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S.A. Grigoryan
T.V. Tonev

Shift-invariant Uniform Algebras on Groups

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Authors:

Suren A. Grigoryan
Chebotarev Institute for
Mathematics and Mechanics
Kazan State University
Universitetskaya 17
Kazan 420008, Tatarstan
Russia
e-mail: Suren.Grigorian@ksu.ru

Thomas V. Tonev
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812-0864
USA
e-mail: TonevTV@mso.umt.edu

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Preface

Shift-invariant algebras are uniform algebras of continuous functions defined on compact connected groups, that are invariant under shifts by group elements. They are outgrowths of generalized analytic functions, introduced almost fifty years ago by Arens and Singer, and are the central object of this book. Associated algebras of almost periodic functions of real variables and of bounded analytic functions on the unit disc are also considered and carried along within the shift-invariant framework. The adopted general approach leads to non-standard perspectives, never-asked-before questions, and unexpected properties.

The book is based mainly on our quite recent, some even unpublished, results. Most of its basic notions and ideas originate in [T2]. Their further development, however, can be found in journal or preprint form only.

Basic terminology and standard properties of uniform algebras are presented in Chapter 1. Associated algebras, such as Bourgain algebras, polynomial extensions, and inductive limit algebras are introduced and discussed. At the end of the chapter we present recently found conditions for a mapping between uniform algebras to be an algebraic isomorphism. In Chapter 2 we give fundamentals, various descriptions and standard properties of three classical families of functions – almost periodic functions of real variables, harmonic functions, and H^p-functions on the unit circle. Later on, in Chapter 7, we return to some of these families and extend them to arbitrary compact groups. Chapter 3 is a survey of basic properties of topological groups, their characters, dual groups, functions and measures on them. We introduce also the instrumental for the sequel notion of weak and strong hull of a semigroup.

Chapter 4 is devoted to shift-invariant algebras. We describe the spaces of automorphisms and of peak subgroups of shift-invariant algebras, and show that the algebraic properties of the generating semigroup S have a significant impact on the properties of the associated shift-invariant algebra A_S. For example, whether analogues of the classical Radó's theorem for null-sets of analytic functions, and of Riemann's theorem for removable singularities hold in a shift-invariant algebra A_S depends on specific algebraic properties of the generating semigroup S. Asymptotically almost periodic functions on $\mathbb{R}$, which share many properties with almost periodic functions, are introduced at the end of the chapter. Extendability of linear multiplicative functionals from smaller to larger shift-invariant algebras is the focal point of Chapter 5. The subject is naturally related with the extendability of non-negative semicharacters from smaller to larger semigroups and, equivalently, of their logarithms, called also additive weights. We give necessary and sufficient conditions for extendability of individual weights, as well as of the entire family of weights on a semigroup. These conditions imply various corona-type theorems. For instance, if S is a semigroup of $\mathbb{R}$ containing the origin, then the algebra of almost periodic functions in one real variable with spectrum in S does not have a $\mathbb{C}_+$-corona if and only if all non-negative semicharacters on S are monotone decreasing, or equivalently, if and only if the strong hull of S coincides with the positive half of the group envelope of S. On the other hand, the same conditions imply necessary and sufficient conditions for the related subalgebra of bounded analytic functions on the unit disc $\mathbb{D}$ to possess a $\mathbb{C}_+$-corona and a $\mathbb{D}$-corona. In Chapter 6 we discuss big disc algebras of generalized analytic functions on a compact abelian group G, an important class of shift-invariant algebras, also known as G-disc algebras. We describe their Bourgain algebras, orthogonal measures and primary ideals.

In Chapter 7 we extend the notion of harmonic and H^p-functions to compact abelian groups, and present corresponding Fatou-type theorems. In Chapter 8 we utilize inductive limits of classical algebras to study and generalize shift-invariant algebras on G-discs. In particular, we show that any sequence Φ of inner functions on the unit disc generates an inductive limit algebra, $H^\infty(\mathcal{D}_\Phi)$, of so called Φ-hyper-analytic functions on the associated big disc $\mathcal{D}_\Phi$. They are generalizations of hyper-analytic functions from [T], and similarly to them do not have a G-disc-corona, i.e. there exists a standard dense embedding of the big disc $\mathcal{D}_\Phi$ into the maximal ideal space of $H^\infty(\mathcal{D}_\Phi)$. We introduce also the class of Blaschke algebras, which are inductive limits of sequences of disc algebras connected with finite Blaschke products.

The selection of topics depended entirely on our own research interests. Many other related topics could not be included, or even mentioned. All chapters are provided with historical notes, references, brief remarks, comments, and unsolved problems. We do not necessarily claim credit for any uncited result. It may be an immediate consequence of previous assertions, or, part of the common mathematical knowledge, or, may have a history difficult to be traced.

The book is addressed primarily to those interested in analytic functions and commutative Banach algebras, though it could be useful to a wide range of research mathematicians and graduate students, familiar only with the fundamentals of complex and functional analysis.

Over the years our thinking in the area has been stimulated and encouraged by discussions and communication with several experts, among which we would like to mention Hugo Arizmendi, Richard Aron, Andrew Browder, Joseph Cima, Brian Cole, Joseph Diestel, Evgeniy Gorin, Farhad Jafari, Krzysztof Jarosz, Paul Muhly, Rao Nagisetty, Scott Saccone, Sadahiro Saeki, Anatoly Sherstnev, Andrzej Sołtysiak, Edgar Lee Stout, John Wermer, and Wiesław Żelazko. Special thanks are due to the participants – current and former – of the Analysis seminar at the University of Montana: Gregory St.George, Karel Stroethoff, Elena Toneva, George Votruba, and Keith Yale for their encouragement and support. We also mention with pleasure and gratitude the contribution of our students Tatyana Ponkrateva from Kazan State University, Aaron Luttman and John Case from the University of Montana, and especially Scott Lambert, who read the entire text and suggested many improvements.

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Chapter 1

Banach algebras and uniform algebras

In this chapter we present a part of the uniform algebra theory we will need, including several important algebraic constructions. Basic notations, terminology, and selected auxiliary results concerning commutative Banach algebras and uniform algebras are presented in the first two sections. The inductive and projective limits of algebras, introduced in more detail, are very convenient tools for describing the structure and revealing the hidden features of specific uniform algebras. Bourgain algebras and polynomial extensions provide powerful methods for constructing new classes of algebras. Further we discuss isomorphisms and homomorphisms between uniform algebras.

1.1 Commutative Banach algebras

A *Banach space* B over the field of complex numbers $\mathbb{C}$ is a linear space over $\mathbb{C}$ (thus, in B there are defined two operations — addition, and multiplication by complex scalars) which is provided with a *norm*, i.e. a non-negative function $\| \,.\, \|\colon\ B \longrightarrow \mathbb{R}_+ = [\,0, \infty)$ with the following properties:

(i) $\|\lambda a\| = |\lambda| \|a\|$ for each $a \in B$ and any complex scalar $\lambda \in \mathbb{C}$.

(ii) $\|a + b\| \leq \|a\| + \|b\|$ for each $a, b \in B$.

(iii) 0 is the only element in B whose norm is zero.

(iv) B is a *complete* space with respect to the topology generated by the norm $\| \,.\, \|$.

By completeness we mean that every Cauchy sequence $\{a_n\}_{n=1}^{\infty}$ of elements in B is convergent.

A Banach space B over $\mathbb{C}$ is called a *Banach algebra*, if B is provided with an associative operation (called *multiplication*) which is distributive with respect to addition, and if the inequality

(v) $\|ab\| \leq \|a\|\|b\|$

holds for every $a, b \in B$. A Banach algebra is *commutative* if its multiplication is commutative, and *with unit* if it possesses a unit element with respect to multiplication (denoted usually by e, or, by 1) such that

(vi) $\|e\| = 1$.

Let B be a commutative Banach algebra with unit. An element f in B is said to be *invertible* if there exists a g in B such that $fg = e$. The element g with this property is uniquely defined. It is denoted by f^{-1} and is called the *inverse element* of f. Hence we have $f^{-1}f = e$ for any invertible element f in B. The set B^{-1} of all invertible elements of B under multiplication is a subgroup of B. A simple example of a commutative Banach algebra with unit is the set of complex numbers $\mathbb{C}$.

Proposition 1.1.1. *Let B be a commutative Banach algebra with unit e. Every element of the open unit ball centered at e is invertible, i.e.*

$$\{h \in B\colon \|h - e\| < 1\} \subset B^{-1}.$$

Proof. Let $\|f\| < 1$, and let $g_n = \sum_{k=0}^{n} f^k$, where $f^0 = e$. If $m < n$, then by (ii) and (v) from the above we have that

$$\begin{aligned}\|g_n - g_m\| &= \Big\| \sum_{k=m+1}^{n} f^k \Big\| \leq \sum_{k=m+1}^{n} \|f^k\| \leq \sum_{k=m+1}^{n} \|f\|^k \\ &= \frac{\|f\|^{m+1} - \|f\|^{n+1}}{1 - \|f\|} \leq \frac{\|f\|^{m+1}}{1 - \|f\|}.\end{aligned}$$

Hence for any $\varepsilon > 0$ and n, m big enough, we have $\|g_n - g_m\| < \varepsilon$, since by $\|f\| < 1$ we have $\lim_{k\to\infty} \|f^{k+1}\| \leq \lim_{k\to\infty} \|f\|^{k+1} = 0$. Thus, $\{g_n\}$ is a Cauchy sequence, and by the completeness of B it converges to an element $g \in B$, i.e. $g = \lim_{n\to\infty} g_n = \sum_{k=0}^{\infty} f^k$.

In addition,

$$\begin{aligned}g\,(e - f) &= \Big(\sum_{n=0}^{\infty} f^n\Big)(e - f) = \Big(\lim_{k\to\infty} \sum_{n=0}^{k} f^n\Big)(e - f) \\ &= \lim_{k\to\infty} \sum_{n=0}^{k} (f^n - f^{n+1}) = \lim_{k\to\infty} (e - f^{k+1}) = e - \lim_{k\to\infty} f^{k+1} = e,\end{aligned}$$

since $\lim_{k\to\infty} \|f\|^{k+1} = 0$. Hence $e - f$ is an invertible element of B, as claimed. □

Definition 1.1.2. The *spectrum* of an element f in a commutative Banach algebra B is the set

$$\sigma(f) = \{\lambda \in \mathbb{C}\colon \lambda e - f \notin B^{-1}\}. \tag{1.1}$$

Corollary 1.1.3. *The spectrum $\sigma(f)$ is contained in the disc $\overline{\mathbb{D}}(\|f\|) = \{z \in \mathbb{C}\colon |z| \leq \|f\|\}$ with radius $\|f\|$, centered at* 0.

