2.4}$$

With this identity one can give an elementary proof of Sturm's comparison Theorem B* which we now give. Suppose $p(x) \geq P(x)$, $Q(x) \geq q(x)$ with $Q(x_0) > q(x_0)$ at some x_0, α, β are consecutive zeros of a nontrivial solution u of $(pu')' + qu = 0$, and that v is a solution of $(Pv')' + Qv = 0$ with no zeros in the open interval (α, β). Note the quotient $u(x)/v(x)$ has a limit at the endpoints. For example the limit at α is zero if $v(\alpha) \neq 0$, and the limit is $u'(\alpha)/v'(\alpha)$ if $v(\alpha) = 0$. Integration of (2.4) over $[\alpha, \beta]$ yields that the left-hand side integrates to zero while the right-hand side integrates to a positive number. This contradiction proves the theorem.

Another major advance was made by Prüfer [84] with the use of trigonometric substitution. In the equation $(pu')' + (q + \lambda w)u = 0$, he made the substitution

$$u = \rho \sin\theta, \qquad pu' = \rho\cos\theta,$$

and then proved that ρ, θ satisfy the differential equations

$$\theta' = \frac{1}{p}\cos^2\theta + (q + \lambda w)\sin^2\theta, \quad \rho' = (\frac{1}{p} - q - \lambda w)(\sin\theta\cos\theta)\rho.$$

The zeros of the solution u are given by the values of x such that $\theta(x) = n\pi$ for some integer n. The equation for θ is independent of ρ, and by using a first-order comparison theorem for nonlinear equations, it is possible to establish Sturm's comparison theorem. Prüfer used the equation for θ to establish the link stated in Theorem D between the number of zeros of an eigenfunction and the corresponding eigenvalue. These equations can also be used to prove the existence of infinitely many eigenvalues. This is the method used in most textbooks today for the proof of Theorem D.

Note that with Prüfer's transformation, the equation $(py')' + qy = 0$, $a \leq x < \infty$, is oscillatory if and only if $\theta(x) \to \infty$ as $x \to \infty$. It also follows easily from this transformation that

$$\int_a^\infty \left[\frac{1}{p} + |q|\right] dx < \infty \Rightarrow \text{ non-oscillation,}$$

$$\int_a^\infty \left[\frac{1}{p} + |q|\right] dx < \pi \Rightarrow \text{ disconjugacy.}$$

Kamke [56] used the trigonometric substitution technique to prove a Sturm type comparison theorem for a system of first-order equations

$$y' = Py + Qz, \quad z' = Ry + Sz$$

where the coefficients are continuous functions.

Klaus and Shaw [57] used the Prüfer transformation to study the eigenvalues of a Zakharov-Shabat system. One of their results shows that the first-order system

$$v_1' = sv_1 + q(t)v_2, \quad v_2' = -sv_2 - q(t)v_1,$$

is (in our terminology) right disfocal on $-d \leq t \leq d$ if $\int_{-d}^{d} |q(t)|dt \leq \pi/2$; moreover the constant $\pi/2$ is sharp. Extension is then made to the interval $(-\infty, \infty)$ and for complex-valued q. Application is made to the nonexistence of eigenvalues (s is the eigenparameter) of the Zakharov-Shabat system, and hence to the nonexistence of soliton solutions of an associated nonlinear Schrödinger equation.

Sturm's comparison Theorem B* has been generalized to include integral comparisons of the coefficients. Consider the two equations, for $a \leq x < \infty$,

$$y'' + q_1(x)y = 0, \tag{2.5}$$

$$y'' + q_2(x)y = 0. \tag{2.6}$$

Then we may phrase Sturm's comparison theorem by:

If $q_1(x) \leq q_2(x)$, $a \leq x < \infty$, then (2.6) disconjugate $\Rightarrow$ (2.5) disconjugate.

This result was extended by Hille [50] (as generalized by Hartman [44, p. 369]) to read:

$$\text{If } \int_t^\infty q_1(x)dx \le \int_t^\infty q_2(x)dx,\ a \le t < \infty,$$

$$\text{then (2.6) disconjugate} \Rightarrow \text{ (2.5) disconjugate.}$$

Further results of this nature were given by Levin [67] and Stafford and Heidel [92].

