Computational Earthquake Physics: Simulations, Analysis and Infrastructure, Part I

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Computational Earthquake Physics

PART I: Introduction

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Large earthquakes are terrible natural disasters which usually cause massive casualties and huge property loss. In the beginning of the new century, large earthquakes violently struck the world, especially the Asia-Pacific region. Nearly 300,000 people were killed by the magnitude 9.0 Northern Sumatra earthquake and tsunami, and the magnitude 7.8 Pakistan earthquake of October 8th, 2005 resulted in 90,000 deaths. In the meantime, there has been great progress in computational earthquake physics. New understanding of earthquake processes, numerous ideas on earthquake dynamics and complexity, new numerical models and methods, higher performance super-computers, and new data and analysis methods are emerging. These include the LSM (Lattice Solid particle simulation Model) Australian Computational Earth Systems Simulator (ACcESS), Japan’s Earth Simulator, GeoFEM, GeoFEST, QuakeSim, SERVO grid, iSERVO, LURR (Load-Unload Response Ratio), PI (Pattern Informatics), Critical Sensitivity, earthquake Critical Point Hypothesis, the friction law and seismicity, tremor, the Virtual California model, interaction between faults and the conversation of earthquakes, ROC (Relative Operating Characteristic), MFEM (Multiscale Finite-Element Model), etc. Most of these are the outcome of ACES-related research and activities, and will be presented in this volume.

The APEC Cooperation for Earthquake Simulation (ACES) [1], endorsed by APEC (Asia-Pacific Economic Cooperation) in 1997, capitalizes on this new...
opportunity and the complementary strengths of the earthquake research programs of individual APEC member economies via collaboration towards development of such models, the necessary research infrastructure to enable large-scale simulations, and to assimilate data into the models.

The inaugural workshop, the second and the third workshop of ACES were held in 1999, 2000 and 2002, respectively [2–5]. During the week of July 9–14, 2004, China hosted the 4th ACES workshop [6] in Beijing. The 4th ACES Workshop was a milestone for ACES as unanimous agreement was reached for the follow-on to ACES, the ACES-iSERVO [7] International Institute (International Solid Earth Research Virtual Observatory Institute). A colloquium on iSERVO was held at the 4th ACES Workshop leading to broad endorsement for establishment of the iSERVO Institute by the international group of over 100 scientists in attendance, and subsequent signing of a formal agreement – “The Beijing Declaration” [8] – to establish the institute which will be a frontier international research institute on simulating the solid earth. A special issue on Earth Systems Modelling overviewed contributions to the development of the iSERVO institute [7] by its key participants. The institute will consist of a node in each participating economy, and will build on complementary national programs, centers and facilities for solid earth simulation. The institute’s focus will be development of predictive capabilities for solid earth phenomena via simulation and breakthrough science using the computational simulation capabilities aimed at understanding solid earth system complexity.

This special issue is divided into two parts. The first part (part I) incorporates Micro-Scale Simulation, Macro-Scale Simulation and Scaling Physics. Topics covered range from numerical developments, rupture and gouge studies of the particle model, Liquefied Cracks and Rayleigh Wave Physics, studies of catastrophic failure and critical sensitivity, numerical and theoretical studies of crack propagation, development in finite-difference methods for modeling faults, long time scale simulation of interacting fault systems, modeling of crustal deformation, through to mantle convection. The second part (Part II, PAGEOPH Vol 163, No 11/12 (2006)) incorporates Computational Environment and Algorithms, Data Assimilation and Understanding, Model Applications and iSERVO.

The 4th ACES workshop (2004) was planned by the ISB (International Science Board) of ACES, consisting of Peter Mora, Xiang-chu Yin, Mitsuhiro Matsuura, Andrea Donnellan and Jean-Bernard Minster, and was hosted by the Institute of Earthquake Science, China Earthquake Administration and LNM (State Key Laboratory of Nonlinear Mechanics), Institute of Mechanics, Chinese Academy of Sciences.
We appreciate our sponsors including China Earthquake Administration, Chinese Academy of Sciences, Chinese National Natural Science Foundation, Chinese Ministry of Science and Technology, Chinese Ministry of Finance, Australia-China Fund, Australian Research Council, The University of Queensland and Earth Systems Science Computational Centre, Australian Computational Earth Systems Simulator Major National Research Facility, Japan Society for the Promotion of Science, Research Organization for Information Sciences and Technology (RIST), the National Aeronautics and Space Administration (NASA) and the US National Science Foundation (NSF) and the United States Geological Survey (USGS).