Proof. Given an $f \in B$, let s be a complex number with $|s| > \|f\|$. Let $g = f/s = (1/s)f$. By the hypothesis $\|g\| = \|f\|/|s| < 1$. Proposition 1.1.1 implies that the element $e - g$ is invertible, and its inverse element is the sum of the convergent series $\sum_{n=0}^{\infty} g^n$. Thus

$$\begin{aligned} e &= (e-g)\sum_{n=0}^{\infty} g^n = (e - f/s)\sum_{n=0}^{\infty} f^n/s^n \\ &= \big((se-f)/s\big)\sum_{n=0}^{\infty} f^n/s^n = (se-f)\sum_{n=0}^{\infty} f^n/s^{n+1}. \end{aligned}$$

Hence $se - f$ is invertible in B. Therefore, $s \notin \sigma(f)$ whenever $|s| > \|f\|$. Consequently, $\sigma(f) \subset \overline{\mathbb{D}}(\|f\|)$, as claimed. □

Corollary 1.1.3 implies that the spectrum of any element f in B is a bounded set in $\mathbb{C}$, and therefore $\mathbb{C} \setminus \sigma(f) \neq \emptyset$. One can see that B^{-1} is an open subset of B, and the correspondence $f \longmapsto f^{-1}$ is a homeomorphism of B^{-1} onto itself. More precisely, B^{-1} is an open group (under multiplication) in B, and the mapping $f \longmapsto f^{-1}\colon B^{-1} \longrightarrow B^{-1}$ is a group automorphism. The spectrum $\sigma(f)$ is a closed and bounded set, thus a compact subset of $\mathbb{C}$. The number

$$r_f = \max\big\{|z|\colon z \in \sigma(f)\big\}$$

is called the *spectral radius* of $f \in B$. Since $r_f \leq \|f\|$, we have $\sigma(f) \subset \overline{\mathbb{D}}(r_f) \subset \overline{\mathbb{D}}(\|f\|)$. The spectral radius r_f can be expressed explicitly in terms of f (e.g.[G1, S4, T2]). Namely,

$$r_f = \lim_{n\to\infty} \sqrt[n]{\|f^n\|} \leq \lim_{n\to\infty} \sqrt[n]{\|f\|^n} = \|f\|. \tag{1.2}$$

Definition 1.1.4. The *peripheral spectrum* of an element f in a commutative Banach algebra B is the set

$$\sigma_\pi(f) = \big\{z \in \sigma(f) : |z| = r_f\big\} = \sigma(f) \cap \mathbb{T}_{r_f}. \tag{1.3}$$

Any commutative Banach algebra B with unit admits a natural representation by continuous functions on a compact topological space. An important role in this representation, as well as in commutative Banach algebra theory in general, is played by complex-valued homomorphisms, i.e. linear multiplicative functionals of the algebra. A *linear multiplicative functional* of B is called any non-zero complex-valued function φ on B with the following properties:

(i) $\varphi(\lambda a + \mu b) = \lambda\,\varphi(a) + \mu\,\varphi(b)$,

(ii) $\varphi(ab) = \varphi(a)\,\varphi(b)$

for every $a,\ b \in B$, and all scalars $\lambda, \mu \in \mathbb{C}$. The set $\mathcal{M}_B$ of all non-zero linear multiplicative functionals of B is called the *maximal ideal space* (or, the *spectrum*) of B.

For a fixed $a \in B$ with $\varphi(a) \neq 0$ we have $\varphi(a) = \varphi(ea) = \varphi(e)\,\varphi(a)$, thus $\varphi(a)\big(\varphi(e) - 1\big) = 0$. Consequently, $\varphi(e) = 1$ for every linear multiplicative functional φ of B. Since $aa^{-1} = e$ for every $a \in B^{-1}$, we have $1 = \varphi(e) = \varphi(aa^{-1}) = \varphi(a)\,\varphi(a^{-1})$, thus $\varphi(a) \neq 0$ for every invertible element $a \in B$.

Lemma 1.1.5. *Every linear multiplicative functional $\varphi \in \mathcal{M}_B$ is continuous on B, and $\|\varphi\| = 1$.*

Proof. Let $f \in B$, and let $|z| > \|f\|$ for some $z \in \mathbb{C}$. Hence, $ze - f \in B^{-1}$ by Corollary 1.1.3. According to the previous remark, $\varphi(ze - f) \neq 0$, and hence $\varphi(f) \neq z\,\varphi(e) = z$ for every $\varphi \in \mathcal{M}_B$. Consequently, the number $\varphi(f)$ belongs to the disc $\big\{z \in \mathbb{C}\colon |z| \leq \|f\|\big\}$, i.e. $\big|\varphi(f)\big| \leq \|f\|$, and this holds for every $f \in B$. Therefore, the functional φ is bounded, thus continuous, and $\|\varphi\| \leq 1$. By definition, $\|\varphi\|$ is the least number M with $\big|\varphi(f)\big| \leq M\|f\|$ for all $f \in B$. For any such M we have $M \geq 1$, since $1 = \big|\varphi(e)\big| \leq M\|e\| = M$. Hence, $\|\varphi\| \geq 1$, and therefore $\|\varphi\| = 1$. ☐

Example 1.1.6. (a) Let X be a compact Hausdorff set. The space $C(X)$ of all continuous functions on X under the pointwise operations and the uniform norm $\|f\| = \max\limits_{x \in X} \big|f(x)\big|$ is a commutative Banach algebra. One can easily identify some of the linear multiplicative functionals of $C(X)$. Namely, for a fixed $x \in X$ consider the functional "the point evaluation φ_x at x" in $C(X)$, i.e. $\varphi_x(f) = f(x)$ for every $f \in C(X)$. Clearly, $\varphi_x \in \mathcal{M}_{C(X)}$. Actually, one can show that every element in $\mathcal{M}_{C(X)}$ is of type φ_x for some $x \in X$. Consequently, $\mathcal{M}_{C(X)}$ and X are bijective spaces. We usually identify them as sets without mention, and write them as $\mathcal{M}_{C(X)} \cong X$.

(b) Let $\mathbb{D} = \mathbb{D}(1) = \big\{z\colon |z| < 1\big\}$ be the open unit disc in the complex plane $\mathbb{C}$ and let $A(\mathbb{D})$ denote the space of continuous functions in the closed unit disc $\overline{\mathbb{D}} = \big\{z \in \mathbb{C}\colon |z| \leq 1\big\}$ that are analytic in $\mathbb{D}$. Equipped with pointwise operations and the uniform norm $\|f\| = \max\limits_{x \in \overline{\mathbb{D}}} \big|f(x)\big|$, $A(\mathbb{D})$ is a commutative Banach algebra, called the *disc algebra*. One can easily check that $\overline{\mathbb{D}} \subset \mathcal{M}_{A(\mathbb{D})}$. In fact, $\mathcal{M}_{A(\mathbb{D})} \cong \overline{\mathbb{D}}$.

A net $\{\varphi_\alpha\}$ of functionals in $\mathcal{M}_B$ is said to converge pointwise to an element $\varphi \in \mathcal{M}_B$ if $\varphi_\alpha(f) \longrightarrow \varphi(f)$ for every $f \in B$. The pointwise convergence generates a topology on the maximal ideal space $\mathcal{M}_B$ of B, called the *Gelfand topology*. With respect to it $\mathcal{M}_B$ is a closed subset of the unit sphere S_{B^*} of the space B^* dual to B. By the Banach-Alaoglu theorem, S_{B^*} is a compact space in the weak*-topology, which in this case coincides with the pointwise topology. Under it $\mathcal{M}_B$ is a closed subset of S_{B^*}, and therefore a compact and Hausdorff set.

Let f be an element in a commutative Banach algebra B. The *Gelfand transform* of f is called the function $\widehat{f}$ defined on $\mathcal{M}_B$ by

$$\widehat{f}(\varphi) = \varphi(f), \ \varphi \in \mathcal{M}_B. \tag{1.4}$$

The Gelfand transform $\widehat{f}$ of any $f \in B$ is a continuous function on $\mathcal{M}_B$ with respect to the Gelfand topology. Indeed, if $\varphi_\alpha \longrightarrow \varphi$ then $\varphi_\alpha(f) \longrightarrow \varphi(f)$, and therefore, $\widehat{f}(\varphi_\alpha) \longrightarrow \widehat{f}(\varphi)$. The *Gelfand transformation* $\Lambda\colon B \longrightarrow \widehat{B} \subset C(\mathcal{M}_B)$ is a homomorphism of B onto the Gelfand transform $\widehat{B} = \{\widehat{f}\colon f \in B\}$ of B. If $B = C(X)$, then $\widehat{f}(\varphi_{x_0}) = \varphi_{x_0}(f) = f(x_0)$ for every $x_0 \in X$. Hence, if we identify $\mathcal{M}_{C(X)}$ with X, as in Example 1.1.6(a), then $\widehat{f}$ coincides with f.

Observe that if $\mathcal{M}_B$ possesses a closed and open set K, then the *characteristic function* $\varkappa_K$ of K (i.e. $\varkappa_K(x) = 1$ for $x \in K$ and $\varkappa_K(x) = 0$ otherwise) belongs to $\widehat{B}$ by the famous *Shilov idempotent theorem* (see e.g. [GRS]), which asserts that under the hypotheses there exists a unique element $b \in B$ with $b^2 = b$ (i.e. b is an *idempotent* of the algebra B) whose Gelfand transform is precisely the characteristic function of K, i.e. $\widehat{b} = \varkappa_K$.

There is a good reason to call $\mathcal{M}_B$ the set of maximal ideals of B. A subset J of a commutative Banach algebra B is called an *ideal* of B, if J is a linear subset of B which is closed with respect to multiplication with elements in B, i.e. $ab \in J$ for any $a \in B$ and $b \in J$. Any ideal of an algebra is an algebra on it own. An ideal $J \subset B$ is *proper* if it differs from B, and *maximal*, if it is proper and every proper ideal of B containing J, equals J. By Zorn's Lemma, one can show that any proper ideal of B is contained in some maximal ideal of B (cf. [G1, S4, T2]).

