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The Monodromy Group

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Preface

The origins of monodromy theory lie in the works of B. Riemann on functions of complex variables and on complex linear differential equations. Riemann formulated one of the fundamental problems in monodromy theory (now called the Riemann–Hilbert problem): having given singularities and corresponding monodromy transformations, find a differential equation which realizes these data. The monodromy groups of linear differential equations and systems were intensively studied in the nineteen century by F. Klein, G. G. Stokes, H. A. Schwarz, L. Schlesinger, L. Pochhammer, E. Picard, R. Garnier, P. Painlevé, H. Poincaré, R. Fuchs and others. Schwarz associated the monodromy group of the hypergeometric equation with the spherical triangle groups, generated by inversions with respect to the sides of a spherical triangle. Schlesinger investigated deformations of differential systems with fixed monodromy. Fuchs associated the equation Painlevé 6 with the isomonodromic deformation equation. Stokes discovered a strange phenomenon (the Stokes phenomenon) of non-uniqueness of constants in the asymptotic expansions of systems near irregular singularities.

In his talk at the 1900 International Congress of Mathematicians, D. Hilbert included the Riemann problem mentioned above in his famous list of problems for twentieth century mathematics (as the XXI-th). This problem was solved independently by J. Plemelj and by H. Röhrl in the class of systems with regular singularities. Only in the 1980s A. A. Bolibruch discovered that the Riemann–Hilbert problem may have no solutions in the Fuchs class of systems with first order poles. Recently some relations between linear differential systems and quantum field theory was revealed (M. Sato, T. Miwa, M. Jimbo, B. A. Dubrovin).

Near the end of the 19th century, E. Picard and E. Vessiot created an analogue of the algebraic Galois theory in the case of linear differential systems. The corresponding differential Galois group consists of symmetries of the system and is identified with an algebraic subgroup of the linear group of automorphisms of a complex vector space (E. Kolchin). In the regular case the differential Galois group forms the Zariski closure of the monodromy group (Schlesinger). The fundamental result in this theory states that a differential system is solvable in quadratures and algebraic functions iff the identity component of the differential Galois group is solvable. An analogous result holds in the topological Galois theory which is represented by the monodromy group of an algebraic function. A. G. Khovanski generalized the latter result to a wider class of multivalued holomorphic functions.

The Stokes phenomenon which occurs in the case of irregular singularity found complete explanation. Firstly, it was proved that there exists a formal normal form which is diagonal and contains only a finite number of terms with rational powers of the 'time' (M. Hukuhara, A. H. M. Levelt, H. Turrittin). Next the change reducing the system to its formal normal form turns out to be analytic in sectors.

This is done either by solving some integral equation (H. Poincaré, W. Wasow) or by showing that the normalizing series belongs to some Gevrey class and is summable in sectors. The moduli of analytic classification are cocycles in some cohomology group with values in a certain Stokes sheaf and are represented by 'differences' between normalizing maps in adjacent sectors.

After publication of "Analysis situs" by H. Poincaré, investigation of the topology of algebraic varieties began. At that time the first variant of the Picard–Lefschetz formula, describing change of the topology of a family of algebraic varieties as the parameter varies around a critical value, appeared. Further rapid progress in this field occurred in the 1960s–70s. The research proceeded in two parallel streams.

J. Milnor proved that the local level of a holomorphic function near an isolated critical point has the homotopy type of a bucket of spheres. R. Thom, J.-C. Tougeron, J. Martinet and V. I. Arnold developed a theory of normal forms of holomorphic functions and began classification of singularities. The (topological) monodromy groups of some singularities turned out to be isomorphic with certain Coxeter groups generated by reflections. Relations with the classification of semi-simple Lie algebras were revealed.

The other approach was more algebraic and based on the cohomology theory of coherent sheaves developed by J. Leray, A. Grothendieck, P. Deligne and others. Another tool was the theorem about resolution of singularities proved by H. Hironaka. People studied families of algebraic varieties which degenerate as the (complex) parameter approaches a critical value. Here the critical points can be isolated and non-isolated as well. After resolution of the singularity, the singular variety becomes a union of smooth divisors with normal crossings in the ambient complex space. Information about multiplicities of these divisors allows us to describe the action of the (topological) monodromy (C. H. Clemens, N. A'Campo). The fundamental result (the monodromy theorem) provides information about the eigenvalues and the dimensions of Jordan cells of the monodromy operator.

The de Rham cohomologies of non-singular algebraic varieties, from a family, form together a holomorphic vector bundle over the space of non-critical parameters. A similar bundle, the cohomological Milnor bundle, is defined in the local case. The cohomological bundle admits sections, defined by the integer cocycles. This allows introduction of the famous Gauss–Manin connection, such that the integer cocycles represent horizontal sections with respect to it. The holomorphic forms on the ambient space form another class of sections of the cohomological bundle, the geometrical sections. Their integrals along families of integer cycles are holomorphic functions which obey a system of linear differential equations called the Picard–Fuchs equations. The Picard–Fuchs equations are related with the equations for horizontal sections with respect to the Gauss–Manin connection. They have regular singularities (P. Griffiths, N. Katz). Together with the asymptotic of integrals they constitute invariants of the degeneration (B. Malgrange, V. I. Arnold, A. N. Varchenko, J. H. C. Steenbrink). The asymptotic of integrals is closely related with the asymptotic and geometry of oscillating integrals appearing in wave optics and quantum physics (V. I. Arnold, A. N. Varchenko).

The integrals of holomorphic forms along integer cycles found application in the linearized version of the XVI-th Hilbert problem, about limit cycles of polynomial planar vector fields. This problem leads to the problem of estimation of the number of zeroes of certain Abelian integrals (Arnold). Existence of such estimates and some concrete formulas were obtained by A. N. Varchenko, A. G. Khovanski and G. S. Petrov.

Any compact non-singular projective variety admits a so-called Hodge structure, which says that one can represent the cohomology classes as harmonic forms with their division into the (p, q)–types. P. Deligne proved that the non-compact and/or singular variety admits a so-called mixed Hodge structure. It means that there is a weight filtration of the cohomology space by an increasing series of subspaces, such that the quotient spaces are equipped with Hodge structures (arising from some complete smooth variety obtained after resolution of singularities). J. H. C. Steenbrink and W. Schmid proved existence of a mixed Hodge structure in the case of degeneration of algebraic varieties, and Steenbrink constructed such a structure in the fibers of the cohomological Milnor bundle. The latter structure is determined by the Jordan cells structure of the monodromy operator and by the asymptotic of geometrical sections (Varchenko).

In the case of degeneration of algebraic varieties the monodromy and the mixed Hodge structure are related with singularities of the period mapping, from the parameter space to the classifying space of Hodge structures (Griffiths). This leads to the problems of moduli of algebraic varieties and to theorems of Torelli type (about injectivity of the period map on the moduli space).

Besides the linear monodromy theory there is its nonlinear part. It is represented by the holonomy groups of some distinguished leaves of holomorphic foliations. This theory is well developed only in two dimensions, where the foliation is defined as a phase portrait of an analytic vector field (with complex 'time'). The singularities were resolved by Seidenberg and after resolution there remain only foci, nodes, saddles and saddle–nodes. The foci and the nodes were classified analytically by Poincaré. The classification of saddles leads to the classification of the holonomy maps associated with loops in one of its separatrices. If the multiplicator of a germ of a holomorphic one-dimensional diffeomorphism is resonant, then the situation is like the case of Stokes' phenomenon. The functional invariants of the analytic classification were found by J. Ecalle and S. M. Voronin. If the multiplicator is non-resonant, then the analyticity of the formal normal form (which is linear) depends on whether the multiplicator satisfies the so-called Briuno condition (A. D. Briuno, J.-C. Yoccoz). In the case of a saddle–node the functional moduli were described by J. Martinet and J.-P. Ramis. Here the main tool of the proof is certain sectorial normalization which is proved either by means of some functional analytic methods (M. Hukuhara, T. Kimura, T. Matuda) or by means of the Gevrey expansions.