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Exciting developments in earthquake science have benefited from new observations, improved computational technologies, and improved modeling capabilities. Designing a realistic supercomputer simulation model for the complete earthquake generation process is a grand scientific challenge due to the complexity of phenomena and range of scales involved from microscopic to global. The APEC Cooperation for Earthquake Simulation (ACES) aims to develop such models. Since 1997 four ACES Workshops have been held in Brisbane and Noosa in Australia, Tokyo and Haoken in Japan, Maui, Hawaii in USA and Beijing, China on July 10–14, 2004, respectively. The book mainly contains the results presented in the 4th ACES Workshop in Beijing and the new outcomes from 4th ACES Workshop to the present. The book covers: Microscopic simulation of earthquake, scaling physics, macroscopic simulation, computational environment and algorithms, data assimilation and understanding, model applications and iSERVO (International Solid Earth Research Virtual Observation).
Part I of the book focuses on microscopic from numerical and physical simulation, scaling physics, dynamic rapture and wave propagation, earthquake generation, cycle and seismic pattern.
Fracture of a Liquefied Crack and the Physics of Rayleigh Waves

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\textit{Abstract}—The standard free-surface boundary conditions for in-plane crack dynamics are shown to be identical to the conditions for crack dynamics on a liquefied crack. The surfaces of both the free and liquefied cracks do not separate during faulting and hence the static normal stress is not relaxed by the faulting. A crack with either free or liquid boundary conditions deforms in the transverse direction during slip. It follows that both the free and liquefied cracks may represent solutions to the heat-flow paradox. As an application of the proof, we derive a physical understanding of the properties of harmonic Rayleigh waves on a uniform elastic half-space without solving a cubic equation.

\textbf{Key words:} Fracture, faulting, boundary conditions, Rayleigh waves.

\textit{Introduction}

The conventional mathematical model of the dynamics of slip on an in-plane shear fracture on a planar fracture surface with normal in the $z$-direction, involves the solution to the elastic wave equation for the slip under the condition of a sudden decrease in the shear traction. The slip is given by the solution to the elastic wave equation under the free-crack conditions

$$\tau_{zz}^{(1)} = \tau_{xz}^{(2)} = 0, \quad \tau_{xz}^{(1)} = \tau_{zz}^{(2)} = 0$$

at $z = 0$; the superscripts refer to each half-space. (We restrict discussion to the two-dimensional case of crack growth without loss of generality.) Since the conditions (1) are also those for stress-free surfaces, one interpretation of the condition $\tau_{zz} = 0$ has been that faulting develops under a crack-opening process in which slip takes place as though the two surfaces of the fault were separated sufficiently that they are no longer in contact. To account for initiation and evolution of slip by crack opening, hopping and Schallamach modes of deformation have been proposed (see Brune, \textsuperscript{1})

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et al., 1993). However, it is unlikely that an in-plane shear fracture could be
accompanied by a decrease in the component $\tau_{zz}$ even at modest depths in the earth
where the hydrostatic stress is nonzero. One proposal to bypass this difficulty has
been to assume that fluids are present and that faulting takes place when pore
pressures match the normal stress $\tau_{zz}$. Another proposal suggests that slip without
 crack opening takes place under conditions of significant contrast in elastic
properties on the two sides of a fault, a condition under which head waves of slip
propagate along the fault (Andrews and Ben-Zion, 1997). The requirement that
there be crack opening will be shown to be the consequence of a misinterpretation of
the boundary condition $\tau_{zz} = 0$. The proposal for fluidization is a vital step,
completely consistent with the conditions (1), and neither crack opening nor material
contrast across the fault are needed to find a solution to the problem of in-plane
faulting dynamics.

\textit{Slip on a Liquefied Crack}

Consider the problem of antisymmetric slip on a fault imbedded in a
homogeneous, elastic medium; the surfaces slip in opposite directions equally.
Near the advancing edge of a fracture there must be a gradient of the slip with
increasing distance from the edge. Thus the region of gradient whose direction of
slip is toward the edge, must be under horizontal dynamical compressional strain
$\epsilon_{xx} < 0$, and the opposing surface must be under horizontal dynamical tensional
strain $\epsilon_{xx} > 0$ (Fig. 1). The gradients of horizontal deformation of the surface
must be accompanied by a deformation in the z-direction. Since one surface is in
horizontal compression and the other in tension, the vertical deformation of the
two opposing surfaces must be in the same direction, i.e., one must deform
inward as an indentation into one half-space and the other must deform outward
as a bulge into the other half-space. (We defer discussion of the sign of the
vertical component of motion to the next section of this paper.) For perfectly
antisymmetric slip, the amplitudes of the two vertical components of deformation
are equal. Hence $u_z$ is continuous across the boundary $z = 0$ and the two slipping
surfaces are in contact throughout the slip history; $u_z$ is symmetric in coordinate
$z$, although it is antisymmetric with respect to the outward drawn normals to the
two surfaces. It follows that the stress $\tau_{zz}$ is also continuous across the boundary.
Thus the appropriate formulation of the boundary conditions for slip on a crack in
an infinite medium is

$$\tau_{zx}^{(1)} - \tau_{zx}^{(2)} = 0, \quad u_z^{(1)} - u_z^{(2)} = 0, \quad \tau_{zz}^{(1)} = \tau_{zz}^{(2)} = 0 \quad (2)$$

at $z = 0$. The whole-space conditions (2) are similar to the half-space conditions (1)
except a) there is a new condition describing the continuity of $u_z$, and b) the
condition $\tau_{zz} = 0$ in the half-space case is relaxed to the less stringent continuity condition on $\tau_{zz}$.