The sets $\{0\}$, B and $aB = \{ab : b \in B\}$ for a fixed $a \in B$, are all ideals. If φ is a linear multiplicative functional of B, then the *null-set* of φ, $\mathrm{Null}\,(\varphi) = \big\{f \in B\colon \varphi(f) = 0\big\}$ is an ideal of B. Indeed, for every $a \in B$ and $b \in \mathrm{Null}\,(\varphi)$, $\varphi(ab) = \varphi(a)\,\varphi(b) = 0$, i.e. $ab \in \mathrm{Null}\,(\varphi)$. Since $\varphi(e) = 1$ we have that $e \notin \mathrm{Null}\,(\varphi)$, and therefore, $\mathrm{Null}\,(\varphi)$ is a proper ideal of B.

The unit e does not belong to any proper ideal J of B, since by assuming the opposite, i.e. $e \in J$, we get $a = ea \in J$ for all $a \in B$, thus $J = B$. The same argument applies to check that proper ideals J do not contain invertible elements of B, i.e. $B^{-1} \cap J = \emptyset$ for any proper ideal J of B. An ideal of B is proper if and only if a is an invertible element of B, since if $a \in B^{-1}$, then $e = a\,a^{-1} \in aB$, a contradiction.

One can easily see that the null-set $\mathrm{Null}\,(\varphi)$ of any linear multiplicative functional φ is a maximal ideal (e.g. [G1, S4, T2]). Actually, every maximal ideal M of B is of type $\mathrm{Null}\,(\varphi_M)$ for some $\varphi_M \in \mathcal{M}_B$, i.e. the set of maximal ideals of B is bijective to the family of null-sets of linear multiplicative functionals on B.

Proposition 1.1.7. *The spectrum of any element f of B coincides with the range of its Gelfand transform $\widehat{f}$, i.e.*

$$\sigma(f) = \widehat{f}(\mathcal{M}_B) = Ran\,(\widehat{f}). \tag{1.5}$$

Proof. Let $z \in \widehat{f}(\mathcal{M}_B)$ and let $\widehat{f}(\varphi) = z$ for some $\varphi \in \mathcal{M}_B$. Hence $z - \varphi(f) = zf(e) - f(\varphi) = 0$, thus $\varphi(ze - f) = 0$, and therefore $ze - f \notin B^{-1}$, as shown prior to Lemma 1.1.5. Consequently $z \in \sigma(x)$. Conversely, if $z \in \sigma(x)$ then $ze - f \notin B^{-1}$ and hence $J = (ze - f)\,B$ is a proper ideal of B by the above remarks. If M is a maximal ideal containing J, then for the corresponding functional $\varphi_{_M}$ we have $\text{Null}\,(\varphi_{_M}) = M \supset J \ni ze - f$, thus $\varphi_{_M}(ze - f) = 0$. Therefore, $z = \varphi_{_M}(ze) = \varphi_{_M}(f) = \widehat{f}(\varphi_{_M})$. □

As a corollary we see that $\sigma(f + g) = (\widehat{f} + \widehat{g})(\mathcal{M}_B) \subset \widehat{f}(\mathcal{M}_B) + \widehat{g}(\mathcal{M}_B) = \sigma(f) + \sigma(g)$, and, similarly, $\sigma(fg) \subset \sigma(g)\,\sigma(g)$ for every $f, g \in B$.

By Proposition 1.1.7 $\max\limits_{z\in\sigma(f)} |z| = \max\limits_{z\in\widehat{f}(\mathcal{M}_B)} |z| = \max\limits_{x\in\mathcal{M}_B} \left|\widehat{f}(x)\right|$, which yields the formula

$$r_{_f} = \max_{x\in\mathcal{M}_B} \left|\widehat{f}(x)\right| = \|\widehat{f}\|_{C(\mathcal{M}_B)}$$

for the spectral radius $r_{_f}$ of any element $f \in B$. Combined with formula (1.2) this identity yields

$$\|\widehat{f}\|_{C(\mathcal{M}_B)} = \max_{x\in\mathcal{M}_B} \left|\widehat{f}(x)\right| = r_{_f} = \lim_{n\to\infty} \sqrt[n]{\|f^n\|}. \tag{1.6}$$

Proposition 1.1.7 implies the following description of the peripheral spectrum (1.3):

$$\sigma_\pi(f) = \left\{\widehat{f}(x) : |\widehat{f}(x)| = r_{_f},\ x \in \mathcal{M}_A\right\}.$$

By the well-known *maximum modulus principle* for analytic functions, the functions in the disc algebra $A(\mathbb{D})$ assume their maximum modulus only at the points in the unit circle $\mathbb{T}$, i.e. the topological boundary $\mathbb{T} = b\mathbb{D}$ of $\overline{\mathbb{D}} \cong \mathcal{M}_{A(\mathbb{D})}$. Sets of this kind are of special interest for commutative Banach algebras. A subset E in the maximal ideal space of a commutative Banach algebra B is called a *boundary* of B if the Gelfand transform $\widehat{f}$ of every element f in B attains the maximum of its modulus $\max\limits_{m\in\mathcal{M}_B} \left|\widehat{f}(m)\right| = \|\widehat{f}\|_{C(\mathcal{M}_B)}$ in E. In other words, E is a boundary for B if for every $f \in B$ there exists a $\varphi_0 \in E$ such that $\left|\widehat{f}(\varphi_0)\right| = \max\limits_{\varphi\in\mathcal{M}_B} \left|\widehat{f}(\varphi)\right|$, i.e. the equality

$$\max_{\varphi\in E} \left|\widehat{f}(\varphi)\right| = \max_{\varphi\in\mathcal{M}_B} \left|\widehat{f}(\varphi)\right|$$

holds for every $f \in B$. Clearly, the maximal ideal space $\mathcal{M}_B$ is a boundary of B. The celebrated Shilov theorem asserts that the intersection ∂B of all closed boundaries of a commutative Banach algebra B is again a boundary, called the

Shilov boundary of B (e.g. [G1, S4, T2]). Clearly, ∂B is the smallest closed boundary of B. This minimal property of the Shilov boundary implies the following characterization of its points.

Corollary 1.1.8. *A point m_0 in $\mathcal{M}_B$ belongs to the Shilov boundary ∂B of a commutative Banach algebra B if and only if for each neighborhood U of m_0 in $\mathcal{M}_B$ there exists a function f in B such that* $\max_{m\in\overline{U}} |\widehat{f}(m)| > \max_{m\in\mathcal{M}_B\setminus U} |\widehat{f}(m)|$.

As it is not hard to see, $\partial C(X) = X$. The maximum modulus principle, mentioned above, shows that $\mathbb{T}$ is a boundary for the disc algebra $A(\mathbb{D})$. In fact, $\partial A(\mathbb{D}) \cong \mathbb{T}$.

Let $\mathbb{B}^n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon \|(z_1, z_2, \ldots, z_n)\| < 1\}$ be the unit ball in $\mathbb{C}^n$ with radius 1 centered at the origin $(0, 0, \ldots, 0) \in \mathbb{C}_n$, let $\mathbb{D}^n$ be the n-polydisc $\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon |z_j| \leq 1,\ 1 \leq j \leq n\}$, and let $\mathbb{T}^n = \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \colon |z_j| = 1,\ 1 \leq j \leq n\}$ be the n-dimensional torus in $\mathbb{C}^n$, i.e. the *distinguished boundary* of $\mathbb{B}^n$. The Shilov boundary of the *ball algebra* $A(\mathbb{B}^n)$ is homeomorphic to the unit sphere in $\mathbb{C}^n$, which is the topological boundary of $\mathbb{B}^n$, while the Shilov boundary for the *polydisc algebra* $A(\mathbb{D}^n)$ is homeomorphic to $\mathbb{T}^n$, which is a proper subset of the topological boundary $b\mathbb{D}^n$ of $\mathbb{D}^n$.

1.2 Uniform algebras

Algebras of continuous functions have many useful properties. They play a major role in this book. A commutative Banach algebra A over $\mathbb{C}$ is said to be a *uniform algebra* on a compact Hausdorff space X if:

(i) A consists of continuous complex-valued functions defined on X, i.e. $A \subset C(X)$.

(ii) A contains all constant functions on X. In particular the identically equal to 1 function on X belongs to A.

(iii) The operations in A are the pointwise addition and multiplication.

(iv) A is closed with respect to the *uniform norm* in $C(X)$,

$$\|f\| = \max_{x\in X} |f(x)|,\ f \in A. \tag{1.7}$$

(v) *A separates the points of X*, i.e. for every two points in X there is a function in A with different values at these points.

A uniform algebra A is said to be *antisymmetric* if there are no real-valued functions in A besides the constants. A is a *maximal algebra on X* if there is no proper intermediate uniform algebra on X between A and $C(X)$. A is a *maximal algebra* if the restriction algebra $A|_{\partial A}$ is a maximal algebra on ∂A. According to the celebrated Wermer's maximality theorem, the disc algebra $A(\mathbb{D})$ is a maximal algebra.

A uniform algebra A is called a *Dirichlet algebra* if the space $\operatorname{Re}\left(A\big|_{\partial A}\right)$ of real parts of its elements is uniformly dense in $C_{\mathbb{R}}(\partial A)$, i.e. if every real continuous function on the Shilov boundary ∂A can be approximated on ∂A by real parts of functions in A. An example of a Dirichlet algebra is, for instance, the disc algebra $A(\mathbb{D})$. Indeed, $\operatorname{Re} A(\mathbb{D})$ consists of all real-valued continuous functions on $\overline{\mathbb{D}}$ that are harmonic on $\mathbb{D}$ and the harmonic conjugates of which are extendable continuously on $\mathbb{T}$. Consequently, $\operatorname{Re} A(\mathbb{D})$ contains all continuously differentiable functions on $\mathbb{T}$, and these are dense in $C_{\mathbb{R}}(\mathbb{T})$.

Let $\varphi \in \mathcal{M}_A$. A non-negative Borel measure μ on X for which the equality

$$\varphi(f) = \int\limits_X f(x)\, d\mu(x)$$

holds for every $f \in A$ is called a *representing measure* for φ on X. Note that

$$\int\limits_X f(x)\, g(x)\, d\mu(x) = \varphi\,(fg) = \int\limits_X f(x)\, d\mu(x) \int\limits_X g(x)\, d\mu(x)$$

for any $f, g \in A$, i.e. μ is a multiplicative measure for A. Any representing measure μ of φ on X satisfies the equalities

$$\|\mu\| = \int\limits_X d\mu = \varphi(1) = 1.$$

By the Hahn-Banach theorem the set M_φ of all representing measures for a $\varphi \in \mathcal{M}_A$ is nonempty. Actually, M_φ is isomorphic to the set of all norm-preserving extensions of $\varphi \in \mathcal{M}_A$ from $A \subset C(X)$ onto $C(X)$ (e.g. [G1]).

