Statistical Inference, Econometric Analysis and Matrix Algebra

Bernhard Schipp • Walter Krämer
Editors

Statistical Inference, Econometric Analysis and Matrix Algebra

Festschrift in Honour of Götz Trenkler

Springer

Editors

Prof. Dr. Bernhard Schipp
TU Dresden
Fakultät Wirtschaftswissenschaften
Professur für Quantitative Verfahren, insb. Ökonometrie
Mommsenstr. 12
01062 Dresden
Germany
bernhard.schipp@tu-dresden.de

Prof. Dr. Walter Krämer
TU Dortmund
Fakultät Statistik
Lehrstuhl für Wirtschafts- und Sozialstatistik
Vogelpothsweg 78
44221 Dortmund
Germany
walterk@statistik.tu-dortmund.de

ISBN: 978-3-7908-2120-8 e-ISBN: 978-3-7908-2121-5

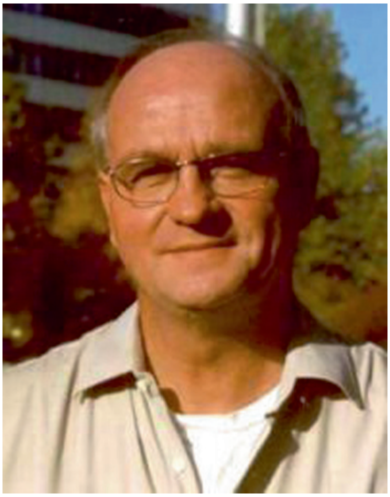
Library of Congress Control Number: 2008936140

Cover design: WMXDesign GmbH, Heidelberg

Printed on acid-free paper

9 8 7 6 5 4 3 2 1

springer.com



Prof. Dr. Götz Trenkler

Preface

This Festschrift is dedicated to Götz Trenkler on the occasion of his 65th birthday.

As can be seen from the long list of contributions, Götz has had and still has an enormous range of interests, and colleagues to share these interests with. He is a leading expert in linear models with a particular focus on matrix algebra in its relation to statistics. He has published in almost all major statistics and matrix theory journals. His research activities also include other areas (like nonparametrics, statistics and sports, combination of forecasts and magic squares, just to mention a few).

Götz Trenkler was born in Dresden in 1943. After his school years in East Germany and West-Berlin, he obtained a Diploma in Mathematics from Free University of Berlin (1970), where he also discovered his interest in Mathematical Statistics. In 1973, he completed his Ph.D. with a thesis titled: *On a distance-generating function of probability measures*. He then moved on to the University of Hannover to become Lecturer and to write a habilitation-thesis (submitted 1979) on alternatives to the Ordinary Least Squares estimator in the Linear Regression Model, a topic that would become his predominant field of research in the years to come.

In 1983 Götz Trenkler was appointed Full Professor of Statistics and Econometrics at the Department of Statistics at the University of Dortmund, where he continues to teach and do research until today. He served as dean of the department from 1987 to 1990 and declined an offer from Dresden University of Technology in 1993. He has been visiting Professor at the University of California at Berkeley, USA, and the University of Tampere, Finland, and is a regular contributor to international conferences on matrix methods in statistics. Currently, he is the Coordinating Editor of *Statistical Papers*, Associate Editor of several other international journals and recently the twice-in-a-row recipient of the best-teacher-award of the department.

Among Götz Trenkler's extracurricular activities are tennis, chess and the compilation of a unique collection of *Aphorisms in Statistics*, samples of which can be found at the beginning of the chapters of this book. He certainly would do the scientific community a great service by having them published at some time.

The editors are grateful to all contributors, many of whom are not only scientific colleagues but also his personal friends.

We express our appreciation for editorial and LaTeX-assistance to Sabine Hegewald, and in particular to Matthias Deutscher, who managed to edit successfully almost 30 manuscripts that were characterized by a great variety of individual preferences in style and layout, and to Alice Blanck and Werner A. Müller from Springer Publishing for their support.