Since the quantity $u_z$ is an antisymmetric function and $u_x$ a symmetric function of coordinate $z$ across $z = 0$, it follows that $\tau_{zz}$ is antisymmetric and $\tau_{xz}$ is symmetric. From the antisymmetry with respect to $z = 0$, $\tau_{zz}^{(1)} = -\tau_{zz}^{(2)}$ while from the condition of continuity in (2), $\tau_{zz}^{(1)} = \tau_{zz}^{(2)}$. Thus

$$\tau_{zz}^{(1)} = \tau_{zz}^{(2)} = 0,$$

on $z = 0$, which is the result obtained by Das and Ak1 (1977). Thus a set of boundary conditions on $z = 0$ equivalent to (2) is

$$\tau_{zz}^{(1)} = \tau_{zz}^{(2)} = 0, \quad u_z^{(1)} - u_z^{(2)} = 0, \quad \tau_{xz}^{(1)} = \tau_{xz}^{(2)} = 0. \quad (2')$$

The condition $\tau_{zz} = 0$ in (1) is a property of conditions (2) as well.

The correspondence to the usual conditions (1) is now complete. The continuity condition $u_z^{(1)} = u_z^{(2)}$ must be added as in (2') to conditions (1). In-plane antisymmetric slip on a liquid interface is accompanied by a transverse motion of the fault. The fallacy in assuming that there is crack-opening during faulting is in the assumption of an inappropriate symmetry: the surfaces do not move apart; instead they remain in contact at all times. Thus neither crack-opening and its variations, nor exotic combinations of properties are needed to understand faulting.

Since the stress $\tau$ is derived from the elastic wave equation and is therefore associated with the dynamics of faulting, the dynamic constraint $\tau_{zz}^{(1)} = \tau_{zz}^{(2)} = 0$ on
$z = 0$ is a condition that the static stress before fracture $T_{zz}$ be unchanged during the dynamic rupture. Hence faulting can take place in the presence of significant hydrostatic stress without reduction of the static prestress. The conditions (2) $\equiv (2')$ require liquidity of the interface. The formulation of the free-crack problem (1) on each of the two half-spaces abutting the crack is identical to the liquid interface formulation for faulting in an infinite medium for antisymmetric slip, (1) $\equiv (2')$. From the equivalence (1) $\equiv (2)$, the conditions (1) are also the conditions for liquidity of the interface.

Rayleigh wave motions with the usual phase velocity are not only a property of motions on a half-space with a stress-free surface (1), but as long as faulting is antisymmetric, they are also a property of a whole space with an internal fault surface with liquid boundary conditions (2'), since the conditions $\tau_{xz} = \tau_{zz} = 0$ on $z = 0$ appear in both. The addition of the continuity condition on the normal displacement $u_x$ to the usual conditions (1) does not change our expectation of an important Rayleigh wave component to the slip on the surface of the fracture.

The infinite medium Green’s function for dynamic fracture on a liquefied crack is easily obtained from the solution by Richards (1979) where the terms of oppositely directed tangential forces are selected, corresponding to the application of a traction $T_{xz}$ at each surface. The single-couple nature of this source function is not in conflict with its double-couple body force equivalence in an unfaulted medium.

The generation of frictional heat during faulting should be minimal during fracture on a crack liquefied with an ideal frictionless liquid. It follows from the demonstration above, that the same result should hold for a crack with free surface conditions. These observations may represent a resolution of the heat flow paradox.

The Physics of Rayleigh Waves

Questions concerning the physics of the propagation of a Rayleigh wave on an elastic homogeneous half-space are often answered with the irrelevant mathematical response that the Rayleigh wave is the solution to a familiar cubic equation. The cubic equation arises from the mathematics of wave propagation with boundary conditions at the surface $\tau_{nn} = \tau_{nt} = 0$ where $n$ is the normal to the surface of the half-space and $t$ is the tangential direction, taken to be the direction of wave propagation. The reliance on mathematics is in direct contrast to the immediacy and transparency with which compression and shear waves can be understood in terms of a simple physical picture of deformation of an elastic solid. The minimal physical argument of the preceding section can be used as the basis for understanding the simple physics of deformation in Rayleigh wave propagation at the surface of a homogeneous half-space. It will be shown without difficult mathematics that the wave velocity is less than that of S waves, that the
two components of motion are Hilbert transforms of one another, that the motions will be retrograde elliptical on the surface and prograde elliptical at depth for sinusoidal Rayleigh wave motions, as well as other familiar properties. Ours will be a qualitative theory. We amend arguments presented elsewhere (KNOPOFF, 2001).

Consider an elastic half-space deformed dynamically by horizontal displacements at the planar surface \( z = 0 \) (Fig. 2); the deformation moves to the right with Rayleigh wave velocity. To the left of point A, the surface is displaced uniformly to the right, while to the right of point B there is no displacement of the surface. Since there is no compression or extension of the undeformed regions at the far left and right, there is no vertical component of the displacement in these regions. Vertical displacement of the surface in the central region AB is associated with the gradient of the horizontal component of the displacement as in the first part of this paper.