If A is a Dirichlet algebra, then every $\varphi \in \mathcal{M}_A$ has a unique representing measure on ∂A, i.e. M_φ is a single-point set for every $\varphi \in \mathcal{M}_A$. If not, the difference of every two representing measures of φ will vanish on A, hence on $\operatorname{Re} A$, hence on $C_{\mathbb{R}}(X)$ and therefore it will be the zero measure.

Proposition 1.2.1. *Let A be a uniform algebra on a compact set X. If there is a representing measure μ for some $\varphi \in \mathcal{M}_A$, such that* $\operatorname{supp}(\mu) = X$*, then A is an antisymmetric algebra.*

Proof. Assume that μ is a representing measure for some $\varphi \in A$ with $\operatorname{supp}(\mu) = X$. Let f be a non-constant real-valued function in A, and let $t_1, t_2 \in f(X) \subset \mathbb{R}$, $t_1 \neq t_2$. Without loss of generality we can assume that $t_1 > 0$. Let F be a closed neighborhood of t_1 in $\mathbb{R}_+$, which does not contain t_2. There exists a function $g \in C_{\mathbb{R}}\big(f(X)\big)$ such that $\sup\limits_X |g| = 1$, $g \equiv 1$ on F, and $g < 1$ on $f(X) \setminus F$.

Note that g is a uniform limit of polynomials on $f(X) \subset \mathbb{R}$. Hence, the function $g \circ f$ belongs to A. Since $\operatorname{supp}(\mu) = X$ and $\|\mu\| = \int_X g \circ f \, d\mu = 1$, we have that $0 < \int_X g \circ f \, d\mu = c < 1$. Since μ is a multiplicative measure, then

$$\lim_{n\to\infty} \int_X (g \circ f)^n d\mu = \Big(\int_X \lim_{n\to\infty} (g \circ f)^n d\mu \Big) = \lim_{n\to\infty} c^n = 0.$$

On the other hand, the assumed property $\operatorname{supp}(\mu) = X$ implies that

$$\lim_{n\to\infty} \int_X (g \circ f)^n \, d\mu = \int_{f^{-1}(F)} d\mu > 0,$$

in contradiction with the previous equality. Therefore, every real-valued function in A is constant, and consequently, A is an antisymmetric algebra. $\square$

The space $C(X)$ for a compact Hausdorff set X is a uniform algebra. Let K be a compact subset of the maximal ideal space $\mathcal{M}_A$ of a uniform algebra A on X. Consider the algebra $\widehat{A}\big|_K$ of restrictions of Gelfand transforms $\widehat{f}$, $f \in A$ on K. In general this is not a closed subalgebra of $C(K)$, and therefore $\widehat{A}\big|_K$ is not always a uniform algebra. However, the closure A_K of $\widehat{A}\big|_K$ in $C(K)$ is a uniform algebra with $\mathcal{M}_{A_K} \subset \mathcal{M}_A$. If $\mathcal{M}_{A_K}$ does not meet ∂A, then $\partial A_K = b(\mathcal{M}_{A_K})$, the topological boundary of $\mathcal{M}_{A_K}$ with respect to the Gelfand topology, which is an immediate corollary of the following.

Theorem 1.2.2 (Rossi's Local Maximum Modulus Principle). *If U is an open subset of $\mathcal{M}_A$, then*

$$\sup_{m\in U} \big|\widehat{f}(m)\big| = \max_{m\in bU\cup(\partial A\cap U)} \big|\widehat{f}(m)\big|$$

for every function $f \in A$.

Let A be a uniform algebra on X. As we know from section 1.1, the maximal ideal space $\mathcal{M}_A$ of A is a compact set. Since the point evaluation $\varphi_x : f \longmapsto \varphi(x)$ at any point of X is a linear multiplicative functional, then $\varphi_x \in \mathcal{M}_A$ for every $x \in X$. This allows us to consider X as a subspace of $\mathcal{M}_A$. The Gelfand transform $\widehat{f}$ of an $f \in A$ is continuous on $\mathcal{M}_A$. For any point of $x \in X$ we have $\widehat{f}(\varphi_x) = \varphi_x(f) = f(x)$, and therefore $\widehat{f}$ can be interpreted as a continuous extension of f on $\mathcal{M}_A$. Moreover, in a certain sense M_A is the largest set for natural extension of all functions in A. Recall that according to Lemma 1.1.5 the norm of any $\varphi \in \mathcal{M}_A$ is 1. Therefore, $\big\|\varphi(f)\big\| \le \|\varphi\|\|f\| = \|f\|$. It follows that the Gelfand transformation $\Lambda : \ A \longrightarrow \widehat{A} \subset C(\mathcal{M}_A) : \ f \longmapsto \widehat{f}$ is an isometric isomorphism. Consequently, the algebra A and its Gelfand transform $\widehat{A}$ are isometrically isomorphic, and hence $\widehat{A}$ is closed in $C(X)$. Since the algebra $A|_{\partial A}$ of restrictions of elements in A on the Shilov boundary ∂A is also isometrically isomorphic to A, we have $A \cong \widehat{A} \cong A\big|_{\partial A}$. For this reason we will not distinguish, for example, the disc algebra $A(\mathbb{D})$ from its restriction algebra $A(\mathbb{T}) = A(\mathbb{D})\big|_{\mathbb{T}}$ on the Shilov boundary $\partial A(\mathbb{D}) = \mathbb{T}$.

Observe, that $\big\|(m_1 - m_2)(f)\big\| = \big\|m_1(f) - m_2(f)\big\| \le \big(\|m_1\| + \|m_2\|\big)\|f\| \le 2\,\|f\|$ for every $m_1, m_2 \in \mathcal{M}_A$, and $f \in A$. Consequently, the norm $\|m_1 - m_2\|$ of the linear functional $m_1 - m_2 \in A^*$ does not exceed 2. Therefore, the diameter of the set $\mathcal{M}_A \subset A^*$ is not greater than 2. The property $\|m_1 - m_2\| < 2$ generates a transitive relation in $\mathcal{M}_A$. It is easy to check that this is an equivalence relation (e.g. [G1],[S4]). The equivalent classes of the set $\mathcal{M}_A$ with respect to this relation are called *Gleason parts* of A. It is clear that points on the extreme ends of a diameter, i.e. for which $\|m_1 - m_2\| = 2$, belong to distinct Gleason parts.

A homomorphism $\Phi\colon A \longrightarrow B$ between two uniform algebras naturally generates an adjoint continuous map $\Phi^*\colon \mathcal{M}_B \longrightarrow \mathcal{M}_A$ between their maximal ideal spaces, defined by

$$\big(\Phi^*(\varphi)\big)(f) = \varphi\left(\Phi(f)\right), \; f \in A, \; \varphi \in \mathcal{M}_B.$$

If $\Phi : A \longrightarrow B$ preserves the norm, i.e. if

$$\big\|\Phi(f)\big\|_B = \|f\|_A$$

for every $f \in A$, then Φ is called an *embedding* of A into B. Clearly, $\Phi^*(\partial B) \subset \mathcal{M}_A$.

Proposition 1.2.3. *Let A and B be uniform algebras, and let $\Phi\colon A \longrightarrow B$ be a homomorphism that does not increase the norm, i.e. for which $\big\|\Phi(f)\big\|_B \le \|f\|_A$, $f \in A$. Then Φ is an embedding of A into B if and only if the range $\Phi^*(\partial B)$ of Φ^* contains the Shilov boundary ∂A.*

Proof. For every $f \in A$ we have

$$\begin{aligned}\max_{m \in \Phi^*(\partial B)} \big|m(f)\big| &= \max_{\varphi \in \partial B} \big|\big(\Phi^*(\varphi)\big)(f)\big| = \max_{\varphi \in \partial B} \big|\varphi\big(\Phi(f)\big)\big| \\ &= \max_{\varphi \in \partial B} \big|\widehat{(\Phi(f))}(\varphi)\big| = \big\|\Phi(f)\big\|_B.\end{aligned} \tag{1.8}$$

If $\partial A \subset \Phi^*(\partial B)$, then

$$\|f\|_A = \max_{\varphi \in \partial A} \big|\widehat{f}(\varphi)\big| = \max_{\varphi \in \partial A} \big|\varphi(f)\big| \le \max_{\varphi \in \Phi^*(\partial B)} \big|\varphi(f)\big| = \big\|\Phi(f)\big\|_B.$$

Therefore, $\|f\|_A = \big\|\Phi(f)\big\|_B$, i.e. Φ preserves the norm.

Conversely, if $\Phi : A \longrightarrow B$ is an isometry, then $\Phi^*(\partial B)$ is a boundary for A, since by (1.8)

$$\max_{\varphi \in \Phi^*(\partial B)} \big|\widehat{f}(\varphi)\big| = \max_{\varphi \in \Phi^*(\partial B)} \big|\varphi(f)\big| = \big\|\Phi(f)\big\|_B = \|f\|_A.$$

Consequently, $\partial A \subset \Phi^*(\partial B)$. ☐

Corollary 1.2.4. *A homomorphism Φ of A onto B which does not increase the norm is an embedding if and only if $\Phi^*(\partial B) = \partial A$.*

Proof. The arguments from the proof of Proposition 1.2.3 show that it is enough to show that $\Phi^*(\partial B) \subset \partial A$. Suppose that, on the contrary, $\Phi^*(\partial B) \supsetneq \partial A$, and let $\varphi_0 \in \Phi^*(\partial B) \setminus \partial A$. According to Corollary 1.1.8, there is a neighborhood U of φ_0 in $\mathcal{M}_A \setminus \partial A$, such that for every function $f \in A$,

$$\max_{m \in \overline{U}} \left|\widehat{f}(m)\right| \le \max_{m \in \mathcal{M}_A \setminus U} \left|\widehat{f}(m)\right|.$$

In particular,

$$\max_{\Phi^*(\varphi) \in \overline{U}} \left|\widehat{f}\big(\Phi^*(\varphi)\big)\right| \le \max_{\Phi^*(\varphi) \in \mathcal{M}_A \setminus U} \left|\widehat{f}\big(\Phi^*(\varphi)\big)\right|.$$

Since $\widehat{f}\big(\Phi^*(\varphi)\big) = \big(\Phi^*(\varphi)\big)(f) = \varphi\big(\Phi(f)\big) = \widehat{\Phi(f)}(\varphi)$, we have

$$\max_{\varphi \in (\Phi^*)^{-1}(\overline{U})} \left|\widehat{\Phi(f)}(\varphi)\right| \le \max_{\varphi \in (\Phi^*)^{-1}(\mathcal{M}_A \setminus U)} \left|\widehat{\Phi(f)}(\varphi)\right|.$$