The holomorphic foliations exist on algebraic surfaces; they are defined by means of polynomial vector fields. J. P. Jouanolou constructed examples of foliations on the projective plane without algebraic leaves, and A. Lins-Neto proved that such foliations are typical. M. F. Singer proved that if a polynomial planar vector field has a first integral expressed by quadratures, then it has a simple integrating factor (exponent of integral of a rational 1-form). It turns out that, for a typical foliation with the line at infinity invariant, a generic leaf is dense in the projective plane (M. O. Hudai-Verenov) and there are infinitely many limit cycles (Yu. S. Il'yashenko). The latter two results are proved using the monodromy group of the leaf at infinity. This is a subgroup of the group of germs of one-dimensional diffeomorphisms. The abelian and solvable groups of this type were classified (D. Cerveau, R. Moussu) and the non-solvable groups are rigid, in the sense that their formal or topological equivalence implies their analytical equivalence (J.-P. Ramis, A. A. Shcherbakov, I. Nakai).

S. L. Ziglin used the monodromy to prove the non-integrability of certain Hamiltonian systems, e.g., the Poisson–Euler system.

Among modern developments of the classical monodromy theory we cite generalizations of the Euler hypergeometric integrals to the case with more singularities (P. Deligne, G. D. Mostow) and to many dimensions (I. M. Gelfand, A. N. Varchenko). Here the monodromy group realizes a representation of the fundamental group of the complement to the discriminant variety and some classical results (like the theorem of Schwarz) were generalized.

The above outlines the history of the monodromy theory. These topics constitute the rough contents of this book.

The monodromy theory can be called a clever bifurcation theory. In the usual bifurcation theory one investigates some objects (functions, varieties, maps, vector fields) depending on real parameter(s) and their changes as the parameter passes through the critical values. For example, the Morse theory describes degeneration of the hypersurface level of a function as the value tends to a critical value. Usually the objects are defined analytically. In that case clever investigation relies on observing the transformation of the object as the parameter varies along a loop around the critical value (in the complex parameter space). Therefore the complex analogue of the Morse theory is the Picard–Lefschetz theory. The monodromy approach to the bifurcation problems turns out to be very effective. It allows us to obtain results out of reach when using the real methods.

There is something mysterious and undefined in the monodromy theory, at least for non-specialists. Often people use it rather loosely, without providing rigorous definitions.

The aim of this monograph is to introduce the reader into the complex of notions and methods used in the monodromy theory. Because these notions and methods involve large parts of modern mathematics, the book contains a lot of auxiliary mathematical material. It is written to be as self-contained as possible. We strived

not to omit technical parts of the proofs. We have included such elements as a proof of the analytic version of the Hadamard–Perron theorem or the proof of the Thom–Martinet preparation theorem. On the other hand, the results which are not fundamental and constitute generalizations of simpler results are treated more loosely. In these cases we present only ideas and general arguments.
The book touches practically all branches of monodromy theory. But it does not contain all known results. Many theorems are not even mentioned. Also the literature reference list is not complete.

The idea of writing this book appeared in 1996, when the author began to deliver a two-year course at Warsaw University. The subject was continued in seminars. The lectures were written down and constitute essential parts of the book. Therefore the book is addressed mainly to (graduate) students.
Another reason was the author's self-education. It was a great enterprise which consumed much of the author's time and energy during its writing. There is hope that this effort was not useless and will help others to learn relatively quickly techniques of the monodromy theory.
In concluding this preface I would like to express my thanks to my students T. Maszczyk, G. Świrszcz, E. Stróżyna, A. Langer, M. Rams, P. Leszczyński, M. Borodzik, Ł. Wiechecki, P. Goldstein, M. Bobieński and M. Borodzik for their patience during lectures and seminars and for detecting many mistakes. I thank V. Gromak for sending me notes from the lectures of G. Mahoux. I thank A. Maciejewski for drawing my attention to the works of S. L. Ziglin, J. J. Morales-Ruiz and J.-P. Ramis. I thank P. Mormul for showing me some references. I thank F. Loray for giving me preprints of some papers and lecture notes. I would like also to thank A. Weber, Ś. Gal, P. Pragacz, Z. Marciniak, Yu. Il'yashenko, A. Varchenko and J. Steenbrink for their interest in this book.
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Chapter 1

Analytic Functions and Morse Theory

This chapter is special. Its aim is quick introduction of the notion of monodromy in applications to multivalued holomorphic functions and their Riemann surfaces. The classical theorem about monodromy is a theorem about such functions.
The simplest example of a multivalued holomorphic function is the implicit function defined by means of a quadratic equation in two variables. This leads to the Morse lemma in two (and more) dimensions.
We apply the Morse theory (in the real domain) to calculate the self-intersection index of the cycle generating the homology of a noncritical complex level of a quadratic homogeneous function.

§1 Theorem about Monodromy

1.1. Definition of the analytic element and of the Riemann surface. By an **analytic element** we mean a pair (D, f), where $D = D_a \subset \mathbb{C}$ is a disc with center at a and $f = f_a$ is a holomorphic function on D, such that the Taylor series of f at a is convergent in D_a. We say that the analytic element (D_a, f_a) has prolongation to an element (D_b, f_b) along a path $\gamma \subset \mathbb{C}$ if γ can be covered by domains of analytic elements such that the corresponding functions agree at the adjacent intersections. The sum of analytic elements obtained from prolongations of (D, f) forms the **Riemann surface** of the (generally multivalued) holomorphic function f.

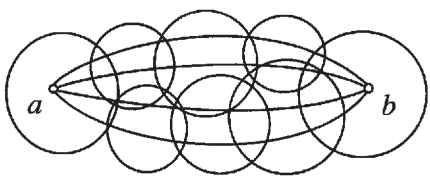

Figure 1

1.2. Theorem about monodromy. *If the paths γ_1, γ_2 joining the points a and b are homotopically equivalent in a domain where the function f is (locally) analytic, then the prolongations of (D_a, f) to $(D_{b,i}, f_i)$ along γ_i's are the same. It means that $f_1 \equiv f_2$ at $D_{b,1} \cap D_{b,2}$.*

Proof. Let γ_s be a 1-parameter family of paths joining a with b and realizing the homotopy between γ_i. Their union spreads over some compact domain E. We cover this domain by analytic elements, starting from (D_a, f), which agree at intersections. In this way we obtain the Riemann surface of f over E. It is clear that this Riemann surface is diffeomorphic to E. Thus f is single-valued on E. □

1.3. Remark. The reader can see that the theorem about monodromy bears a topological character; it informs us about coverings. One can formulate it in the following way:

Let $\pi : Y \to X$ be a covering of topological spaces and let γ_i, $i = 1, 2$, be two paths in X joining the points a and b. If the paths are homotopically equivalent, then the two maps $\pi^{-1}(a) \to \pi^{-1}(b)$, defined by lifts of the paths γ_i to Y, coincide.

If f is a multivalued function on $U \subset \mathbb{C}$ and M is its Riemann surface, then one has the single-valued function $\tilde{f}$ on M,

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & \mathbb{C} \\ \pi \downarrow & & \uparrow f \\ U & = & U \end{array}$$

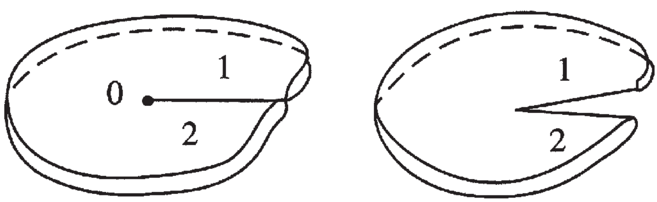

Figure 2

Example. $f(z) = \sqrt{z}$, $z \neq 0$. The prolongation of this function around 0 does not give the same value, but after two turns around 0 we get the same function. In order to get the Riemann surface of $\sqrt{z}$, we take two copies of the complex plane and cut them along the closed positive axis $z \geq 0$.
We put these sheets one above another, turn the upper one along the real axis and glue the boundaries. The Riemann surface of $\sqrt{z}$ is equal $\mathbb{C} \setminus 0$ and we have the diagram

$$\begin{array}{ccc} \mathbb{C}\backslash 0 & \xrightarrow{x} & \mathbb{C} \\ \pi \downarrow & & \uparrow \sqrt{z} \\ \mathbb{C}\backslash 0 & = & \mathbb{C}\backslash 0 \end{array}$$

We can prolong π to the map on $\mathbb{C}$, $\pi(x) = x^2$. Then we say that π is a **ramified covering**; with one *ramification point* $x = 0$.