Again we choose the coordinates (\( x, z \)) to correspond to the directions (\( t, n \)). The sign of the vertical component of the displacement \( u_z \) in the transition interval AB between the two undeformed regions follows from the condition that the shear stress

Figure 2
a) Wave of horizontal displacements at the surface of an elastic half-space. The displacements are positive and uniform to the left of A and positive and uniform to the right of B. b) \( u_x(x) \) at \( z = 0 \). c) \( u_z(x) \) at \( z = 0 \).
τ_{xz} vanishes at the surface. We assume that the horizontal component of the
displacements of the elastic medium resulting from the deformation of the surface
must decrease with increasing depth, and vanishes at infinite depth. For \( u_x > 0 \) in
AB, as in the figure, then \( \partial u_x / \partial z < 0 \). Since the boundary condition for the shear
stress at the surface is \( \tau_{xz} = \mu(\partial u_x / \partial z + \partial u_z / \partial x) = 0 \), then \( \partial u_z / \partial x > 0 \). Hence the
horizontal compression at sites in the transition zone in the region between A and B
must have a corresponding indentation of the surface. Conversely a traveling
horizontal extension must have an outward bulge in the vertical direction. An
intuitive expectation that the surface under compression would be likely to bulge
outward would be correct if the half-space were incompressible, but under the
condition of zero shear at the surface, it has the opposite sign. We return to this point
below.

There is also a geometrical view of the deflection of the surface. Consider an
element of fault area at the surface at X in the transition zone. As above, \( \partial u_x / \partial u_z < 0 \)
and \( \partial u_z / \partial u_x > 0 \), the two quantities being equal and opposite to each other. These
gradients are shown at the left in Figure 3a. After the mean motion, which is
rightward and downward, is subtracted (Fig. 3b), there remains a residual net

\[
\begin{align*}
\text{a) } & \quad  & \text{b) } & \quad  & \text{c) } & \quad \\
\text{A} & \quad \text{B} & \quad & \quad & \quad & \\
\end{align*}
\]

Figure 3
Deformation of an element of area at point X (see Fig. 2) at the surface of the half-space, under a condition
of vanishing shear stress. a) Deformation has a negative vertical gradient of \( u_x \) and an equal positive
horizontal gradient of \( u_z \). These can be decomposed into a translation and a rotation. b) Vectorial
decomposition as in a). c) Total motion of element showing depression and rotation of surface line
element AB.
rotation of the elemental area in the clockwise direction (Fig. 3c); the surface is indented as before.

Consider the deformation of the half-space by spatially sinusoidal horizontal displacements at the surface (Fig. 4a) in the moving coordinate system. The vertical component of the displacement is also sinusoidal since it is proportional to the gradient of the surface motion; the maxima of the bulges B and the minima of the indentations D in the vertical component of the displacement at the surface must correspond to the extreme values of extension E and compression C in the gradient of the horizontal component of the displacement, respectively (Fig. 4b). A zero value of the vertical component of the displacement corresponds to a zero value of the gradient of the horizontal component. The two components of the displacement at the surface have a 90° phase difference, and hence the motion has an elliptical trajectory. Thus the two components of motion are Hilbert transforms of one another for any function of \((ct - x)\). From the bulge/extension and indentation/compression phase correlations at times (a,b,c,d), the
elliptical motion is retrograde at the surface (Fig. 4c) for either direction of propagation.

To determine that the Rayleigh wave velocity is less than that of S-waves, consider the wave properties of the deformation, characterized by the wave equation. The solutions to the elastic wave equation for horizontally traveling waves are

$$\exp\{i\omega(x/c - t) - \omega\eta_{ps} z\},$$

where $c$ is the velocity of Rayleigh waves. The vertical spatial decay rates $\eta_s$ and $\eta_p$ for the shear and compression wave components of the motion satisfy

$$\frac{1}{c^2} = \frac{1}{(\alpha^2, \beta^2)} + \eta_{ps}^2,$$

where $\alpha > \beta$ are the P- and S-wave velocities. By inspection, $c < \beta$, a direct consequence of the vanishing of the motion at infinite depth.

The P-wave component of the motion decays more rapidly with depth than the S-wave component, $\eta_p > \eta_s$. The displacement vectors and stress and strain components, all propagate parallel to the surface with phase velocity $c$, and all have amplitudes that vary with depth as the sum

$$(Ae^{-\alpha_p z} + Be^{-\alpha_s z})e^{i\omega(x/c - t)}.$$  

The invariant volumetric strain $\epsilon_{kk} = \epsilon_{xx} + \epsilon_{zz}$ is an exception to this rule since it can have no dependence on the shear properties of the system. The volumetric strain must fall off with depth as $e^{-\alpha_p z}$, i.e., $B = 0$. There is a 90° phase shift between the components of the pairs $(u_x, u_z), (\epsilon_{xx}, \epsilon_{zz}),$ etc.; the vector displacement and strain are elliptically polarized.

The normal stress $\tau_{zz} = (\lambda + 2\mu)\partial u_x/\partial x + \lambda \partial u_x/\partial x = 0$ at the surface. Hence

$$\partial u_z/\partial z = -\frac{\lambda}{\lambda + 2\mu} \partial u_x/\partial x$$

at $z = 0$. A site of maximum horizontal compression at the surface is also a site of maximum amplitude of the vertical component of displacement at the surface. At this site $\partial u_x/\partial x < 0$, and thus $\partial u_z/\partial z > 0$. Thus the vertical component of the displacement at the surface increases with depth. At greater depth the vertical component of displacement must decrease to zero steadily with depth. Hence $\partial u_z/\partial z < 0$ at great depth below a surface site of maximum horizontal compression.