3. Some basic oscillation results

The first major attack on problem (2.1) seems to have been made in 1883 by Kneser [59] who studied the higher-order equation $y^{(n)} + qy = 0$, and proved that all solutions oscillate an infinite number of times provided that $x^m q(x) > k > 0$ for all sufficiently large values of x, where $n \ge 2m > 0$ and n is even. Of course for $n = 2$, this follows immediately from the Sturm comparison theorem applied to the oscillatory Euler equation $y'' + kx^{-2}y = 0$, $k > 1/4$, since $k/x^2 \le k/x$ for $x \ge 1$. Hubert Kalf has noted that Weber [98] refined Kneser's result to decide on oscillation or non-oscillation in the case where $x^2 q(x)$ tends to a limit as x tends to infinity. The Kneser criterion has recently been extended by Gesztesy and Ünal [36].

A result which subsequently received a lot of attention was proved by Fite [33] in studying the equation $y^{(n)} + py^{(n-1)} + qy = 0$ on a ray $x \ge x_1$. Fite's result was if $q \ge 0$, $\int_{x_1}^\infty q dx = \infty$ and y is a solution of $y^{(n)} + qy = 0$, then y must change sign an infinite number of times in case n is even, and in case n is odd such a solution must either change sign an infinite number of times or not vanish at all for $x \ge x_1$. For $n = 2$ we then have a sufficient condition for (2.1), i.e.,

$$q(x) \ge 0, \quad \int_{x_1}^\infty q(x)dx = \infty \Rightarrow y'' + qy = 0 \text{ is oscillatory.}$$

This theme of $q(x)$ being sufficiently large has reoccurred in oscillation theory in many situations. The first improvement of the Fite result was due to Wintner [101] who removed the sign restriction on $q(x)$ and proved the stronger result

$$t^{-1} \int^t q(x)(t-x)dx \to \infty \text{ as } t \to \infty \Rightarrow y'' + qy = 0 \text{ is oscillatory.}$$

Independently Leighton [64] proved, for $(py')' + qy = 0$, that

$$\int^\infty \frac{dx}{p(x)} = \infty, \quad \int^\infty q(x)dx = \infty \Rightarrow (py')' + qy = 0 \text{ is oscillatory.}$$

Again there is no sign restriction on $q(x)$.

An elegant proof of this Fite-Wintner-Leighton result has been given by Coles [23]. We give this proof since it a good illustration of Riccati equation techniques.

Suppose that $\int^\infty p^{-1}\,dx = \infty$, $\int^\infty q\,dx = \infty$, and that u is a non-oscillatory solution of $(pu')' + qu = 0$, say $u(x) > 0$ on $[b, \infty)$. Define $r = pu'/u$. Then a calculation shows that $r' = -q - r^2/p$, and hence for large x, say $x \geq c$,

$$r(x) + \int_b^x \frac{r^2}{p}\,dt = r(b) - \int_b^x q\,dx < 0.$$

This implies that $r(x) < -\int_b^x p^{-1}r^2\,dt$. Thus defining $R(x) = \int_b^x p^{-1}r^2\,dt$, one has that for $x \geq c$, $R' = r^2/p \geq R^2/p$. Integration of this inequality gives

$$\int_c^x \frac{1}{p}\,dt \leq \int_c^x \frac{R'}{R^2}\,dt = \frac{1}{R(c)} - \frac{1}{R(x)} \leq \frac{1}{R(c)}$$

which is contrary to $\int^\infty p^{-1}\,dx = \infty$.

Related to the above result of Wintner is that of Kamenev [55] who showed that if for some positive integer $m > 2$,

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_a^t (t-s)^{m-1} q(s)\,ds = \infty,$$

then the equation $y'' + qy = 0$ is oscillatory on $[a, \infty)$. The Kamenev type results have been extended to operators with matrix coefficients and Hamiltonian systems by Erbe, Kong, and Ruan [30], Meng and Mingarelli [75], and others.

The mid-twentieth century saw a large number of papers written on problems (2.1) and (2.2). We mention a small sampling of these results.

Theorem 3.1 (Hille, 1948). *If $q(x) \geq 0$ is a continuous function on $I = [a, \infty)$, such that $\int_a^\infty q < \infty$, and*

$$g_* := \liminf_{x\to\infty} x \int_x^\infty q(t)dt, \quad g^* := \limsup_{x\to\infty} x \int_x^\infty q(t)dt,$$

then $g^* > 1$ or $g_* > 1/4$ implies $y'' + qy = 0$ is oscillatory, and $g^* < 1/4$ implies $y'' + qy = 0$ is non-oscillatory.

Hille's results have been extended to equations with matrix coefficients and linear Hamiltonian systems by Sternberg [93] and Ahlbrandt [3].