We can compactify the complex plane to the projective plane (or the Riemann sphere)

$$\mathbb{C} \cup \infty = \mathbb{C}P^1 = \overline{\mathbb{C}} \simeq S^2$$

and we can also compactify the Riemann surface $M \to M \cup \infty \simeq \mathbb{C}P^1$. The point ∞ is also a ramification point, because after the change of variables we have $1/x \to 1/z = (1/x)^2$. In Figure 3 the pole ∞ is sent to the pole ∞ and the indicated circles are mapped with degree 2.

We shall study the multivalued holomorphic functions and their algebraic and topological invariants in Chapter 11.

§2 Morse Lemma

Let $U \subset \mathbb{C}$ be a domain containing 0 and let $f : U \to \mathbb{C}$ be a holomorphic function.

1.4. Definition. We say that the point 0 is **critical** for f iff $f'(0) = 0$. The critical point 0 is called **non-degenerate** iff $f''(0) \neq 0$. The value $f(0)$ is called the **critical value** of f.

The Morse Lemma in one dimension. *Let 0 be a non-degenerate critical point of f. Then there exists a local holomorphic change $x = \varphi(y)$, $x(0) = 0$ such that*

$$f(\varphi(y)) = f(0) + y^2.$$

Proof. We can assume that $f(0) = 0$.

The Hadamard lemma. *If $f(0) = 0$, then $f(x) = xg(x)$ with some analytic function g.*

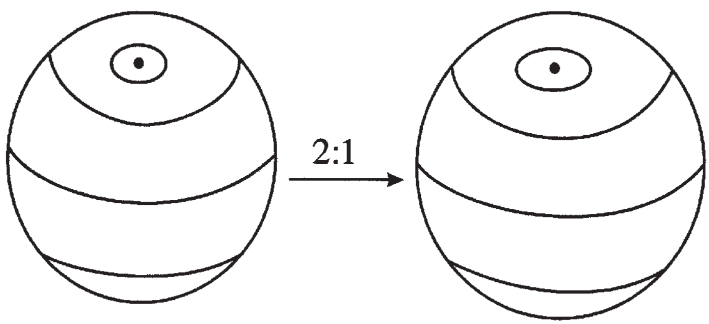

Figure 3

Proof. We have $f(x) = f(x) - f(0) = \int_0^1 [\frac{d}{dt} f(tx)] dt = x \int_0^1 f'(tx) dt.$ □

We have $g(0) = f'(0) = 0$. We apply the Hadamard lemma again and we obtain $g(x) = xh(x)$ and $f(x) = x^2 h(x)$, where $h(0) = \frac{1}{2} f''(0) \neq 0$. We put $y = x\sqrt{h(x)}$, where we can choose one of the two unique branches of the square root. □

Consider now the multidimensional case. Let $f : U \to \mathbb{C}$ be a holomorphic function, $U \subset \mathbb{C}^n$, $0 \in U$.

1.5. Definition. The point 0 is **critical** iff $Df(0) = 0$. It is **non-degenerate** iff the Hessian matrix $D^2 f(0)$ is non-singular or (equivalently) iff $\det \frac{\partial^2 f}{\partial x_i \partial x_j}(0) \neq 0$. The value $f(0)$ is called the **critical value** of f.

1.6. Morse Lemma. *If* 0 *is not a degenerate critical point of the function* f*, then there exists a holomorphic change of variables* $x = \varphi(y)$, $y = (y_1, \ldots, y_n)$, $\varphi(0) = 0$, *such that*

$$f(\varphi(y)) = f(0) + y_1^2 + \ldots + y_n^2.$$

1.7. Remark. In the real case the thesis of the Morse Lemma says that

$$f(\varphi(y)) = f(0) + y_1^2 + \ldots + y_k^2 - y_{k+1}^2 - \ldots - y_n^2.$$

Using the Morse Lemma we shall investigate the level surfaces of a holomorphic function in a neighborhood of the non-degenerate critical point. Any such level surface forms an analytic subvariety in $\mathbb{C}^n$ of complex codimension 1, or a codimension 2 real subvariety in $\mathbb{R}^{2n}$. Let

$$g(y) = y_1^2 + \ldots + y_n^2.$$

The case $n = 1$. Here $\{g(y) = c\}$ is either one point 0 for $c = 0$ or two points $\pm\sqrt{c}$ otherwise.

The case $n = 2$. We have $y_2 = \sqrt{c - y_1^2}$. The level surface $\{g = c\}$ is the Riemann surface of the function $\sqrt{c - y_1^2}$.
Let $c = 0$. Then $y_2 = \pm\sqrt{-1}y_1$. We get two complex lines joined at one point. Topologically it is diffeomorphic to the cone as in Figure 4.
If $c \neq 0$, then the function $\sqrt{c - y_1^2}$ has two branching points $y_1 = \pm\sqrt{c}$. When the variable y_1 varies, turning once around one branching point, then we arrive at the other sheet of the Riemann surface. When y_1 runs around both ramifications, then we arrive at the same place. In order to get the Riemann surface of $y_2(y_1)$ we take two copies of the complex plane cut along the segment joining the ramification points. We put one sheet above another, turn the upper sheet around the line passing through the branching points and glue the two sheets along the cuts (see Figure 5). We obtain a surface diffeomorphic to an infinite cylinder. The image of the cut forms a closed curve Δ; it is a cycle generating the homology group of this surface in dimension 1.

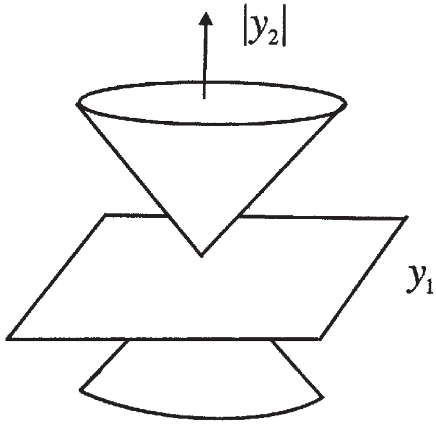

Figure 4

1.8. Proposition (Topology of levels of a Morse function).

(a) *If $c \neq 0$ then the surface $\{g = c\}$ is diffeomorphic to TS^{n-1} (i.e. the tangent bundle to the unit sphere in $\mathbb{R}^n$) and the zero section Δ of this bundle is a cycle generating the reduced homology groups of this surface.*

(b) *Moreover, the space $\{x : 0 \leq g(x) \leq 1\}$ is homotopically equivalent to the space $\{g(x) = 1\} \cup D^n$, where D^n is a ball glued to $\Delta \subset \{g = 1\}$ along the boundary. The deformation retraction of $\{0 \leq g \leq 1\}$ to $\{g = 1\} \cup D^n$ can be realized in such way that the part of $\{0 \leq g \leq 1\}$ outside some neighborhood of 0 is sent to the analogous part of $\{g = 1\}$.*

1.9. Remark. In Chapter 3 below we give definitions of the homology groups and other notions from algebraic topology which will be used in monodromy theory.

Proof of Proposition 1.8. (a) For $n = 1$ this is obvious. For $n = 2$ this follows from Figure 5. Below we present the formulas realizing this diffeomorphism.
Denote $y_1 = u_1 + iv_1$, $y_2 = u_2 + iv_2$, $u = (u_1, u_2)$, $v = (v_1, v_2)$. Assume that $c > 0$. The equation $y_1^2 + y_2^2 = c$ means that

$$u_1^2 + u_2^2 = c + v_1^2 + v_2^2, \ u_1v_1 + u_2v_2 = 0.$$

The latter equation means that the vectors u and v are orthogonal, $\langle u, v \rangle = 0$. The diffeomorphism is

$$(u, v) \to \left(u/\sqrt{c + |v|^2}, v \right).$$

(The first component lies in S^1, the second component lies in the linear subspace of $\mathbb{R}^2$ orthogonal to the first component, i.e. tangent to S^1).

If $c = |c|e^{i\theta} \not> 0$ then we apply the transformation

$$(y_1, y_2) \to (e^{i\theta/2}y_1, e^{i\theta/2}y_2),$$

which is a diffeomorphism, and use the above arguments.