Although both the volumetric and shear strains vanish at depth, at very large but finite depth the volumetric strain is much smaller when compared with the shear strain since $\eta_p \geq \eta_s$. We approximate $\epsilon_{kk} = 0$ at large but finite depth, and obtain $\partial u_x/\partial x > 0$, i.e., there is a reversal of the horizontal component of the displacement.
at depth relative to that at the surface. Thus the elliptical polarization at depth is prograde. As a consequence, there must be a crossover depth at which the horizontal component of sinusoidal displacements is zero. Near the surface, the motion is controlled by the vanishing of the shear strain while at depth the motion is dominated by the vanishing of the volumetric strain. The general shape of the depth dependence in both components of the motion can now be sketched as in Figure 5.

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Experimental Evidence of Critical Sensitivity in Catastrophe

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Abstract—The paper presents an experimental study on critical sensitivity in rocks. Critical sensitivity means that the response of a system to external controlling variable may become significantly sensitive as the system approaches its catastrophic rupture point. It is found that the sensitivities measured by responses on three scales (sample scale, locally macroscopic scales and mesoscopic scale) display increase prior to catastrophic transition point. These experimental results do support the concept that critical sensitivity might be a common precursor feature of catastrophe. Furthermore, our previous theoretical model is extended to explore the fluctuations in critical sensitivity in the rock tests.

Key words: Critical sensitivity, catastrophe, damage fraction, acoustic emission, digital speckle correlation method, fluctuation.

1. Introduction

Many observational evidences show that there are some precursors prior to a major earthquake, such as accelerating moment release (AMR) or power-law increase in the number of intermediate-size events (Jaume et al., 1999; Bowman et al., 1998; Rundle et al., 2000a), anomalously high values of Lurr (Yin et al., 2000), etc.

Recently, based on statistical mesoscopic damage mechanics, Xia et al. (Xia et al., 2002; Zhang et al., 2004) suggested that the sensitivity of response to controlling variable might be an effective variable to characterize a system with macroscopic uncertainty. In particular, a properly defined sensitivity of the system may display significant increase as the system approaches its catastrophic point, i.e., the transition point from damage accumulation to catastrophic rupture. Such a behavior is called critical sensitivity. The underlying mechanism behind critical sensitivity is the
coupling effect between disordered heterogeneity on multiple scales and dynamical nonlinearity due to damage-induced stress redistribution (Stein, 1999; Xia et al., 2000, 2002). Critical sensitivity may provide a clue to prediction of catastrophic transition, such as material failure or great earthquakes, provided the sensitivity of the system is measurable or can be monitored.

To validate the concept of critical sensitivity, a series of experiments have been conducted. 167 gabbro samples were tested under uniaxial compression. The deformation and damage processes were observed with acoustic emission (AE) and white Digital Speckle Correlation Method (DSCM) (Peter et al., 1981; Ma et al., 2004) synchronously. Then, the sensitivities characterizing the evolution of damage, surface displacement pattern and AE energy induced by boundary displacement can be obtained. The experimental results show that these three kinds of responses of the samples become significantly sensitive to the controlling variable, i.e., boundary displacement, as the samples approach catastrophe. This clearly indicates that the experimental results do support the concept of critical sensitivity reasonably.

In addition, it can be seen that the sensitivities observed in the experiments display fluctuations, while the sensitivity obtained from the previous theoretical approximation (Zhang et al., 2004) is monotonic and smooth. In order to understand the mechanism governing the fluctuations in sensitivity, a model with multi-peak structure in the distribution function of mesoscopic units’ threshold is introduced. It is found that the multi-peak structure might be responsible for the fluctuations shown in critical sensitivities.

2. Observations of Critical Sensitivity in Rock Failure

2.1 Experimental Method

In our tests, rectangular gabbro samples with dimensions of $5 \times 5 \times 13$ mm$^3$, were compressed uniaxially with a MTS810 testing machine. The loading mode is boundary-displacement control with velocity of 0.02 mm/min. The displacement was measured by an extensometer with resolution of 3 $\mu$m and an offset of 1 kN load.

The surface of the specimen was illuminated by a white luminescence and the speckle images were captured and transferred to a computer by a CCD camera. After the experiment, the speckle images were analyzed with DSCM, thus, both displacement and strain fields during the loading process were obtained.