By the assumed $\Phi(A) = B$, we see that

$$\max_{\varphi \in (\Phi^*)^{-1}(\overline{U})} \left|\widehat{g}(\varphi)\right| \le \max_{\varphi \in (\Phi^*)^{-1}(\mathcal{M}_A \setminus U)} \left|\widehat{g}(\varphi)\right|$$

for every $g \in B$. Consequently, $(\Phi^*)^{-1}(\mathcal{M}_A \setminus U)$ is a closed boundary of B, and $(\Phi^*)^{-1}(\varphi_0) \subset (\Phi^*)^{-1}(U) \subset \mathcal{M}_B \setminus (\Phi^*)^{-1}(\mathcal{M}_A \setminus U) \subset \mathcal{M}_B \setminus \partial B$, in contradiction with the initially assumed property $\varphi_0 \in \Phi^*(\partial B)$. Hence $\Phi^*(\partial B) \subset \partial A$. □

Every embedding $\Phi\colon\ A(\mathbb{T}) \longrightarrow A(\mathbb{T})$ of the disc algebra onto itself is an isometric isomorphism between $A(\mathbb{T})$ and $\Phi\big(A(\mathbb{T})\big)$. Consequently, the adjoint map $\Phi^*\colon \mathcal{M}_{\Phi(A(\mathbb{T}))} \longrightarrow \mathcal{M}_{A(\mathbb{T})}$ generates a homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{\mathbb{D}}$, and $\Phi^*\big(\partial(\Phi(A(\mathbb{T}))\big) = \partial A(\mathbb{T}) = \mathbb{T}$, i.e. $\Phi^*(\mathbb{T}) = \mathbb{T}$, $\Phi^*(\overline{\mathbb{D}}) = \overline{\mathbb{D}}$, and hence the function Φ^* is a *finite Blaschke product* (cf. [G2]) on $\mathbb{D}$, i.e.

$$B(z) = e^{i\theta} \prod_{k=1}^{n} \left(\frac{z - z_k}{1 - \overline{z}_k z}\right) \text{ for some } z_k,\ 0 < |z_k| < 1,\ k = 1, 2, \dots, n.$$

Therefore, for any embedding $\Phi\colon A(\mathbb{T}) \longrightarrow A(\mathbb{T})$ of $A(\mathbb{T})$ onto itself there exists a finite Blaschke product $B(z)$ on $\mathbb{D}$ with $\Phi \circ f = f \circ B$, i.e. such that

$$\Phi\big(f(z)\big) = (f \circ \Phi^*)(z) = f\big(B(z)\big) \text{ for every } f \in A(\mathbb{T}). \tag{1.9}$$

Let $A \subset C(X)$ be a uniform algebra on a compact set X. One can easily identify certain points as elements of the Shilov boundary ∂A of a uniform algebra A. A point $x_0 \in X$ is called a *peak point* of a uniform algebra A if there exists a function f in A such that $f(x_0) = 1$ and $\left|\widehat{f}(x)\right| < 1$ for every $x \in \mathcal{M}_A \setminus \{x_0\}$. Clearly, every peak point belongs to the Shilov boundary ∂A. In general the set of peak points is not a boundary for A. However, for algebras with metrizable maximal ideal spaces it is (e.g. [G1, S4]). Moreover, in this case the set of peak points is the *minimal boundary* for A, i.e. it is contained in every boundary of A.

An element $f \in A$ is called a *peaking function of* A if $\|f\| = 1$, and either $\widehat{f}(x) = 1$, or, $|\widehat{f}(x)| < 1$ for any $x \in \mathcal{M}_A$. In this case the set $P(f) = \{x \in \mathcal{M}_A : \widehat{f}(x) = 1\} = \widehat{f}^{-1}(1)$ is called the *peak set* (or, *peaking set*) of A corresponding to $\widehat{f}$. Clearly, every peak point is a peak set of A, and f is a peaking function if and only if $\sigma_\pi(f) = \{1\}$. If $K \subset \mathcal{M}_A$ is such that $K = P(f)$ for some peaking function f, we say that $\widehat{f}$ *peaks on* K. Clearly, K is a peak set if there is a function$f \in A$, such that $\widehat{f}|_K \equiv 1$, and $|\widehat{f}(m)| < 1$ whenever $m \in \mathcal{M}_A \setminus K$.

A point $x \in \mathcal{M}_A$ is called a *generalized peak point of* A (or, a *p-point of* A) if it coincides with the intersection of a family of peak sets of A. Equivalently, x is a p-point of A if for every neighborhood V of x there is a peaking function f with $x \in P(f) \subset V$. The *Choquet boundary* (or, the *strong boundary*) δA of A is the set of all generalized peak points of A. It is a boundary of A, and its closure coincides with the Shilov boundary ∂A of A, i.e. $\overline{\delta A} = \partial A$. Unlike δA, the set of peak points of A in general is not dense in ∂A, unless $\mathcal{M}_A$ is metrizable (cf. [G1, S4]).

Till the end of the section we will assume that $A \subset C(X)$ is a uniform algebra on its maximal ideal space $\mathcal{M}_A = X$. Denote by $\mathcal{F}(A)$ the set of all peaking functions of A. For a fixed point x in X by $\mathcal{F}_x(A)$ denote the set of all peaking functions of A by $P(f) \ni x$, i.e. with$\widehat{f}(x) = 1$.

Lemma 1.2.5. *Let $A \subset C(X)$ be a uniform algebra. If $f, g \in A$ are such that $\|fh\| \leq \|gh\|$ for all peaking functions $h \in \mathcal{F}(A)$, then $|f(x)| \leq |g(x)|$ on ∂A.*

Proof. Assume that $\|fh\| \leq \|gh\|$ for every $h \in \mathcal{F}(A)$, but $|f(x_0)| > |g(x_0)|$ for some $x_0 \in \partial A$. Without loss of generality we may assume that $x_0 \in \delta A$. Choose a $\gamma > 0$ such that $|g(x_0)| < \gamma < |f(x_0)|$, and choose an open neighborhood V of x_0 in X so that $|g(x)| < \gamma$ on V. Let $h \in \mathcal{F}_{x_0}(A)$ be a peaking function of A on X with $P(h) \subset V$. By choosing a sufficiently high power of h we can assume from the beginning that $|g(x)h(x)| < \gamma$ for every $x \in \partial A \setminus V$. Since this inequality obviously holds also on V, we deduce that $\|gh\| < \gamma$. Hence,

$$|f(x_0)| = |f(x_0)h(x_0)| \leq \|fh\| \leq \|gh\| < \gamma.$$

Therefore, $|f(x_0)| < \gamma$ in contradiction with the choice of γ. Consequently, $|f(x)| \leq |g(x)|$ on ∂A. □

Corollary 1.2.6. *If the functions $f, g \in A$ satisfy the equality $\|fh\| = \|gh\|$ for all peaking functions $h \in \mathcal{F}(A)$, then $|f(x)| = |g(x)|$ on ∂A.*

Lemma 1.2.7. *If the functions* $f, g \in A$ *satisfy the inequality*

$$\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) \leq \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$$

for all $k \in A$*, then* $|f(x)| \leq |g(x)|$ *for every* $x \in \partial A$*.*

Proof. The proof follows the line of proof of Lemma 1.2.5.
Assume that $\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) \leq \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$ for every $k \in A$, but $|f(x_0)| > |g(x_0)|$ for some $x_0 \in \partial A$. Without loss of generality we may assume that $x_0 \in \delta A$. Choose a $\gamma > 0$ such that $|g(x_0)| < \gamma < |f(x_0)|$, and choose an open neighborhood V of x_0 in X so that $|g(x)| < \gamma$ on V. Let $R > 1$ be such that $\|f\| \leq R$ and $\max_{\xi \in \partial A} |g(\xi)| \leq R$. Let $k \in \mathcal{F}_{x_0}(A)$ be a peaking function for A with $P(k) \subset V$. By choosing a sufficiently high power of k we can assume from the beginning that $|g(x)| + |Rk(x)| < R + \gamma$ for every $x \in \partial A \setminus V$. Since this inequality holds also on V, we deduce that $|g(x)| + |Rk(x)| < R + \gamma$ for every $x \in \partial A$. Hence,

$$\begin{aligned} |f(x_0)| + R &= |f(x_0)| + |Rk(x_0)| \\ &\leq \max_{\xi \in \partial A} \big(|f(\xi)| + |Rk(\xi)|\big) \leq \max_{\xi \in \partial A} \big(|g(\xi)| + R|k(\xi)|\big) < R + \gamma. \end{aligned}$$

Therefore, $|f(x_0)| < \gamma$ in contradiction with the choice of γ. Consequently, $|f(x)| \leq |g(x)|$ for every $x \in \partial A$. □

Corollary 1.2.8. *If the functions* $f, g \in A$ *satisfy the equality*

$$\max_{\xi \in \partial A} \big(|f(\xi)| + |k(\xi)|\big) = \max_{\xi \in \partial A} \big(|g(\xi)| + |k(\xi)|\big)$$

for all $k \in A$*, then* $|f(x)| = |g(x)|$ *for every* $x \in \partial A$*.*

The following lemma, due to Bishop, helps to localize elements of uniform algebras.

Lemma 1.2.9 (Bishop's Lemma). *If* $E \subset X$ *is a peak set for* A*, and* $f \not\equiv 0$ *on* E *for some* $f \in A$*, then there exists a peaking function* $h \in \mathcal{F}(A)$ *which peaks on* E *and such that*

$$|f(x)h(x)| < \max_{\xi \in E} |f(\xi)| \tag{1.10}$$

for any $x \in X \setminus E$*.*

Proof. If $f \in A$ and $\max_{\xi \in E} |f(\xi)| = M > 0$. For any natural $n \in \mathbb{N}$ define the set

$$U_n = \big\{x \in X \colon |f(x)| < M\big(1 + 1/2^{n+1}\big)\big\}.$$

Clearly, $E \subset U_n \subset U_{n-1}$ for every $n > 1$. Choose a function $k \in \mathcal{F}(A)$ which peaks on E, and let k_n be a big enough power of k so that $|k_n(x)| < \dfrac{1}{2^n}$ on $X \setminus U_n$.