Let $n > 2$. It turns out that the formulas obtained in the previous case are in use in the general situation. Let $y_j = u_j + \sqrt{-1}v_j$, $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$. If $c > 0$ then the equation $y_1^2 + \ldots + y_n^2 = c$ means that

$$|u|^2 = c + |v|^2, \ \langle u, v\rangle = 0.$$

The map

$$(u, v) \to \left(u/\sqrt{c + |v|^2}, v\right)$$

transforms the level surface $\{g = c\}$ to TS^{n-1}, the tangent space to the unit $(n-1)$-dimensional sphere in $\mathbb{R}^n$. We treat the case $\arg c \neq 0$ in the same way as before.

Here also we have the $(n-1)$-dimensional cycle Δ, the preimage of the zero section of this tangent bundle. It generates the reduced homology group of the surface $\{g = c\}$ in dimension $n - 1$. As $c \to 0$ the cycle Δ tends to the critical point.

(b) We note that the space TS^{n-1} is contractible, along fibers, to S^{n-1}. Thus the set $\{0 \leq g \leq 1\}$ is contractible to a disc, the sum of spheres $S_r^{n-1} \subset \{g = r^2\}$ with radii $r \in [0, 1]$. This is the ball D^n from Proposition 1.8(b). Also it is not difficult to construct the deformation retraction as in the thesis of Proposition 1.8(b). □

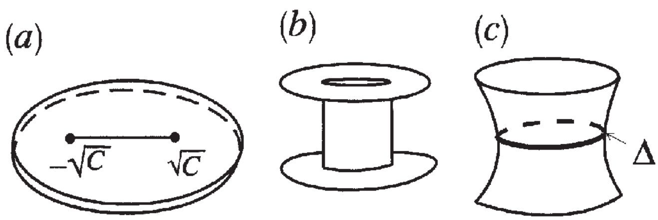

Figure 5

1.10. Definition. The cycle Δ is called the **vanishing cycle.**

Proof of the Morse Lemma. Here I present the proof suggested to me by T. Maszczyk. Firstly we need a multidimensional version of the Hadamard lemma.

The Hadamard lemma. *If $f(x)$, $x \in (\mathbb{C}^n, 0)$ is a germ of an analytic function such that $f(0) = 0$, then $f(x) = \sum x_i g_i(x)$ with analytic functions g_i.*

Let 0 be a non-degenerate critical point of the function f. We can assume that $f(0) = 0$. We apply the Hadamard lemma two times, to f and to the g_i's, and

obtain $f(x) = \sum h_{ij}(x)x_i x_j$ where h_{ij} are analytic functions. Applying the change $(h_{ij}) \to (\frac{1}{2}(h_{ij} + h_{ji}))$ we can assume that the matrix $[h_{ij}(x)]$ is symmetric. It is non-singular for small x (because $2[h_{ij}(0)] = D^2 f(0)$).
Consider the quadratic form

$$\xi \to \Phi(\xi) = \sum h_{ij}(x)\xi_i\xi_j.$$

It is diagonalizable, which means that there exists a linear transformation $(\xi_i) \to (\eta_i = \sum \alpha_{ij}(x)\xi_j)$ such that $\Phi(\xi) = \sum \eta_i^2$. Now it is enough to put $y_i = \sum \alpha_{ij}(x)x_j$.
□

§3 The Morse Theory

Consider the tangent bundle to the sphere TS^n. Let $\Delta = \{(x, v) : v = 0\}$ be its zero section representing a closed n-dimensional cycle. Is it possible to perturb slightly the cycle Δ, to some cycle Δ_1 in such a way that $\Delta \cap \Delta_1 = \emptyset$?
In order to study this problem we reformulate it. We can consider the cycle Δ as a map $S^n \to TS^n$,

$$\Delta : x \to (x, 0).$$

Its perturbation also will be a map from S^n to its tangent bundle of the form

$$\Delta_1 : x \to (y(x), v(x)),$$

where $\sup_x\{|y(x) - x| + |v(x)|\} < \epsilon$. It is clear that we can assume that $y(x) \equiv x$. In such case we have a vector field $\{v(x)\}$ on the sphere and the previous problem follows: can we comb the sphere? (Is there a vector field on S^n non-vanishing at any point?)
If $n = 1$, then the vector field $(x_1, x_2) \to (x_2, -x_1)$provides an example. Generally, if n is odd, then the answer is positive: $v(x_1, x_2, x_3, x_4, \ldots, x_{2k-1}, x_{2k}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{2k}, -x_{2k-1})$.

If n is even, then the answer is negative.

To show this we need some new notions. Let M be a real n–dimensional manifold and let $\{v(x), x \in M\}$ be a vector field on it, $v(x) \in T_x M$.

1.11. Definition. A point $x_0 \in M$ is called a **singular** (or critical or equilibrium) point iff $v(x_0) = 0$. Assume that x_0 is an isolated singular point.
Take a small sphere $S(x_0, \epsilon)$ around x_0 of radius ϵ and consider the map

$$S(x_0, \epsilon) \ni x \overset{\phi}{\longmapsto} \frac{v(x)}{|v(x)|} \in S^{n-1}.$$

The degree of the map ϕ is called the **index** of the singular point x_0 and is denoted by $i_{x_0}v$.

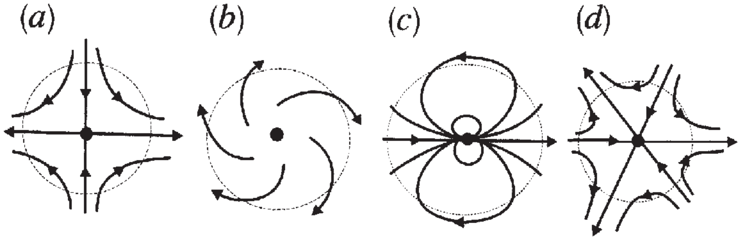

Figure 6

1.12. Remark. If a vector field has non-isolated critical points, then it can be perturbed, in the class of differentiable vector fields, to such with only isolated critical points. It is done using the Sard theorem as follows.
The singular point x_0 is called **degenerate** iff $\det Dv(x_0) = 0$. The non-isolated singular points are degenerate. The degenerate singular points for $v(x)$ are the critical points for the maps $x \to v(x)$. Because the critical values form a set of Lebesque measure zero (the Sard theorem), the vector field $v(x) - w$ for suitably small $w \in \mathbb{R}^n$ does not have degenerate singular points (in a chart of M diffeomorphic to a subset of $\mathbb{R}^n$).

1.13. Remark. The **degree of a map** f between differentiable oriented manifolds M and N of the same dimensions is defined in Chapter 3 below in homological terms. Here we give the analytic definition.
If the map is sufficiently regular then the degree is calculated as follows. Let $y \in N$ and $f^{-1}(y) = \{x_1, \ldots, x_k\}$. Then we have

$$\deg f = \sum_{i=1}^{k} \pm 1,$$

where the sign is $+$, when $Df(x_i)$ preserves the orientation and is $-$, if it reverses the orientation.
If the map f is not regular, then we approximate it by a regular map f_ϵ and put $\deg f = \deg f_\epsilon$.

Examples. For the vector field $\dot{x} = x, \dot{y} = -y$ (or $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$) the index at 0 is -1. For the vector field $\dot{x} = x + y$, $\dot{y} = -x + y$ the index is 1. The examples of vector fields with index 2 and -2 provide the fields $\dot{z} = z^2$ and $\dot{z} = \bar{z}^2$ respectively (written using the complex variable $z = x + iy$).
Generally, for a linear vector field, given by the non-singular matrix A, the index at 0 is $\operatorname{sign}\det A$.
Consider the vector field $\dot{z} = 1$ in $\mathbb{C} \simeq \mathbb{R}^2$. It prolongs itself to a vector field in the Riemann sphere with index at ∞ equal to 2. Indeed, in the variable $\xi = 1/z$ we get $\dot{\xi} = -\xi^2$, which is (up to sign) the same as the vector field from one of the previous examples.

For the vector field $\dot{z} = z$ the index at $z = 0$ is 1 and the index at $z = \infty$ is also 1. Thus the sum of indices is the same for both vector fields and is equal to 2.