Moreover, two AE sensors were fixed on two sides of a sample with a specially designed clamp. The resonant frequencies of the sensors are 140 kHz and 250 kHz, respectively. The AE signals were recorded and processed by an AE21C system produced by the Institute of Computer Technology of Shenyang, China. As well known, AE is an effective method to detect damage process of rock, so the AE series, such as AE energy, can provide statistical information on damage evolution.
2.2 Definitions of Critical Sensitivity Adopted in Rock Experiments

To deal with experimental data, a dimensionless boundary displacement \( U \) is adopted, i.e., \( U = \frac{U^*}{13} \), where \( U^* \) is the actual boundary displacement. (Hereafter, symbols without superscript * will represent either dimensionless (like stress, displacement, length and energy, etc.) or normalized (like strain) variables, while those with * mean dimensional or non-normalized ones), and factor 13 is the length of the sample along the loading axis with unit of millimeter. Figure 1 shows the processed curves of actual nominal stress \( \sigma^* \) versus dimensionless boundary displacement \( U^P \) for 151 gabbro samples (in the following text, symbols with superscript P demonstrate the processed experimental variables. See the processing method in Section 2.2.1). The catastrophe appears at the end of each curve. Clearly, it is hard to forecast when catastrophe will occur beforehand. In particular, the maximum and the failure stresses show large diversity. This macroscopic uncertainty results in great difficulty in rupture prediction.

In order to measure the sensitivity of a rock sample to external controlling variable, we define sensitivity \( S \) as

\[
S = \frac{\Delta}{\Delta U} \left( \frac{\Delta R}{\Delta U} \right),
\]

where the dimensionless boundary displacement \( U \) is the external controlling variable and \( R \) is the response of the rock sample. Moreover, in the theoretical model, if the second-order derivative of \( R \), \( \frac{d^2 R}{dU^2} \), exists, Eq. (1) can be written as

\[
S = \frac{d^2 R}{dU^2}.
\]

In the present paper, \( R \) is defined as the accumulative response of the rock sample. Hence, the first-order derivative of \( R \) demonstrates the changing rate of the response

![Figure 1](image)

Processed curves of experimental nominal stress \( \sigma^* \) versus dimensionless boundary displacement \( U^P \) for 151 gabbro samples under uniaxial compression. The symbols × indicate catastrophe points.
with respect to the external controlling variable. In order to measure the sensitivity of the changing rate of the response to the external controlling variable, the second-order derivative of $R$ is adopted as the definition of sensitivity, because it is considerably more sensitive to the external controlling variable than the first-order derivative.

Importantly, $R$ could be chosen from different kinds of responses. In this paper, the responses from the behaviors at different scales is adopted, i.e., the mean damage fraction $D$ at global scale, the distance $\Delta H^*$ between successive patterns of surface displacement related to the behavior at locally macroscopic scale and the AE energy $\Theta^*$ contributed from the events at mesoscopic scale. In data processing, $D$ can be calculated from the experimental nominal stress $\sigma^*$ and strain $\varepsilon^*$ curve, $\Delta H^*$ can be calculated from the surface displacement patterns, and $\Theta^*$ can be obtained directly from the AE system, as discussed later in detail.

Critical sensitivity means that the response to the controlling variable, i.e., boundary displacement $U$, may become significantly sensitive, i.e., $S \gg 1$, prior to the catastrophe. Now, we focus on whether critical sensitivity is a common precursor to final rupture in rock experiments.

2.2.1 Sensitivity calculated from damage evolution

At the initial part of the raw experimental nominal stress $\sigma^*$ and strain $\varepsilon^*$ curve (Fig. 2(a), solid line), the slope of $\sigma^*(\varepsilon^*)$ curve, $\Delta \sigma^*/\Delta \varepsilon^*$ (Fig. 2(a), bulk solid line), increases with increasing $\varepsilon^*$ due to the closure of micro-cracks. For simplicity, without regard to healing process of micro-cracks, only the weakness induced by
damage is considered based on damage mechanics (Jayatilaka, 1979). Then, the
global damage fraction $D$ can be calculated from the processed $\sigma^{*p} (\varepsilon^{*p})$ curve
obtained by the following steps (Xu et al., 2004): (1) Calculate the slope of $\sigma^{*}(\varepsilon^{*})$
curve, $\Delta \sigma^{*}/\Delta \varepsilon^{*}$ (Fig. 2(a), bulk solid line), and suppose $E_0^{*}$, which equals to the
maximum value of $\Delta \sigma^{*}/\Delta \varepsilon^{*}$ (at point O in Fig. 2(a)), as the initial elastic modulus of
the rock sample. (2) Draw a straight line with slope $E_0^{*}$ from the origin
($\sigma^{*p} = \varepsilon^{*p} = 0$) to $O'$ ($O'$ and O locate at the same nominal stress). (3) Parallelly
shift the O-F-part of the raw stress-strain curve $\sigma^{*}(\varepsilon^{*})$ to point $O'$. Then, the entire
processed experimental nominal stress-strain curve $\sigma^{*p}(\varepsilon^{*p})$ (dashed line in Fig. 2(a))
is obtained. Similarly, parallelly draw the processed curve of nominal stress $\sigma^{*p}$
versus boundary displacement $U$ (solid line in Fig. 2(b)) from the origin ($\sigma^{*p} = U^{p}
= 0$). Then, the processed experimental nominal stress $\sigma^{*p}$ versus dimensionless
boundary displacement $U^{p}$ curve (dashed line in Fig. 2(b)) can be obtained.