Theorem 3.2 (Hartman, 1948). *If $y'' + qy = 0$ is non-oscillatory on $[a, \infty)$, then there are solutions u, v of $y'' + qy = 0$ such that*

$$\int^\infty u^{-2}(t)\,dt < \infty \quad \textit{and} \quad \int^\infty v^{-2}(t)\,dt = \infty.$$

Theorem 3.3 (Wintner, 1951). *The equation $y'' + qy$ is non-oscillatory on $[a, \infty)$ if $\int_x^\infty q(t)\,dt$ converges and either $-3/4 \leq x\int_x^\infty q(t)\,dt \leq 1/4$ or $[\int_x^\infty q(t)\,dt]^2 \leq q(x)/4$.*

Theorem 3.4 (Nehari, 1954). *If $I = [a, \infty)$ and $\lambda_0(b)$ is the smallest eigenvalue of*

$$-y'' = \lambda c(x)y, \quad y(a) = y'(b) = 0,$$

where $c(x) > 0$ is continuous on I, then $y'' + c(x)y = 0$ is non-oscillatory on I iff $\lambda_0(b) > 1$ for all $b > a$.

Theorem 3.5 (Hartman-Wintner, 1954). *The equation $y''+qy=0$ is non-oscillatory on $[a,\infty)$ if $f(x)=\int_x^\infty q(t)dt$ converges and the differential equation $v''+4f^2(x)v=0$ is non-oscillatory.*

Theorem 3.6 (Hawking-Penrose, 1970). *If $I=(-\infty,\infty)$ and $q(x)\geq 0$ is a continuous function on I such that $q(x_0)>0$ for some x_0, then $y''+q(x)y=0$ is not disconjugate on I.*

A particularly simple proof of this result has been given by Tipler [97] which we now present. Suppose y is the unique solution of $y''+q(x)y=0$ with the initial conditions $y(x_0)=1$, $y'(x_0)=0$. Then $y''(x_0)=-q(x_0)y(x_0)<0$, and further $y''(x)\leq 0$ as long as $y(x)\geq 0$. Since $y'(x_0)=0$, this concavity of y implies that y eventually has a zero both to the right and to the left of x_0.

Many results on oscillation can be expanded by making a change of independent and dependent variables of the form $y(x)=\mu(x)z(t)$, $t=f(x)$, where $\mu(x)$ and $f'(x)$ are nonzero on the interval I. In the case of $(py')'+qy$, this leads to

$$(py')'+qy=(\gamma/\mu)[\dot{w}+Qz],\quad w=P\dot{z},\quad \gamma(x)=f'(x),$$

where $\dot{z}=dz/dt$ and

$$P(t)=p(x)\mu^2(x)\gamma(x),\quad Q(t)=\frac{\mu(x)}{\gamma(x)}\left[(p\mu')'+q\mu\right].$$

Applications of these ideas can be found in Ahlbrandt, Hinton, and Lewis [5].

To return to the concept of disconjugacy and the link to the calculus of variations, it was in 1837 that Jacobi [54] gave his sufficient condition for the existence for a (weak) minimum of the functional

$$J[y]=\int_a^b f(x,y,y')dx \tag{3.1}$$

over the class of admissible functions y defined as those sufficiently smooth y satisfying the endpoint conditions $y(a)=A$, $y(b)=B$. A necessary condition for an extremal is the vanishing of the first variation, $dJ(y+\epsilon\eta)/d\epsilon\rfloor_{\epsilon=0}$, for sufficiently smooth variations η satisfying $\eta(a)=\eta(b)=0$. This leads to the Euler-Lagrange equation $f_y-d(f_{y'})/dx=0$ for y. A sufficient condition for a weak minimum is that the second variation

$$\delta^2J(\eta)=\int_a^b\left[p\eta'^2+q\eta^2\right]dx \tag{3.2}$$

be positive for all nontrivial admissible η where $p=f_{y'y}$ and $q=f_{yy}-d(f_{y'y})/dx$. Jacobi discovered that the positivity of (3.2) was related to the oscillation properties of $-(py')'+qy=0$. In particular he discovered (3.2) is positive if $-(py')'+qy=0$ has a solution y which is positive on $[a,b]$. The condition of (3.2) being positive is equivalent to $-(py')'+qy=0$ being disconjugate on $[a,b]$. This is the principal connection of oscillation theory to the calculus of variations. This connection may be proved with Picone's identity as we now demonstrate.