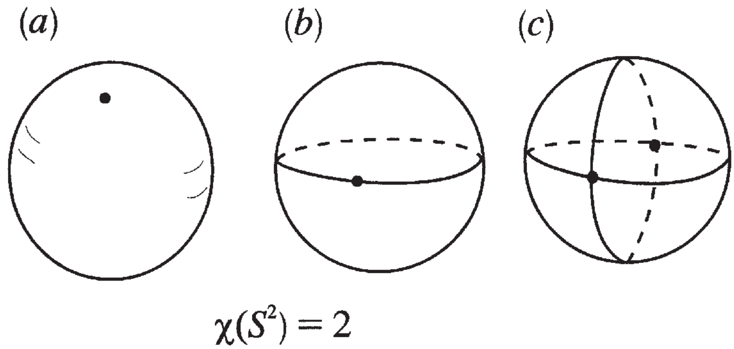

Figure 7

1.14. The Poincaré–Hopf theorem. *If the vector field $v(x)$ has only isolated singular points $x_1, \ldots, x_r$, then*

$$\sum i_{x_j} v = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of the manifold M.

1.15. Remark. The Euler characteristic is calculated as follows. We take a partition of the manifold M into cells (or simplices) and we obtain a complex which consists of m_0 0-dimensional cells, m_1 1-dimensional cells etc. Then we define

$$\chi(M) = m_0 - m_1 + m_2 - m_3 + \ldots \pm m_n.$$

Proof of Theorem 1.14. We sketch the proof of the Poincaré–Hopf theorem based on Morse's Lemma. Let $\mathcal{X}$ be the space of all differentiable vector fields (equipped with some natural topology). Let $\mathcal{X}_0 \subset \mathcal{X}$ consist of vector fields with only **non-degenerate** singular points, i.e. such that $\det Dv(x_i) \neq 0$. Note that then $i_{x_i} v = \mathrm{sign} \det Dv(x_i)$.

The set $\mathcal{X}_0$ is open and dense in $\mathcal{X}$ (see Remark 1.12 above). The function $v \to \sum$(indices) is locally constant at $\mathcal{X}_0$. It is enough to show that it is the same at the boundary of $\mathcal{X}_0$.

Let $v \in \mathcal{X} \setminus \mathcal{X}_0$ be a vector field with degenerate critical points but isolated and with finite indices. We take some small perturbation $v_\epsilon \in \mathcal{X}_0$ of v. The field v_ϵ has only non-degenerate critical points, where some of them may coalesce as $v_\epsilon \to v$. From Figure 8(b) it is seen that the sums of indices of v and of its perturbation are the same; (the index calculated along the outer cycle is equal to the sum of indices calculated along the inner circles).

Therefore it remains to calculate the sum of indices of some particular vector field from $\mathcal{X}_0$.

1.16. Definition. A twice differentiable function $f : M \to \mathbb{R}$ which has only non-degenerate critical points and different critical values is called the **Morse function.**

Assume that M is a Riemannian manifold, i.e. it is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_x$. (If M is compact and smooth, then using the partition of unity 1 can always construct such a metric.) This metric tensor defines the isomorphism $T_x M \ni w \to \langle \cdot, w \rangle \in T_x^* M$. If f is a function on M, then applying the inverse of this isomorphism to $df(x) \in T_x^* M$ we obtain the *gradient vector field* $v(x) = \text{grad} f(x) = \nabla f(x)$. In local coordinates with the euclidean metric, we have $\dot{x}_i = \partial f / \partial x_i$ and $Dv = (\partial^2 f / \partial x_i \partial x_j)$. In particular, the index of the gradient vector field at a critical point x_0 is $\text{sign} \det(\partial^2 f / \partial x_i \partial x_j)$. In the case of general metric $\langle \cdot, \cdot \rangle_x = (A(x) \cdot, \cdot)$ we have $\nabla f = A^{-1} \partial f / \partial x$, and the same formula for index holds.

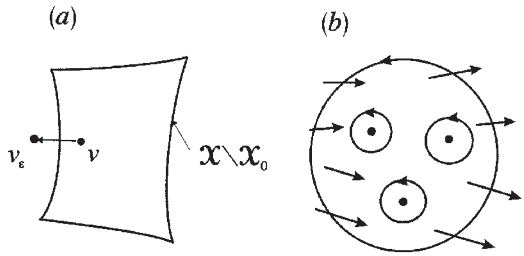

Figure 8

Let $f : M \to \mathbb{R}$ be a Morse function. As the model vector field, for calculating the sum of indices, we take $\nabla f(x)$.

Now we present the Morse theory about determination of the topology of a manifold using its Morse function (see [**Mil1**]). Its main ingredient is the behavior of the level surfaces of the function f in neighborhoods of its critical points.

By the real Morse Lemma it is enough to study bifurcations of the level surfaces for the function

$$f(x) = f(0) + x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2,$$

a constant plus a quadratic form. The **Morse index of quadratic form** is equal to the number of its minuses; we call it the **Morse index of the critical point** of the function f.

We investigate the sets $\{f = c\}$ and $\{f \leq c\}$ as c varies from its minimal value to its maximal value. Altogether we construct some partition of M into cells.

If $k = n$, then the Morse index of the critical point is 0 and we have a local minimum. The sets $\{f \leq c\}$ are discs. We associate with each critical point x_j of index 0 a 0-dimensional cell $\sigma_j^0 = \{x_j\}$ of the promised cell complex.

If $k = n - 1$, then we observe the following bifurcation. Locally the sets $\{f \leq c\}$, $c < f(0)$ consist of two pieces; they are diffeomorphic to $D^n \times S^0$, where D^k denotes the k-dimensional ball with the boundary $\partial D^k = S^{k-1}$. (It is homotopically equivalent to S^0.) After bifurcation, $c > f(0)$, the two components became connected which (up to homotopy) means adding a segment joining the components. With each critical point of Morse index 1 we associate a 1-dimensional cell (a closed connected curve) σ_j^1, which is adjoined to the 0-dimensional skeleton in the way the bifurcation of passing through this critical value says: $\partial\sigma_j^1 = S^0 \subset \{f < f(0) - \epsilon\}$. Generally, near any critical point of Morse index $i = n - k$ the sets $\{f \leq c\}$, $c < f(0)$ are diffeomorphic to $D^{n-i+1} \times S^{i-1} \simeq S^{i-1}$. The bifurcation is equivalent to adjoining to this set the handle $D^{n-i} \times D^i \simeq D^i$. So we add to our complex an i-dimensional cell σ_l^i glued along the boundary to the $(i-1)$-dimensional skeleton (see Figure 9).

Of course, the Euler characteristic of M is equal to the number of cells associated with critical points of index 0 minus the number of cells associated with points of index 1 plus, etc. This completes the proof of the Poincaré–Hopf theorem. □

1.17. The self-intersection of the cycle Δ. We have $\chi(S^n) = 0$ if n is odd, and $= 2$ if n is even. By the Poincaré–Hopf theorem this means that the odd-dimensional spheres can have empty intersections with their deformations in their tangent spaces and the even-dimensional spheres do not have this property.

The sum of indices of a vector field on S^n can be treated as the index of self-intersection of this sphere in its tangent bundle. We proved that this number is equal to

$$(\Delta, \Delta) = 1 + (-1)^n.$$

We shall use these facts in the sequel.

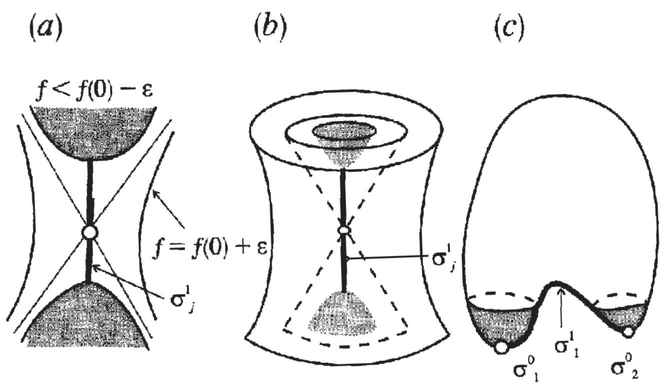

Figure 9

Chapter 2

Normal Forms of Functions

In this chapter we present elements of the theory of singularities of holomorphic functions. We introduce notions of multiplicity, stability, versal deformation, and normal form, and we describe their main properties. We present also the beginning of the list of normal forms for singularities.

This subject is rather standard and well elaborated in many sources. We follow mainly the first volume of the book of V. I. Arnold, A. N. Varchenko and S. M. Gusein-Zade [**AVG**].