Theoretically, global damage fraction $D$, a macroscopic variable characterizing
damage evolution, can be obtained from the processed experimental nominal stress
$\sigma^{*p}$ and strain $\varepsilon^{*p}$ curve based on mean field (MF) approximation. According to
damage mechanics and MF approximation, the constitutive relation can be

$$\sigma^{*p} = \varepsilon^{*p} E_0^{*} (1 - D).$$

Then, global damage fraction $D$ can be calculated by

$$D = 1 - \frac{\sigma^{*p}}{\varepsilon^{*p} E_0^{*}}.$$  \hspace{1cm} (3)

The global damage fraction $D$ of rock sample calculated from $\sigma^{*p}(\varepsilon^{*p})$ curve is
shown in Figure 3(a). When choosing the response of $R$ to be damage $D$, the
sensitivity is denoted by $S_D$. Figure 3(b) shows the curve of $S_D$ versus the boundary
displacement $U$ for a sample.

In order to compare the sensitivity series of different samples, a normalized
boundary displacement $U_0$ is adopted,

$$U_0 = U / U_c,$$  \hspace{1cm} (4)

where $U_c$ is the dimensionless boundary displacement of a sample at its catastrophic
point. Figure 3(c) shows the curves of $S_D$ versus $U_0$ for 151 samples.

2.2.2 Sensitivity calculated from the distance between successive patterns of surface
displacement

Since the length-pixel ratio of the imaging system is about 0.028 mm/pixel in
DSCM system, the obtained displacement can be understood as an average over the
area of $28 \times 28 \mu m^2$. The surface strain pattern can be calculated from the surface
displacement pattern. Owing to damage evolution, the surface strain pattern becomes
inhomogeneous on multi-scales, and later strain localization appears. Then, new
scales between the pixel scale and the sample scale emerge. These emerging scales are called locally macroscopic scales in this paper. In order to describe the change of response of the system at the locally macroscopic scales, distance between the successive patterns of surface displacement is introduced. The distance between two patterns of surface displacement is defined as

$$\Delta H_i^* = \frac{1}{N_{\text{eff}}} \sum_{x=1}^{131} \sum_{y=1}^{401} \left| \Delta u_i^*(x,y) \right|, \quad i = 1, 2,$$

where $\Delta u_i^*(x,y)$ is the increment of surface displacement vector $u_i^*$ at point $(x,y)$ along axis $i$ ($i = 2$ indicates loading direction whereas $1$ the direction vertical to
loading), 131 and 401 are the number of points in the surface displacement pattern along axis 1 and axis 2, respectively, and $N_{\text{eff}}$ is the number of points effective for DSCM calculation.

The curve of $\Delta H^*_i$ versus $U^p$ is shown in Figure 4(a), and the sensitivity $S_{H_i}$ calculated from the distance between surface displacement patterns is shown in Figure 4(b) for a single sample and Figure 4(c) for 143 samples.

### 2.2.3 Sensitivity calculated from AE energy

Acoustic emission, resulted from mesoscopic damage evolution, is a response of rock sample to boundary displacement on the mesoscopic scale. AE energy is a

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**Figure 4**

(a) The distance between surface displacement patterns $\Delta H^*_i$ versus $U^p$. (b) Sensitivity $S_{H_i}$ calculated from distance between surface displacement patterns versus $U^p$ for a sample. (c) $S_{H_i}$ versus $U_0$ for 143 samples.
proper parameter characterizing damage and the accumulated AE energy $\Theta^*$ can be obtained directly from AE recording, see Figure 5(a).

Then, taking response $R = \Theta^*$, the sensitivity $S_{\Theta^*}$ calculated from AE energy is shown in Figure 5(b) for a single sample and Figure 5(c) for 131 samples.

2.3 Critical Sensitivity in Rock Experiments

According to the above-mentioned method, the experimental results of sensitivity reflecting the responses on three scales are obtained, namely $S_D$, $S_{H_t}$ and $S_{\Theta^*}$ from the responses on sample scale, locally macroscopic scales and mesoscopic scale, respectively.

![Graphs](image)

Figure 5
(a) Accumulative AE energy $\Theta^*$ versus $U^p$. (b) Sensitivity $S_{\Theta^*}$ calculated from AE energy versus $U^p$ for a sample. (c) $S_{\Theta^*}$ versus $U_0$ for 131 samples.
Figures 3(b), 4(b), and 5(b) show the sensitivity $S$ at the three scales for a single sample. At the initial stage, $S_{\Theta}$ is equal to zero since the events of mesoscopic damage are too small to be detected by AE sensors, $S_{H_1}$ and $S_D$ are also equal to zero since the change of the surface displacement patterns is nearly zero and macroscopic damage can be neglected at the initial stage. In other words, at the initial stage, the responses of the system at all scales are not sensitive to the external controlling variable, i.e., the boundary displacement $U$. As the boundary displacement increases, the three sensitivities remain in low level. This means that the system is in a state with low sensitivity. However, the three sensitivities increase significantly prior to the catastrophic transition point. This implies that the system becomes highly sensitive prior to catastrophic point, from mesoscopic scale to macroscopic scale.

Figures 3(c), 4(c), and 5(c) show the three sensitivities for more than 100 samples. Noticeably, the series of sensitivities are different from sample to sample. That is to say, the catastrophic rupture demonstrates sample specificity. But, more importantly, there is a common trend in sensitivity for all samples, namely significantly increasing sensitivity near the catastrophic transition $U_0 = 1$. This is a strong experimental evidence of critical sensitivity prior to catastrophic rupture in heterogeneous rock.