Dresden and Dortmund
July 2008

Bernhard Schipp
Walter Krämer

Contents

Part VI Matrix Algebra and Matrix Computations

Part VII Special Topics

List of Contributors

S. Ejaz Ahmed Department of Mathematics and Statistics, University of Windsor, 401 Sunset Avenue, Windsor, ON, Canada N9B 3P4, seahmed@uwindsor.ca

Oskar Maria Baksalary Faculty of Physics, Adam Mickiewicz University, ul. Umultowska 85, 61-614 Poznań, Poland, baxx@amu.edu.pl

Björn Bornkamp Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, bornkamp@statistik.tu-dortmund.de

Herbert Büning Freie Universität Berlin, 14195 Berlin, Germany, Herbert.Buening@fu-berlin.de

Ka Lok Chu Department of Mathematics, Dawson College, 3040 ouest, rue Sherbrooke, Westmount, QC, Canada H3Z 1A4, ka.chu@mail.mcgill.ca

Richard William Farebrother 11 Castle Road, Bayston Hill, Shrewsbury SY3 0NF, UK, R.W.Farebrother@Manchester.ac.uk

Roland Fried Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, fried@statistik.tu-dortmund.de

Arno Fritsch Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, arno.fritsch@statistik.tu-dortmund.de

Ursula Gather Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, gather@statistik.tu-dortmund.de

Jürgen Groß Carl von Ossietzky Universität Oldenburg, Fakultät V, Institut für Mathematik, 26111 Oldenburg, Germany, j.gross@uni-oldenburg.de

Christoph Hanck Department Quantitative Economics, Universiteit Maastricht, 6211 LM Maastricht, The Netherlands, c.hanck@ke.unimaas.nl

Joachim Hartung Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, hartung@statistik.tu-dortmund.de

Wilfried Hazod Fakultät für Mathematik, Technische Universität Dortmund, 44221 Dortmund, Germany, wilfried.hazod@mathematik.tu-dortmund.de

Christian Heumann Institut für Statistik, Universität München, 80799 München, Germany, christian.heumann@stat.uni-muenchen.de

Rodrigo Herrera Fakultät Wirtschaftswissenschaften, Technische Universität Dresden, 01062 Dresden, Germany, rherrera@gmx.net

Ludwig A. Hothorn Institut für Biostatistik, Leibniz Universität Hannover, D-30419 Hannover, Germany, hothorn@biostat.uni-hannover.de

Jeffrey J. Hunter Institute of Information & Mathematical Sciences, Massey University, Private Bag 102-904, North Shore Mail Centre, Auckland 0754, New Zealand, j.hunter@massey.ac.nz

Katja Ickstadt Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, ickstadt@statistik.tu-dortmund.de

George G. Judge University of California-Berkeley, 207 Giannini Hall, Berkeley, CA 94720, judge@are.berkeley.edu

Guido Knapp Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, knapp@statistik.tu-dortmund.de

Walter Krämer Fakultät Statistik, TU Dortmund, 44221 Dortmund, Germany, walterk@statistik.uni-dortmund.de

Joachim Kunert Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, kunert@statistik.tu-dortmund.de

Oliver Kuss Institut für Medizinische Epidemiologie, Biometrie und Informatik, Martin-Luther-Universität Halle-Wittenberg, 06097 Halle (Saale), Germany, oliver.kuss@medizin.uni-halle.de

Sandra Leissen Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, leissen@statistik.tu-dortmund.de

Uwe Ligges Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, ligges@statistik.tu-dortmund.de

Antti Liski Tampere University of Technology, Tampere, Finland, Antti.Liski@tut.fi

Erkki P. Liski University of Tampere, Tampere, Finland, Erkki.Liski@uta.fi

Shuangzhe Liu Faculty of Information Sciences and Engineering, University of Canberra, Canberra ACT 2601, Australia, Shuangzhe.Liu@canberra.edu.au

R.J. Martin Wirksworth, DE4 4EB, UK, r.j.martin@sheffield.ac.uk

Ron C. Mittelhammer School of Economic Sciences, Washington State University, 101C Hulbert Hall, Pullman, WA 99164-6210, mittelha@wsu.edu

Oliver Morell Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, morell@statistik.tu-dortmund.de

Thomas Mühlenstädt Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, muehlens@statistik.tu-dortmund.de

Markus Neuhäuser Fachbereich Mathematik und Technik, RheinAhr-Campus Remagen, 53424 Remagen, Germany, neuhaeuser@rheinahrcampus.de