The function $h = \sum_{1}^{\infty} \frac{1}{2^n} k_n$ belongs to $\mathcal{F}(A)$. Moreover, $P(h) = h^{-1}\{R\} = E$, $|h(x)| < 1$ on $X \setminus E$, and $\max_{\xi \in E} \big(|f(\xi)h(\xi)|\big) = M$. We claim that $|f(x)h(x)| < M$ for every $x \notin E$. In what follows, x is a fixed element in $X \setminus E$.

(i) Let $x \notin U_1$. Then $x \notin U_n$ for all $n \in \mathbb{N}$, and hence $|k_n(x)| < \frac{1}{2^n} < 1$ for all $n \in \mathbb{N}$. Hence, $|h(x)| < \sum_{1}^{\infty} \frac{1}{2^n} = 1$, thus $|f(x)h(x)| < M$.

(ii) Let $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for every $1 \le i \le n-1$, and $x \notin U_i$ for all $i \ge n$. Hence $|f(x)| < M\big(1 + 1/2^{i+1}\big)$ for every $1 \le i \le n-1$, and $|k_i(x)| < \frac{1}{2^i}$ for all $i \ge n$. Since $x \in U_{n-1}$, we see that $|f(x)| < M\big(1 + 1/2^n\big)$, and

$$|f(x)h(x)| < M\big(1 + 1/2^n\big)\left(\sum_{i=1}^{n-1} \frac{1}{2^i}|k_i(x)| + \sum_{i=n}^{\infty} \frac{1}{2^i}|k_i(x)|\right).$$

Further,

$$\sum_{i=1}^{n-1} \frac{1}{2^i}|k_i(x)| < \sum_{i=1}^{n-1} \frac{1}{2^i} = \big(1 - 1/2^{n-1}\big), \text{ and}$$

$$\sum_{i=n}^{\infty} \frac{1}{2^i}|k_i(x)| \le \sum_{i=n}^{\infty} \frac{1}{2^i}\left(\frac{1}{2^i}\right) = \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{M}{3 \cdot 4^{n-1}} < \frac{1}{2 \cdot 4^{n-1}} = \frac{1}{2^n \cdot 2^{n-1}}.$$

Consequently,

$$\begin{aligned}|f(x)h(x)| &< M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}} + \frac{1}{2^n 2^{n-1}}\right) \\ &\le M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}}\left(1 - \frac{1}{2^n}\right)\right) < M\big(1 + 1/2^n\big)\left(1 - \frac{1}{2^{n-1}} \cdot \frac{1}{2}\right) \\ &= M\big(1 + 1/2^n\big)\big(1 - 1/2^n\big) = M\big(1 - 1/2^{2n}\big) < M.\end{aligned}$$

(iii) If $x \in \bigcap_{n=1}^{\infty} U_n$, then $|f(x)| \le M$, whence $|f(x)h(x)| < M$ since $|h(x)| < 1$ on $X \setminus E$. $\square$

If also $\sigma_\pi(fh) = \sigma_\pi(gh)$ for all $h \in \mathcal{F}(A)$, then we have a much stronger result than in Corollary 1.2.6. Namely,

Lemma 1.2.10. *If $f, g \in A$ satisfy the equality*

$$\sigma_\pi(fh) = \sigma_\pi(gh) \tag{1.11}$$

for every peaking function $h \in A$, then $f(x) = g(x)$ on ∂A.

Proof. Clearly, $\|fh\| = \|gh\|$, since $|z| = \|f\|$ for every $z \in \sigma_\pi(f)$. Corollary 1.2.6 yields $|f(x)| = |g(x)|$ on ∂A. Let $y \in \delta A$. If $f(y) = 0$, then $|g(y)| = |f(y)| = 0$ implies that also $g(y) = 0$. Therefore, we can assume without loss of generality that $f(y) \neq 0$. Choose an open neighborhood V of y in X, and a peaking function $k \in \mathcal{F}_y(A)$ with $P(k) \subset V$. Let $|f(x_V)| = \max_{\xi \in P(k)} |f(\xi)|$ for some $x_V \in P(k)$. By Bishop's Lemma there is a peaking function $h \in \mathcal{F}_y(A)$ with $P(h) = P(k)$, so that the functions fh and gh attain the maxima of their modulus only within $P(h)$. Therefore, by (1.11), $f(x_V) = f(x_V)\, h(x_V) \in \sigma_\pi(fh) = \sigma_\pi(gh)$. Hence, there is a $z_V \in P(h)$ so that

$$f(x_V) = g(z_V)\, h(z_V) = g(z_V). \tag{1.12}$$

Since in every neighborhood $V \ni y$ there are points x_V and z_V in V with $f(x_V) = g(z_V)$, then $f(y) = g(y)$ by the continuity of f and g. Consequently, $f = g$ on $\partial A = \overline{\delta A}$. □

The next lemma is an additive version of Bishop's Lemma (Lemma 1.2.9).

Lemma 1.2.11 (Additive analogue of Bishop's Lemma). *If $E \subset X$ is a peak set for A, and $f \not\equiv 0$ on E for some $f \in A$, then there exists a function $h \in \mathcal{F}(A)$ which peaks on E and such that*

$$|f(x)| + N|h(x)| < \max_{\xi \in E} |f(\xi)| + N \tag{1.13}$$

for any $x \in X \setminus E$ and any $N \geq \|f\|$.

Proof. The proof follows the line of proof of Bishop's Lemma 1.2.9. If $\|f\| = \max_{\xi \in X} |f(\xi)| = R$ and $\max_{\xi \in E} |f(\xi)| = M$, then clearly, $0 < M \leq R$. For any natural $n \in \mathbb{N}$ define the set

$$U_n = \big\{x \in X : |f(x)| < M\big(1 + 1/2^{n+1}\big)\big\}.$$

Clearly, $E \subset U_n \subset U_{n-1}$ for every $n > 1$. Choose a function $k \in \mathcal{F}(A)$ which peaks on E, and let k_n be a big enough power of k so that $R\,|k_n(x)| < \dfrac{M}{2^n}$ on $X \setminus U_n$. The function $h = \sum_1^\infty \dfrac{1}{2^n} k_n$ belongs to $\mathcal{F}(A)$. Moreover, $P(h) = h^{-1}\{R\} = E$, $|h(x)| < 1$ on $X \setminus E$, and $\max_{\xi \in E} \big(|f(\xi)| + R\,|h(\xi)|\big) = M + R$. We claim that $|f(x)| + R\,|h(x)| < M + R$ for every $x \notin E$. In what follows, x is a fixed element in $X \setminus E$.

(i) Let $x \notin U_1$. Then $x \notin U_n$ for all $n \in \mathbb{N}$, and hence $R\,|k_n(x)| < \dfrac{M}{2^n} < M$ for all $n \in \mathbb{N}$. Hence, $R\,|h(x)| < \sum_1^\infty \dfrac{M}{2^n} = M$, thus $|f(x)| + R\,|h(x)| < R + M$.

(ii) Let $x \in U_{n-1} \setminus U_n$ for some $n > 1$. Then $x \in U_i$ for all $1 \le i \le n-1$, and $x \notin U_i$ for each $i \ge n$. Hence $|f(x)| < M\big(1 + 1/2^{i+1}\big)$ for all $1 \le i \le n-1$, and $R|k_i(x)| < \dfrac{M}{2^i}$ for each $i \ge n$. Since $x \in U_{n-1}$, we see that $|f(x)| < M\big(1 + 1/2^n\big)$, and

$$|f(x)| + R\,|h(x)| < M\big(1 + 1/2^n\big) + \sum_{i=1}^{n-1} \frac{R}{2^i}|k_i(x)| + \sum_{i=n}^{\infty} \frac{R}{2^i}|k_i(x)|.$$

Further,

$$\sum_{i=1}^{n-1} \frac{R}{2^i}|k_i(x)| < \sum_{i=1}^{n-1} \frac{R}{2^i} = R\big(1 - 1/2^{n-1}\big), \text{ and}$$

$$\sum_{i=n}^{\infty} \frac{R}{2^i}|k_i(x)| \le \sum_{i=n}^{\infty} \frac{1}{2^i}\left(\frac{M}{2^i}\right) = M \sum_{i=n}^{\infty} \frac{1}{4^i} = \frac{M}{3 \cdot 4^{n-1}} < \frac{M}{2 \cdot 4^{n-1}} = \frac{M}{2^n \cdot 2^{n-1}}.$$

Consequently,

$$\begin{aligned} |f(x)| + R\,|h(x)| &< M\big(1 + 1/2^n\big) + R\big(1 - 1/2^{n-1}\big) + \frac{M}{2^n 2^{n-1}} \\ &\le M + R\left(\frac{1}{2^n} + 1 - \frac{1}{2^{n-1}} + \frac{1}{2^n \cdot 2^{n-1}}\right) \\ &= M + R\left(1 - \frac{1}{2^n} + \frac{1}{2^n \cdot 2^{n-1}}\right) < M + R. \end{aligned}$$

(iii) If $x \in \bigcap_{n=1}^{\infty} U_n$, then $|f(x)| \le M$, whence $|f(x)| + R\,|h(x)| < M + R$ since $|h(x)| < 1$ on $X \setminus E$.

Actually, (1.13) holds with any $N > R$ for the function h constructed above. Indeed,

$$\begin{aligned} |f(x)| + N|h(x)| &= |f(x)| + R\,|h(x)| + (N - R)\,|h(x)| \\ &< \max_{\xi \in E} |f(\xi)| + R + (N - R) = \max_{\xi \in E} |f(\xi)| + N. \end{aligned}$$

□

Corollary 1.2.12. *Let E be a peak set of A, $x_0 \in E$, $f \in A$, $N \ge \|f\|$, and $\alpha \in \mathbb{T}$ be such that $|f(x_0)| = \max_{\xi \in E} |f(\xi)| > 0$ and $f(x_0) = \alpha\,|f(x_0)|$. If h is the peaking function of A with $P(h) = E$, constructed in Lemma 1.2.11, then*

(a) $\big|f(x) + \alpha N h(x)\big| \le |f(x)| + N|h(x)| < \|f + \alpha N h\| = \big|f(x_0) + \alpha N h(x_0)\big| = \big|f(x_0)\big| + N$ *for all $x \in X \setminus E$, and*

(b) $\|f + \gamma N h\| \le \|f + \alpha N h\|$ *for every $\gamma \in \mathbb{T}$.*

Proof. (a) Lemma 1.2.11 implies that $\left|f(x)+\alpha N h(x)\right| \leq |f(x)|+N|h(x)| < \max_{\xi\in E}\left(|f(\xi)|+N\right) = |f(x_0)|+N = \left|f(x_0)+\alpha N h(x_0)\right|$ for all $x \in X\setminus E$. Hence, $\|f+\alpha N h\| = \max_{\xi\in E}\left|f(\xi)+\alpha N h(\xi)\right| = \left|f(x_0)+\alpha N h(x_0)\right| = |f(x_0)|+N$, i.e. (a) holds.