§1 Tougeron Theorem

2.1. Notations and definitions. By $\mathcal{O}_{x_0} = \mathcal{O}_{x_0}(\mathbb{C}^n)$ we denote the **local ring** of **germs** at x_0 of holomorphic functions, i.e. functions holomorphic in some neighborhood of x_0. Two functions, f at U and g at V, are equivalent iff $f \equiv g$ at $U \cap V$. Usually we put $x_0 = 0$ and write $\mathcal{O}$ or $\mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$, instead of $\mathcal{O}_0$. It is usual to write $f : (\mathbb{C}^n, 0) \to \mathbb{C}$.

By $\mathfrak{m}$ we denote the **maximal ideal** of the ring $\mathcal{O}$, $\mathfrak{m} = \{f : f(0) = 0\}$. The ideal $\mathfrak{m}$ is generated by $x_1, \ldots, x_n$ (Hadamard's lemma).

By $j^k f = j^k f(0)$, i.e. the k-th **jet** of f, we denote the Taylor series of f up to order k. By J^k we denote the space of k-jets.

The **gradient ideal** of the germ f is generated by $\partial f / \partial x_i$ and is denoted by

$$I_f = (\partial f / \partial x_1, \ldots, \partial f / \partial x_n).$$

The **local algebra** of the germ f is

$$A_f = \mathcal{O} / I_f.$$

The **Milnor number**, or the **multiplicity**, of the germ f is

$$\mu = \dim A_f.$$

Examples. 1. Let $f(x) = x^{n+1}$. Then $I_f = (x^n)$, the set of polynomials with zero first $n-1$ derivatives at $x = 0$. The local algebra is generated by the monomials $1, x, x^2, \ldots, x^{n-1}$ and $\mu(f) = n$. The functions x^{n+1} form the series $\mathbf{A}_n$ of simple singularities (see below).

2. Let $f(x, y) = x^3 + y^4$, i.e. the simple singularity $\mathbf{E}_6$ (see Theorem 2.38 below). Its gradient ideal is generated by x^2 and y^3. In order to calculate the local algebra of this function we present the situation graphically.

At Figure 1(a) we have the lattice $\mathbb{Z}_+^2$ consisting of points of the plane with non-negative integer coordinates (i,j). Each such point represents the monomial $x^iy^j \in \mathcal{O}$. The ideal I_f contains all monomials from the set represented by

$$\left((2,0)+\mathbb{Z}_+^2\right) \cup \left((0,3)+\mathbb{Z}_+^2\right),$$

i.e. we add to the basic points all the uppe-right quarters of $\mathbb{Z}_+^2$. The remaining points from the lattice represent the basis of the local algebra. Its dimension is 6.

3. Let $f(x,y)=x^2y+y^3$, i.e. the simple singularity $\mathbf{D}_4$ (see Theorem 2.38 below). Then the generators of the gradient ideal $(2xy, x^2+3y^2)$ are represented by: the point $(1,1)$ and by two points $(2,0)$, $(0,2)$ which are associated one with another (see Figure 1(b)).

Of course, I_f contains $(1,1)+\mathbb{Z}_+^2$. It is also clear that the monomials represented by $(0,0),(1,0),(0,1)$ are outside I_f and form a part of the basis of the local ring A_f.

The two points $(2,0),(0,2)$ cannot lie simultaneously in the ideal as well as cannot be simultaneously outside of it, (they are dependent in A_f). So we add one of them, e.g. $(0,2)$, to the basis of A_f. Considering the quadratic parts of the Taylor expansions of the functions from our (preliminary) basis and from the ideal I_f, we see that the monomials $(0,0),(1,0),(0,1),(0,2)$ are independent in A_f. The rest is in the gradient ideal, which means that $J=\mathbb{C}+x\mathbb{C}+y\mathbb{C}+y^2\mathbb{C}+I_f=\mathcal{O}$.

To prove this it is enough to show that the monomials x^i, y^j are in J. But $x^2 = (x^2+3y^2)-3(y^2) \in I_f + y^2\mathbb{C}$ and $x^i = x^{i-2}(x^2+3y^2)+3x^{i-3}y(xy) \in I_f$. Similarly we treat the monomials y^j.

Therefore $\mu(f)=4$.

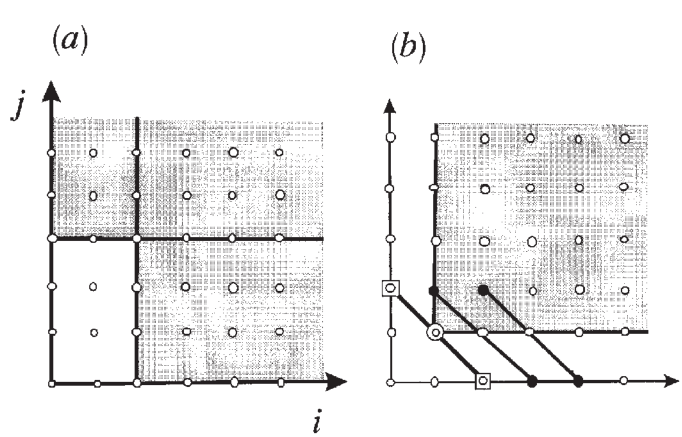

Figure 1

4. *Problem*: show that $\mu(x^2y+x^{k-1})=k$.

2.2. Theorem (Isolated critical points). *The Milnor number* $\mu < \infty$ *iff* $x = 0$ *is an isolated critical point of the function* f.

Parallel with the multiplicity of a function one can define the multiplicity of germs of vector fields.
Let $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, $F = (f_1, \dots, f_n)$ be a germ of a holomorphic map. Let $A_F = \mathcal{O}/(f_1, \dots, f_n)$ be the local algebra of the germ F. Then $\mu(F) = \dim A_F$ is the multiplicity of the germ F.

2.3. Theorem. *The multiplicity* $\mu = \dim A_F < \infty$ *iff* $x = 0$ *is an isolated solution of the equation* $F = 0$.

2.4. Theorem (Index and multiplicity). *When we treat* F *as a vector field, then* $\mu(F) = i_0 F$ *where* $i_0 F$ *is the index of the singular point* $x = 0$ *of* F.

The above two theorems are proved in the next section.

2.5. Theorem of Tougeron. *Let* f *be a germ of a holomorphic function such that* $\mu = \mu(f) < \infty$. *Then there exists an analytic change of variables* $y = h(x)$ *such that* $f \circ h = j^{\mu+1} f(0)$.

2.6. Remark. A jet $j^k f$ is called **sufficient** iff any two germs with this k-th jet are analytically equivalent. It means stability with respect to high order perturbations. The theorem of Tougeron says that the jet $j^{\mu+1} f$ is sufficient.

Proof of Theorem 2.5. Unfortunately here we cannot repeat the proof of the Morse lemma. We follow the book of Arnold, Varchenko and Gusein-Zade [**AVG**].
Assume that f has a critical point at 0 of multiplicity μ and let $\phi \in \mathfrak{m}^{\mu+2}$. We shall show that $f + \phi \sim f$.
We use the *homotopy method.* Namely we join the functions f and $f + \phi$ by an arc in a functional space of functions and we seek a one-parameter family of diffeomorphisms realizing equivalences with f. In other words, we try to solve the equation

$$(f + t\phi) \circ h_t(x) \equiv f(x), \; t \in [0, 1], \tag{1.1}$$

where h_t is unknown.
Introduce the non-autonomous vector field $v_t(x)$ by the formula

$$dh_t/dt = v_t(h_t(x)).$$

We shall find the vector field v_t first and then, integrating the latter equation, we shall find the diffeomorphisms h_t.
Differentiating (1.1) with respect to t we get the equation $\phi \circ h_t + (f + t\phi)_* \cdot v_t \circ h_t \equiv 0$. Thus we have to solve the equation

$$(f + t\phi)_* \cdot v_t = -\phi \tag{1.2}$$

with respect to v_t.

2.7. Lemma. *We have* $\mathfrak{m}^\mu \in (\partial(f+t\phi)/\partial x_1, \dots, \partial(f+t\phi)/\partial x_n)$, *which means that any monomial of sufficiently high degree lies in the gradient ideal of the function* $f + t\phi$.

Example. For non-degenerate critical point the gradient ideal coincides with the maximal ideal and $\mu = 1$.