Since the value of sensitivity of different samples displays large diversity, only a qualitative analysis can be given in present paper. In our later research, more quantitative work will be done.
3. Theoretical Analysis of Critical Sensitivity

However, it is noticeable that the sensitivity shows severe fluctuations in experiments (Figs. 3–5). In order to explore the mechanism underlying the fluctuations of sensitivity in rock experiments, a theoretical model is proposed to explain the phenomenon. Suppose the rock sample and the MTS tester, be two parts in series and driven by boundary displacement quasi-statically. According to MF approximation, the boundary displacement equals to

\[ U^* = L^* \varepsilon^* + L^*_c \varepsilon^*_c, \]

where \( \varepsilon^* \) and \( \varepsilon^*_c \) are nominal strains of the rock sample and the elastic tester respectively, while \( L^* \) and \( L^*_c \) are the corresponding initial length along the loading axis of the two parts. Under uniaxial monotonic loading, the equilibrium condition between the rock sample and the elastic part can be written as

\[ \sigma^* = \varepsilon^*_c E^*_c = \varepsilon^* E^*_0(1 - D), \]

where \( \sigma^* \) is the nominal stress, \( E^*_c \) is the elastic modulus of the tester, and \( E^*_0 \) is the initial elastic modulus and \( D \) is the damage of the rock sample. According to damage mechanics, the true stress \( \sigma^*_s \) and true strain \( \varepsilon^*_s \) of the rock sample are respectively given by

\[ \sigma^*_s = \frac{\sigma^*}{1 - D} \quad \text{and} \quad \varepsilon^*_s = \varepsilon^*. \]

After taking dimensionless stress \( \sigma \) and normalized strain \( \varepsilon \),

\[ \sigma = \frac{\sigma^*}{\eta^*}, \quad \varepsilon = \frac{\varepsilon^*_c}{\eta^*} \quad \text{and} \quad \varepsilon_c = \frac{\varepsilon^*_c}{\eta^*}, \]

where \( \eta^* \) is the position factor of Weibull distribution (WEIBULL, 1951) (see Eq. (28)). According to Eqs. (7)–(9), the relations between the nominal and true variables, i.e., stress, strain and damage of the damage part, are

\[ \sigma = \varepsilon(1 - D), \]

\[ \sigma = \sigma_s(1 - D) \quad \text{and} \quad \varepsilon = \varepsilon_s, \]

\[ \sigma_s = \varepsilon_s. \]

According to Eqs. (7) and (9), the relations between dimensionless nominal stress and normalized strain \( \varepsilon_c \) of the elastic part can be

\[ \sigma = \varepsilon_c. \]

According to Eqs. (6) and (9), the dimensionless boundary displacement \( U \) can be derived as
where \( k \) is the ratio between the rigidity of the elastic part and the initial rigidity of the rock sample

\[
k = \frac{E^*/L^*}{E^0*/L^0*}
\] (15)

As soon as damage occurs in heterogeneous elastic-brittle medium, some stored energy will be released. Since the elastic energy of the elastic-brittle (without residual deformation) model under MF approximation

\[
\Theta_{\text{el}}(\varepsilon) = \frac{1}{2} \sigma \varepsilon = \frac{1}{2} (1 - D) \varepsilon^2,
\] (16)

and the increment of external work on rock sample is

\[
\Delta W = \sigma \Delta \varepsilon = (1 - D) \varepsilon \Delta \varepsilon. \tag{17}
\]

Then, the increment of energy release can be

\[
\Delta \Theta(\varepsilon) = \Delta W(\varepsilon) - \Delta \Theta_{\text{el}}(\varepsilon) = \frac{\varepsilon^2}{2} \Delta D. \tag{18}
\]

Suppose the rock sample can be simplified as a driven, nonlinear threshold system (Rundle et al., 2000b; Zhang et al., 2004), comprising numerous interacting and nonlinear mesoscopic units, which fails when the force acting on it reaches a predefined threshold. In the present model, it is assumed that all units have the same elastic modulus \( E^* \) but different breaking stress threshold \( \sigma_c^* \). Hence, firstly, each unit remains elastic until its own \( \sigma_c^* \), i.e.,

\[
\sigma_s^* = \varepsilon_s^* E_0^*, \tag{19}
\]

where \( \sigma_s^* \) and \( \varepsilon_s^* \) are mesoscopic stress and strain of each unit, i.e., true stress and true strain as mentioned before, respectively. According to Eq. (19), the stress threshold \( \sigma_c^* \) and strain threshold \( \varepsilon_c^* \) of the mesoscopic unit should also follow

\[
\sigma_c^* = \varepsilon_c^* E_0^*. \tag{20}
\]

According to definitions (9), the relations between the dimensionless stress threshold \( \sigma_c \) and normalized strain threshold \( \varepsilon_c \) can be

\[
\sigma_c = \varepsilon_c. \tag{20a}
\]

Secondly, as soon as \( \sigma_s^* \) reaches \( \sigma_c^* \) on a unit, the unit will be broken and can never support load, i.e.,