(b) By Lemma 1.2.11 and (a), we have

$$\begin{aligned}\|f+\gamma N h\| &= \max_{\xi\in X}\left|f(\xi)+\gamma N h(\xi)\right| \\ &\leq \max_{\xi\in X}\left(|f(\xi)|+N|h(\xi)|\right) = |f(x_0)|+N = \|f+\alpha N h\|.\end{aligned}$$ □

If $\sigma_\pi(f+h)=\sigma_\pi(g+h)$ for all $h\in A$, then we have a much stronger result than in Corollary 1.2.8. Namely,

Lemma 1.2.13. *If* $f,g\in A$ *satisfy the equalities*

(a) $\sigma_\pi(f+h)=\sigma_\pi(g+h)$, *and*

(b) $\max_{\xi\in\partial A}\left(|f(\xi)|+|h(\xi)|\right) = \max_{\xi\in\partial A}\left(|g(\xi)|+|h(\xi)|\right)$

for every $h\in A$, *then* $f(x)=g(x)$ *for every* $x\in\partial A$.

Proof. The proof follows the line of proof of Lemma 1.2.10. Let $f,g\in A$ and let $\|f\|=\|g\|=R$. Equality (b) and Corollary 1.2.8 imply that $|f(x)|=|g(x)|$ on ∂A. Let $y\in\delta A$. If $f(y)=0$, then by $|g(y)|=|f(y)|=0$ we see that $g(y)=0$ too. Suppose now that $f(y)\neq 0$. Choose an open neighborhood V of y in X, and a peaking function $k\in\mathcal{F}_y(A)$ with $P(k)\subset V$. There is an $x_V\in P(k)$ so that $|f(x_V)| = \max_{\xi\in P(k)}|f(\xi)| = M \leq R$. Let $f(x_V)=\alpha_V M$ for some $\alpha_V\in\mathbb{T}$. By the additive version of Bishop's Lemma we can choose a peaking function $h\in\mathcal{F}_y(A)$ with $P(h)=P(k)$ and such that the function $|f(x)|+|Rh(x)|$ attains its maximum only within $P(h)$. Hence

$$\begin{aligned}&|f(x_V)|+R = M+R = \left|\alpha_V(M+R)\right| = \left|f(x_V)+\alpha_V\, R\right| \\ &= \left|f(x_V)+\alpha_V\, Rh(x_V)\right| \leq \|f+\alpha_V Rh\| = \max_{\xi\in\partial A}\left|\big(f+Rh\big)(\xi)|\right) \\ &\leq \max_{\xi\in\partial A}\left(|f(\xi)|+|Rh(\xi)|\right) = \max_{\xi\in P(h)}\left(|f(\xi)|+|Rh(\xi)|\right) \\ &= \max_{\xi\in P(h)}\left(|f(\xi)|+R|\right) = |f(x_V)|+R,\end{aligned}$$

and therefore,

$$\left|f(x_V)+\alpha_V\, R\right| = \max_{\xi\in\partial A}\left(|f(\xi)|+|Rh(\xi)|\right) = \|f+\alpha_V Rh\|, \tag{1.14}$$

and, by equality (a), $f(x_V)+\alpha_V R\in\sigma_\pi(f+\alpha_V Rh)=\sigma_\pi(g+\alpha_V Rh)$. Hence, there is a $z_V\in X$ so that

$$f(x_V)+\alpha_V R = g(z_V)+\alpha_V Rh(z_V). \tag{1.15}$$

We may assume that $z_V \in \partial A$, since $\big|g(z_V) + \alpha_V Rh(z_V)\big| = |f(x_V) + \alpha_V R| = \big|f(x_V) + \alpha_V Rh(x_V)\big|$ is the maximum modulus of both functions $g + \alpha_V Rh$, $f + \alpha_V Rh$, and as a peaking set of A (cf. [L1]), the preimage $\big(g + \alpha_V Rh\big)^{-1}\big(g(z_V) + \alpha_V Rh(z_V)\big)$ of the number $g(z_V) + \alpha_V Rh(z_V)$ under the function $g + \alpha_V Rh$ necessarily meets ∂A. By (1.14) and Corollary 1.2.8 we have

$$\begin{aligned}
&\big|g(z_V) + \alpha_V Rh(z_V)\big| \le |g(z_V)| + |Rh(z_V)| \\
&= |f(z_V)| + |Rh(z_V)| \le \max_{\xi \in \partial A} \big(|f(\xi)| + |Rh(\xi)|\big) = \|f + \alpha_V Rh\| \\
&= \max_{\xi \in \delta A} \big(\big|g(\xi) + \alpha_V Rh(\xi)\big|\big) = \big|g(z_V) + \alpha_V Rh(z_V)\big|.
\end{aligned}$$

Hence, $|g(z_V)| + |Rh(z_V)| = \max_{\xi \in \delta A} \big(\big|g(\xi) + \alpha_V Rh(\xi)\big|\big)$. Since the function $\big|g(\xi) + \alpha_V Rh(\xi)\big|$ attains its maximum only within $P(h)$ it follows that $z_V \in P(h)$, thus $h(z_V) = 1$. The equality (1.15) now becomes $f(x_V) + \alpha_V R = g(z_V) + \alpha_V R$, thus $f(x_V) = g(z_V)$. Since in every neighborhood $V \ni y$ there are points x_V and z_V in V with $f(x_V) = g(z_V)$, then $f(y) = g(y)$ by the continuity of f and g. Consequently, $f = g$ on $\partial A = \overline{\delta A}$. □

1.3 Inductive and inverse limits of algebras and sets

In this section we introduce the notion of inductive and inverse systems and their limits, which are used to construct associated algebras. Since we need the technique in some special cases only, we do not present it in its general form, which can be found elsewhere (e.g. [L1], [ES]).

Consider a family $\{A^\alpha\}_{\alpha \in \Sigma}$ of uniform algebras. Suppose that the index set Σ is directed, i.e. Σ is a partially ordered set, and every pair α, β of elements of Σ has a common successor $\gamma \succ \alpha, \beta$ in Σ. Suppose also that for every pair $A^\alpha, A^\beta,\ \alpha \prec \beta$, of algebras there is an algebraic homomorphism $\iota_\alpha^\beta \colon A^\alpha \longrightarrow A^\beta$. The family $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha \in \Sigma}$ is called an *inductive system* (or, inductive spectrum, direct spectrum) of algebras A^α with connecting homomorphisms ι_α^β, if

(i) ι_α^α is the identity mapping on A^α, and

(ii) $\iota_\beta^\gamma \circ \iota_\alpha^\beta = \iota_\alpha^\gamma$ whenever $\alpha \prec \beta \prec \gamma$.

A *chain* of the system $\{A^\alpha, \iota_a^\beta\}_{\alpha \in \Sigma}$ is called any set of type $\nu = \{p^\alpha \colon p^\alpha \in A^\alpha\}_{\alpha \succ \alpha_\nu}$, such that $\iota_\alpha^\beta(p^\alpha) = p^\beta$ for every $\alpha, \beta \succ \alpha_\nu$. Let $\mathcal{N}$ denote the set of all chains of the system $\{A^\alpha, \iota_a^\beta\}_{\alpha \in \Sigma}$. Consider the following equivalence relation in $\mathcal{N}$: If $\nu_1 = \{p^\alpha\}_{\alpha \succ \alpha_{\nu_1}}$ and $\nu_2 = \{q^\alpha\}_{\alpha \succ \alpha_{\nu_2}} \in \mathcal{N}$, then $\nu_1 \sim \nu_2$ if there exists a $\beta \in \Sigma,\ \beta \succ \alpha_{\nu_1}, \alpha_{\nu_2}$, such that $p^\sigma = q^\sigma$ for every $\sigma \succ \beta$. The set A of equivalence classes of $\mathcal{N}$ with respect to this relation is called the *inductive limit* of the system $\{A^\alpha, \iota_\alpha^\beta\}$, and is denoted by $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha \in \Sigma}$. The equivalent class of a given chain $\nu = \{p^\alpha\}_{\alpha \succ \alpha_\nu} \in \mathcal{N}$ consists of all chains $\eta = \{q^\alpha\}_{\alpha \succ \alpha_\eta} \in \mathcal{N}$ whose coordinates q^α coincide eventually with the coordinates p^α of ν.

Example 1.3.1. (a) Let $\{A^\alpha\}_{\alpha\in\Sigma}$ be a family of uniform algebras, such that $A^\alpha \subset A^\beta$ whenever $\alpha \prec \beta$. Let ι_α^β be the inclusion mapping of A^α into A^β, i.e. $\iota_\alpha^\beta(a) = a \in A^\beta$ for every $a \in A^\alpha$. It is easy to see that in this case the family $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is an inductive system, and $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma} = \bigcup_{\alpha\in\Sigma} A^a$.

(b) Given a uniform algebra A, and an index set Σ, consider the inductive system $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, where $A^\alpha = A$ for every $\alpha \in \Sigma$, and each of ι_α^β, $\alpha, \beta \in \Sigma$, is the identity mapping on A. It is easy to check that the limit $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ of this system is isomorphic to A.

Every coordinate algebra A^β of an inductive system can be mapped naturally into the inductive limit $\varinjlim\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ by a mapping $\iota_\beta\colon A^\beta \longrightarrow \varinjlim\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$, defined as follows: if $a^\beta \in A^\beta$, then $\iota_\beta(a^\beta)$ is the equivalent class of the chain $\{\iota_\beta^\gamma(a^\beta)\}_{\gamma\succ\beta} \in \mathcal{N}$.

Let, for instance, $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be the inductive system from Example 1.3.1(a), where $A^\alpha \subset A^\beta$ whenever $\alpha \prec \beta$, and ι_α^β is the inclusion mapping of A^α into A^β. By definition, the inclusion mapping ι_β of a fixed coordinate algebra A^β into $\bigcup_{\alpha\in\Sigma} A^\alpha = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ maps every $a \in A^\beta$ to the equivalence class of the stationary chain $\{a^\gamma\}_{\gamma\succ\beta}$ with $a^\gamma = a$. Since this class is uniquely defined by the element a, it can be identified by a itself, and henceforth $\iota_\beta(a) = a$ for every $a \in A^\beta$.