From Lemma 2.7 the theorem of Tougeron follows. Indeed, because $\phi \in \mathfrak{m}^{\mu+2}$, the equation (1.2) has solution $v_t = \sum v_{t,i} \frac{\partial}{\partial x_i}$. Its components $v_{t,i} \in \mathfrak{m}^2$ and hence $v_t(0) = 0$, $Dv(0) = 0$. Moreover v_t depends smoothly on t.
So, in order to find the family of diffeomorphisms h_t, it is sufficient to solve the initial value problem

$$\frac{d}{dt} h_t = v_t(h_t(x)), \; h_0 = Id.$$

The assumption $\phi \in \mathfrak{m}^{\mu+2}$ is needed to ensure the existence and uniqueness of solutions to the latter problem. Because $v_t(0) = 0$, we get $h_t(0) = 0$. □

Proof of Lemma 2.7. Consider firstly the case $\phi \equiv 0$. Let $\phi_1, \dots, \phi_\mu \in \mathfrak{m}$. It is enough to show that $\phi_1 \cdot \ldots \cdot \phi_\mu \in I_f$ where I_f is the gradient ideal.
Consider the series of functions: $\phi_0 = 1$, ϕ_1, $\phi_1\phi_2, \dots$, $\phi_1 \dots \phi_\mu$. They are linearly dependent in the local algebra A_f. So, we have

$$c_0 + c_1\phi_1 + c_2\phi_1\phi_2 + \ldots + c_\mu \phi_1 \ldots \phi_\mu \in I_f$$

for some constants c_j. If c_r is the first nonzero coefficient, then $\phi_1 \dots \phi_r(c_r + \dots) \in I_f$, or $\phi_1 \dots \phi_r \in I_f$. Of course, in this case the product of all ϕ_i's also lies in the gradient ideal.
Consider now the general case $\phi \not\equiv 0$. Let $M_1, \dots, M_r$ be all the homogeneous monomials of degree μ; they form a basis in the space of homogeneous polynomials of degree μ. We know already that $M_j \in I_f$. This means that

$$M_j \quad = \sum \tfrac{\partial f}{\partial x_i} h_{ij} = \sum \tfrac{\partial (f+t\phi)}{\partial x_i} h_{ij} - \sum \tfrac{\partial t\phi}{\partial x_i} h_{ij}.$$

The last sum in the above formulas belongs to $\mathfrak{m}^{\mu+1}$ and can be expressed by means of the monomials M_j (Hadamard's lemma). We get

$$M_j = \sum \frac{\partial (f + t\phi)}{\partial x_i} h_{ij} - t \sum_k M_k \sum_l x_l a_{kl}(x)$$

where $a_{kl} \in \mathfrak{m}$ (by the assumption about ϕ). We can rewrite this system of equations in the matrix form

$$(I - tA)M = B$$

where tA is a small matrix and the components of the vector B belong to the gradient ideal. Because the matrix $I - tA$ is invertible, also the components M_i of the vector M are in this ideal. □

2.8. Corollary *Any germ of a function of finite multiplicity can be replaced by an equivalent polynomial.*

§2 Deformations

We have to introduce some notions concerning actions of infinite-dimensional groups on infinite-dimensional functional spaces. So, firstly we demonstrate them in the finite-dimensional case.
Let M be a manifold (of finite dimension for a while) and let a group G act on it: $(f, g) \to gf$, $f \in M$, $g \in G$. Let $f \in M$. We denote its orbit by $Gf = \{gf : g \in G\}$.

2.9. Definition ([Arn2]). A **deformation** of f is a map $F : \Lambda \to M$, where Λ is the **base** of the deformation with a distinguished point 0 and $F(0) = f$.
Two deformations F, F' are **equivalent** iff there is a family $g(\lambda) \in G$, $\lambda \in \Lambda$, such that

$$F'(\lambda) = g(\lambda)F(\lambda),$$

i.e. the equivalence along the orbits.
If $\phi : (\Lambda', 0) \to (\Lambda, 0)$ is a map between the base spaces, then the **induced deformation** (from F by means of ϕ) is

$$\phi^* F(\lambda') = F(\phi(\lambda')),$$

i.e. a change of parameters.
A deformation F (of f) is called **versal** iff any other deformation of f is equivalent to a deformation induced from the deformation F.
A deformation is called **mini-versal** iff it is versal and the dimension of its base is minimal.

We can say that a deformation is versal iff it intersects all orbits near f (see Figure 2). In particular, the deformation with the base M and identity map is versal; but usually is not mini-versal.

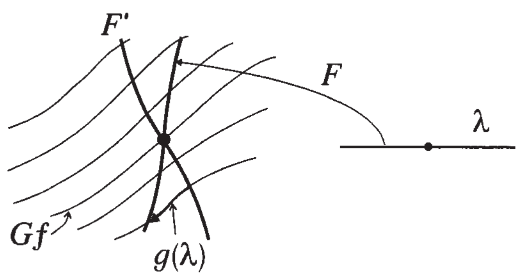

Figure 2

In the singularity theory we deal with the infinite-dimensional situation. The role of the manifold M is played by the space of germs $f = f(x)$ of holomorphic functions and the role of G is played by the group of local analytic diffeomorphisms $h = h(x)$ acting on functions by compositions on the right; it is called the *right equivalence*. However the definitions from 2.9 pass to the infinite-dimensional case unchanged.

A deformation of a germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ is a germ $F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to \mathbb{C}$, $F(x, 0) = f(x)$. The equivalence of two deformations is written as $F'(x, \lambda) = F(h(x, \lambda), \lambda)$ (where $h(\cdot, \lambda) = h_\lambda$ is a family from G) and the induced deformation is given by $F'(x, \lambda') = F(x, \phi(\lambda'))$.

2.10. Definition. We say that a germ f is **stable** iff the orbit of f contains a whole neighborhood of f. We say that a germ f is **simple** iff a neighborhood of f is covered by a finite number of orbits. If a neighborhood of f is covered by l-parameter families of orbits such that $\max l = m$, then we say that the germ f is m-**modal**.
By a **normal form** we mean some (simultaneous) choice of a member from each orbit. This choice is not unique, so one should do it in a way as natural as possible.

2.11. Remarks. (a) In [**AVG**] singularities of other objects are considered: of maps from $\mathbb{C}^n$ to $\mathbb{C}^m$ with the so-called left-right equivalence (when we can make independent changes in the source space and in the target space) and with respect to the so-called V-equivalence (when the change in the target space is linear and depends on x). There analogous definitions (as in the case of functions) are introduced and analogous results are obtained.
(b) The singularity theory is used not only in local analysis. Usually one has a function on a manifold, where it has a finite number of critical points. During deformation of a function the critical points also can move with the parameter. For example $f_\epsilon(x) = x^2 - \epsilon x$ has a critical point at $\epsilon/2$. Therefore it is reasonable to keep some neighborhood of a critical point fixed during the deformation.

The notions of stability and versality have their infinitesimal versions. The infinitesimal stability is obtained from differentiation of the equality $(f + t\phi)(x) = f \circ h_t(x)$ with respect to t at $t = 0$

$$\phi(x) = \sum \frac{\partial f}{\partial x_i} v_i(x). \tag{2.1}$$

2.12. Definition of infinitesimal stability. The germ f is **infinitesimally stable** iff the equation (2.1) has solution (v_i) for every ϕ.

In particular, the proof of the theorem of Tougeron is a proof of the implication: infinitesimal stability $\Rightarrow$ stability (i.e. stability with respect to perturbations of high order).
In fact, the notion of stability and of its infinitesimal version has greater application in the theory of maps, e.g. the Whitney singularities of planar maps (see [**AVG**]):
$(x, y) \to (x^2, y)$ (the **fold**),
$(x, y) \to (x^3 + xy, y)$ (the **cusp**).

Let us differentiate the equation

$$F'(x, \lambda') = F(g(x, \lambda'), \phi(\lambda')),$$

$F' = f(x) + \lambda'\alpha(x)$, $g(x, 0) = x$ with respect to $\lambda' \in \mathbb{C}$. We get

$$\alpha(x) = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i} v_i(x) + \sum_{j=1}^{k} \frac{\partial F}{\partial \lambda_j} c_j. \tag{2.2}$$

2.13. Definition of infinitesimal versality. We say that a deformation $F(x, \lambda)$ is **infinitesimally versal** iff the equation (2.2) has solution $v_i(x) \in \mathcal{O}$, $c_j \in \mathbb{C}$ for any function $\alpha(x) \in \mathcal{O}$.