One can define algebraic operations in an inductive limit of algebras $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ as follows. Let the chains $\nu_1 = \{a^\alpha\}_{\alpha\succ\alpha_1}$ and $\nu_2 = \{b^\alpha\}_{\alpha\succ\alpha_2}$ be representatives of two elements in $\varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$. Let $\gamma \in \Sigma$, $\gamma \succ \alpha_1, \alpha_2$. The sum $\nu_1 + \nu_2$ is defined as the equivalence class of the chain $\{a^\alpha + b^\alpha\}_{\alpha\succ\gamma} \in \mathcal{N}$. The product in A is defined in a similar way. It is easy to see that the inductive limit $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ is an algebra under these operations.

In the case when the index set Σ is the set of natural numbers $\mathbb{N}$ with the natural ordering, $\{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ is called also an *inductive sequence*, and is expressed by the diagram

$$A^1 \xrightarrow{\iota_1^2} A^2 \xrightarrow{\iota_2^3} A^3 \xrightarrow{\iota_3^4} \cdots .$$

The fact that the algebra $A = \varinjlim \{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ is the limit of the inductive sequence $\{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ can be expressed by the diagram

$$A^1 \xrightarrow{\iota_1^2} A^2 \xrightarrow{\iota_2^3} A^3 \xrightarrow{\iota_3^4} \cdots \longrightarrow A.$$

The inverse systems are dual objects to the inductive ones. Consider a family $\{S_\alpha\}_{\alpha\in\Sigma}$ of sets, parametrized by a directed index set Σ. Suppose that for every pair S_α, S_β, $\alpha \prec \beta$, of sets there is a mapping $\tau_\alpha^\beta\colon S_\beta \longrightarrow S_\alpha$. The collection $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is called an *inverse system* (or, inverse spectrum, projective spectrum) of S_α with connecting mappings τ_α^β, if

(i) τ_α^α is the identity on S_α, and

(ii) $\tau_\alpha^\beta \circ \tau_\beta^\gamma = \tau_\alpha^\gamma$ whenever $\alpha \prec \beta \prec \gamma$.

A *chain* of the system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is any element $\{s_\alpha\}_{\alpha\in\Sigma}$ in the Cartesian product $\prod\limits_{\alpha\in\Sigma} S_\alpha$ such that $\tau_\alpha^\beta(s_\beta) = s_\alpha$ whenever $\alpha \prec \beta$. The family of all chains is denoted by $\varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, and is called the *inverse limit* of the system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$. Clearly, S is a subset of the Cartesian product $\prod\limits_{\alpha\in\Sigma} S_\alpha$. The limit $S = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ of an inverse system can be mapped naturally into every coordinate set S_β by the β-coordinate projection $\pi_\beta : S \longrightarrow S_\beta : \pi\big(\{s_\alpha\}_{\alpha\in\Sigma}\big) = s_\beta$.

In the case when $\Sigma = \mathbb{N}$ with the natural ordering, $\{S_n, \iota_n^m\}_{n\in\mathbb{N}}$ is called also *inverse sequence*, and is expressed by the diagram

$$S_1 \xleftarrow{\tau_1^2} S_2 \xleftarrow{\tau_2^3} S_3 \xleftarrow{\tau_3^4} \cdots .$$

The fact that the set S is the limit of the inverse sequence $\{S_n, \iota_n^m\}_{n\in\mathbb{N}}$ can be expressed also by the diagram

$$S_1 \xleftarrow{\tau_1^2} S_2 \xleftarrow{\tau_2^3} S_3 \xleftarrow{\tau_3^4} \cdots \longleftarrow S.$$

Example 1.3.2. (a) Let $\{S_\alpha\}_{\alpha\in\Sigma}$ be a family of sets, such that $S_\alpha \supset S_\beta$ whenever $\alpha \prec \beta$. Let τ_α^β be the inclusion mapping of S_β into S_α, i.e. $\tau_\alpha^\beta(s) = s \in S_\alpha$ for every $s \in S_\beta$. It is easy to see that in this case the family $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is an inverse system, and $\varinjlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma} = \bigcap\limits_{\alpha\in\Sigma} S_a$. By definition, the projection π_α of $\bigcap\limits_{\alpha\in\Sigma} S_a = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ into a fixed coordinate set S_β maps every chain $\{s_\alpha\}_{\alpha\in\Sigma}$ with $s_\alpha = s$ to its β coordinate $s \in S_\beta$, i.e. π_α is the inclusion mapping of $\bigcap\limits_{\alpha\in\Sigma} S_a$ into S_β.

(b) Let S be a set, and let Σ be an index set. Consider the inverse system $\{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$, where $S_\alpha = S$ for $\alpha \in \Sigma$, and every τ_α^β, $\alpha, \beta \in \Sigma$, is the identity mapping on S. It is easy to check that the limit $\varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ of this system is bijective to S.

If all coordinate sets S_α are topological spaces, then the inverse limit $S = \varprojlim \{A^n, \iota_n^m\}_{n\in\mathbb{N}}$ can be equipped with the topology inherited on S from the Cartesian product $\prod\limits_{\alpha\in\Sigma} S_\alpha$. If, in addition the mappings $\tau_\alpha^\beta \colon S_\alpha \longrightarrow S_\beta$ are continuous, then so are all projections $\pi_\alpha \colon S \longrightarrow S_\alpha$. One can show that if all S_α are compact sets, and $\tau_\alpha^\beta \colon S_\beta \longrightarrow S_\alpha$ are continuous mappings, then S is also a compact set. If all sets S_α have a particular algebraic structure, and the mappings τ_α^β respect this structure, then, in principle, the inverse limit S inherits this structure. For instance, if all S_α are groups [resp. semigroups], and all τ_α^β are group [resp.

semigroup] homomorphisms, then $S = \varprojlim \{S_\alpha, \tau_\alpha^\beta\}_{\alpha\in\Sigma}$ is also a group [resp. semigroup], actually a subgroup [resp. subsemigroup] of the Cartesian product $\prod_{\alpha\in\Sigma} S_\alpha$.

Example 1.3.3. Consider the inverse sequence

$$\mathbb{T}_1 \xleftarrow{\tau_1^2} \mathbb{T}_2 \xleftarrow{\tau_2^3} \mathbb{T}_3 \xleftarrow{\tau_3^4} \mathbb{T}_4 \xleftarrow{\tau_4^5} \cdots,$$

where $\mathbb{T}_k = \mathbb{T}$ are unit circles, and $\tau_k^{k+1}(z) = z^2$ on $\mathbb{T}$. Since all $\mathbb{T}_k$ are compact abelian groups, and z^2 is a continuous group homomorphism, the inverse limit $\varprojlim \{\mathbb{T}_{k+1}, z^2\}_{k\in\mathbb{N}}$ is a compact abelian group. Similarly, if $\overline{\mathbb{D}}_k = \overline{\mathbb{D}}$ are closed unit discs, and $\tau_k^{k+1}(z) = z^2$ on $\mathbb{D}$, the limit of the inverse sequence

$$\overline{\mathbb{D}}_1 \xleftarrow{\tau_1^2} \overline{\mathbb{D}}_2 \xleftarrow{\tau_2^3} \overline{\mathbb{D}}_3 \xleftarrow{\tau_3^4} \overline{\mathbb{D}}_4 \xleftarrow{\tau_4^5} \cdots,$$

is a compact abelian semigroup, containing the inverse limit $\varprojlim \{\mathbb{T}_{k+1}, z^2\}_{k\in\mathbb{N}}$.

Let $\{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be an inductive sequence of uniform algebras $A^\alpha \subset C(X_\alpha)$, and let $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ be its limit algebra. The maximal ideal spaces $\mathcal{M}_{A^a}$ can be lined up into an adjoint inverse system, namely $\{\mathcal{M}_{A^\alpha}, (\iota_\alpha^\beta)^*\}_{\alpha\in\Sigma}$, where the mappings $(\iota_\alpha^\beta)^* \colon \mathcal{M}_{A^\beta} \longrightarrow \mathcal{M}_{A^\alpha}$ are the adjoint mappings of $\iota_\alpha^\beta \colon A^\alpha \longrightarrow A^\beta$, defined as $\big((\iota_\alpha^\beta)^*(\varphi)\big)(f) = \varphi\big(\iota_\alpha^\beta(f)\big)$, where $\varphi \in \mathcal{M}_{A^\beta}$, and $f \in A^\alpha$. The inverse limit $\mathcal{M}_A = \varprojlim \big\{\mathcal{M}_{A^\alpha}, (\iota_\alpha^\beta)^*\big\}_{\alpha\in\Sigma}$ of maximal ideal spaces $\mathcal{M}_{A^a}$ is a compact set. Suppose that the adjoint mappings $(\iota_\alpha^\beta)^*$ map the sets $X_\beta \subset \mathcal{M}_{A^\beta}$ onto $X_\alpha \subset \mathcal{M}_{A^\alpha}$ for every $\alpha, \beta \in \Sigma$. There arises an inverse system $\big\{X_\alpha, (\iota_\alpha^\beta)^*|_{X_\alpha}\big\}$, and its limit $X = \varprojlim \big\{X_\alpha, (\iota_\alpha^\beta)^*|_{X_\alpha}\big\}_{\alpha\in\Sigma}$ is a closed subset of $\mathcal{M}_A$.

There is a close relationship between the properties of the limit algebra A and its coordinate algebras A_α (cf. [L1]).

Proposition 1.3.4. *Assume that $\iota_\alpha^\beta(1) = 1$, and that the adjoint mappings $(\iota_\alpha^\beta)^*$ map X_β onto X_α for every $\alpha, \beta \in \Sigma$. Then:*

(i) *$A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ can be viewed as an algebra of continuous functions on X.*

(ii) *The maximal ideal space of the closure $\overline{A}$ of A in $C(X)$ coincides with the set $\mathcal{M}_A$.*

(iii) *If $(\iota_\alpha^\beta)^*$ maps the Shilov boundary ∂A^β onto ∂A^α for every $\alpha \prec \beta$, then the Shilov boundary of $\overline{A}$ is the set $\varprojlim \big\{\partial A^\alpha, (\iota_\alpha^\beta|_{\partial A^a})^*\big\}_{\alpha\in\Sigma}$.*

(iv) *If every A^α is a Dirichlet algebra on X_α, then $\overline{A}$ is a Dirichlet algebra on X.*

Definition 1.3.5. The closure $\overline{A}$ of an inductive limit $A = \varinjlim \{A^\alpha, \iota_\alpha^\beta\}_{\alpha\in\Sigma}$ of algebras is called an *inductive limit algebra.*