2.14. Theorem (Versal deformations). *Any deformation infinitesimally versal is versal.*

Examples. 1. The deformation $F(x, \lambda) = x^2 + \lambda$ is versal because the equation $\alpha(x) = 2xv(x) + c$ has the solution $v(x) = (\alpha(x) - \alpha(0))/2x$, $c = \alpha(0)$.
2. Similarly the deformation $F = x^3 + \lambda_1 x + \lambda_2$ is versal.
3. Generally, if $e_1(x), \ldots, e_\mu(x)$ define a basis of the local algebra A_f of the germ f, then the deformation

$$F(x, \lambda) = f(x) + \lambda_1 e_1(x) + \ldots + \lambda_\mu e_\mu(x)$$

is versal. It is also mini-versal deformation.

2.15. Corollary. *For any germ of finite multiplicity we can choose the function as well as the mini-versal deformation in polynomial forms.*

Remark. In the theory of singularities of functions and maps, theorems about reductions (to a sufficient jet or to a normal form) are formulated in the analytic versions. The corresponding changes of variables are analytic. In particular, the formal classification (reduction by means of formal power series) coincides with the analytic classification of singularities.
As the reader will see this is not the case in differential equations theory and in dynamical systems theory. Very often power series, which reduce some singularity of a vector field or of a diffeomorphism, diverge.

Proof of Theorem 2.14. This proof relies mostly on local algebra.
Let $F(x, \lambda)$ be an infinitesimal deformation of a germ f and let $F'(x, \lambda')$, $\lambda' \in (\mathbb{C}^{k'}, 0)$ be another deformation of f.
We apply a certain trick which allows us to reduce the problem to the case, when F' is a deformation of F with one parameter. Take the function

$$\widetilde{F}(x, \lambda, \lambda') = F(x, \lambda) + F'(x, \lambda') - f(x).$$

It is a deformation of f with parameters (λ, λ') as well as a deformation of F with the parameter λ'.
Any extension of an infinitesimally versal deformation is an infinitesimally versal deformation. Consider the chain

$$\mathbb{C}^k \subset \mathbb{C}^{k+1} \subset \ldots \subset \mathbb{C}^{k+k'}$$

of spaces, which define a chain of deformations with one parameter. Now step-by-step we show equivalence of each of these deformations with some deformations induced from the previous one.

Consider therefore the following special case

$$\Phi(x, \lambda, \mu), \quad \lambda \in \mathbb{C}^l, \ \mu \in \mathbb{C}; \ \ \Phi(x, \lambda, 0) = F(x, \lambda).$$

2.16. Proposition. *The deformation Φ is equivalent to a deformation induced from F.*

Proof. The property that Φ is equivalent to a deformation induced from F can be formulated as follows:

$$\Phi(g_\mu(x, \lambda), \phi_\mu(\lambda), \mu) \equiv F(x, \lambda), \tag{2.3}$$

where $h_\mu(x, \lambda) = (g_\mu(x, \lambda), \phi_\mu(\lambda)$ is a 1-parameter family of local diffeomorphisms. (Apply h_μ^{-1} to (2.3) and you obtain the definition from Definition 2.9). The family h_μ defines the non-autonomous vector field $dh_\mu/d\mu = V_\mu \circ h_\mu$,

$$V_\mu = \sum H_j(x, \lambda, \mu)\frac{\partial}{\partial x_j} + \sum \xi_i(\lambda, \mu)\frac{\partial}{\partial \lambda_i}$$

in $\mathbb{C}^n \times \mathbb{C}^l$, analogously as in the proof of the Tougeron theorem.
Differentiating the identity (2.3) with respect to μ, we get the equation

$$\frac{\partial \Phi}{\partial \mu} + \sum H_j \frac{\partial \Phi}{\partial x_j} + \sum \xi_i \frac{\partial \Phi}{\partial \lambda_i} \equiv 0.$$

As in the proof of the Tougeron theorem the problem reduces to that of solving the equation

$$\alpha(x, \lambda; \mu) = H(x, \lambda; \mu) \cdot \frac{\partial \Phi}{\partial x} + \Xi(\lambda; \mu) \cdot \frac{\partial \Phi}{\partial \lambda} \tag{2.4}$$

for any α.
The assumption of Theorem 2.14, i.e. the infinitesimal versality, ensures existence of a solution to the equation (2.4) for $\lambda = 0, \mu = 0$. We need to extend this solution to a solution of the equation (2.4) in the general case.
In order to pass from a particular solution to a general solution we need some preparation theorems. There are three such theorems: the *Weierstrass Preparation Theorem,* the *Division Theorem* and the *Thom–Martinet Preparation Theorem.* What we need is the Thom–Martinet Preparation Theorem for modules over local rings of holomorphic functions.

2.17. Thom–Martinet Preparation Theorem. *Let $(x, y) \in \mathbb{C}^n \times \mathbb{C}^k$, $\mathcal{O}_{n+k} = \mathcal{O}_0(\mathbb{C}^n \times \mathbb{C}^k)$, $\mathcal{O}_k = O_0(\mathbb{C}^k)$, $\mathcal{O}_n = \mathcal{O}_0(\mathbb{C}^n)$. Let $I \subset \mathcal{O}_{n+k}$ be an ideal and denote $I_{x,0} = \{f(x, 0) : \ f \in I\}$.*

If some elements $e_1, \dots, e_r \in \mathcal{O}_{n+k}$ are such that the functions $e_i(x,0)$ generate the module $\mathcal{O}_n/I_{x,0}$ (over $\mathbb{C}$), then the functions e_i generate the module $\mathcal{O}_{n+k}/I$ over $\mathcal{O}_k$.
In other words, for any $\alpha \in \mathcal{O}_{n+k}$ there exist germs $g_i(y)$ such that

$$\alpha(x,y) = \sum g_i(y) e_i(x,y) \ (\mathrm{mod}\ I).$$

Finishing the proof of Theorem 2.14. We put $y = (\lambda; \mu)$, $I = (\frac{\partial \Phi}{\partial x_1}, \dots, \frac{\partial \Phi}{\partial x_n})$, $e_i = \frac{\partial \Phi}{\partial \lambda_i}$ in Theorem 2.17. Its thesis says that the equation (2.4) has a solution in the class of germs of analytic functions. This gives Proposition 2.16 and then Theorem 2.14. □

Now we make some moves in the direction of the proof of Theorem 2.17. For this we need two other preparation theorems.

2.18. Weierstrass Preparation Theorem. *Let $f(z_1, \dots, z_m; w) = f(z,w)$, $w \in \mathbb{C}$ be a germ of a holomorphic function such that $f(0,w) = w^n + \dots$. Then there exist a holomorphic function $h(z,w)$, $h \neq 0$ and holomorphic functions $a_1(z), \dots, a_n(z)$ such that*

$$f = gh, \quad g(z,w) = w^n + a_1(z) w^{n-1} + \dots + a_n(z).$$

The function g is called the Weierstrass polynomial.

Proof. If we denote by $b_i(z)$ the zeroes of the function f, then we have the representation $f = h \prod (w - b_i(z))$, $h \neq 0$. The coefficients $a_q(z)$ of the Weierstrass polynomial are symmetric polynomials of the zeroes b_i. Moreover the ring of symmetric polynomials is generated by the sums of powers of b_i. The latter are given by the formulas

$$b_1^q + \dots + b_n^q = \frac{1}{2\pi i} \oint_{|w| = const} w^q \cdot \frac{\partial f / \partial w}{f} dw,$$

where the subintegral function is holomorphic in (z,w), if $|w|$ is sufficiently small. Therefore a_q and g are holomorphic functions.
The analyticity of the function h follows from the formula

$$h = \frac{1}{2\pi i} \oint_{|u| = const} \frac{h(z,u) du}{u - w} = \frac{1}{2\pi i} \oint_{|u| = const} \frac{(f/g) du}{u - w},$$

where the subintegral function is holomorphic for small $|w|$ and $|z|$. □

2.19. Division Theorem. *Let $f(z,w)$ be as in Theorem 2.18. Then for any germ $\phi(z,w)$ of a holomorphic function there exist holomorphic germs $h(z,w)$ and $h_i(z)$, $i = 0, \dots, n-1$ such that*

$$\phi = hf + \sum_0^{n-1} h_i(z) w^i.$$