Sévérien Nkurunziza Department of Mathematics and Statistics, University of Windsor, 401 Sunset Avenue, Windsor, ON, Canada N9B 3P4, severien@uwindsor.ca

Ingram Olkin Department of Statistics, Sequoia Hall, 390 Serra Mall, Stanford University, Stanford, CA 94305-4065, USA, iolkin@stat.Stanford.EDU

Tomasz Owsiany Institute of Mathematics, University of Rzeszów, Al. Rejtana 16 A, 35-959 Rzeszów, Poland, towsiany@wp.pl

D.S.G. Pollock Department of Economic, University of Leicester, Leicester LE1 7RH, UK, d.s.g.pollock@le.ac.uk

Simo Puntanen Department of Mathematics and Statistics, University of Tampere, 33014 Tampere, Finland, Simo.Puntanen@uta.fi

Jörg Rahnenführer Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, rahnenfuehrer@statistik.tu-dortmund.de

Sabine Rothe Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, srothe@statistik.tu-dortmund.de

Philip Sander Universität Hamburg, Institut für Statistik und Ökonometrie, Von-Melle-Park 5, 20146 Hamburg, Germany, philip.sander@gmx.de

Razaw al Sarraj Department of Energy and Technology, Box 7032, 750 07 Uppsala, Sweden, Razaw.Al-Sarraj@etsm.slu.se

Bernhard Schipp Fakultät Wirtschaftswissenschaften, Technische Universität Dresden, 01062 Dresden, Germany, bernhard.schipp@tu-dresden.de

Karsten Schmidt Fakultät Wirtschaftswissenschaften, Fachhochschule Schmalkalden, 98574 Schmalkalden, Germany, kschmidt@fh-sm.de

Shalabh Department of Mathematics & Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, India, shalab@iitk.ac.in, shalabh1@yahoo.com

Peter Stahlecker Universität Hamburg, Institut für Statistik und Ökonometrie, Von-Melle-Park 5, 20146 Hamburg, Germany, peter.stahlecker@uni-hamburg.de

Czesław Stępniak, Institute of Mathematics, University of Rzeszów, Al. Rejtana 16 A, 35-959 Rzeszów, Poland, cees@univ.rzeszow.pl

George P.H. Styan Department of Mathematics and Statistics, McGill University, 1005-805 ouest, rue Sherbrooke, Montréal QC, Canada H3A 2K6, styan@math.mcgill.ca

Lauri Tarkkonen Department of Mathematics and Statistics, PO Box 54, University of Helsinki, 00014 Helsinki, Finland, Lauri.Tarkkonen@helsinki.fi

Helge Toutenburg Institut für Statistik, Universität München, 80799 München, Germany, toutenb@stat.uni-muenchen.de

Dietrich Trenkler Fachbereich Wirtschaftswissenschaften, Universität Osnabrück, 49069 Osnabrück, Germany, Dietrich.Trenkler@Uni-Osnabrück.de

Kimmo Vehkalahti Department of Mathematics and Statistics, PO Box 68, University of Helsinki, 00014 Helsinki, Finland, Kimmo.Vehkalahti@helsinki.fi

Dietrich von Rosen Department of Energy and Technology, Box 7032, 750 07 Uppsala, Sweden, dietrich.von.rosen@et.slu.se

Claus Weihs Fakultät Statistik, Technische Universität Dortmund, 44221 Dortmund, Germany, weihs@statistik.tu-dortmund.de

Hans Joachim Werner Wirtschaftswissenschaftlicher Fachbereich, Statistische Abteilung, Universität Bonn, 53113 Bonn, Germany, hjw.de@uni-bonn.de

Part I
Nonparametric Inference

It is proven that the celebration of birthdays is healthy. Statistics show that those people who celebrate the most birthdays become the oldest.

S. den Hartog

Adaptive Tests for the c-Sample Location Problem

Herbert Büning

Abstract This paper deals with the concept of adaptive tests and with an application to the *c*-sample location problem. Parametric tests like the ANOVA F-tests are based on the assumption of normality of the data which is often violated in practice. In general, the practising statistician has no clear idea of the underlying distribution of his data. Thus, an adaptive test should be applied which takes into account the given data set. We use the concept of Hogg [21], i.e. to classify, at first, the unknown distribution function with respect to two measures, one for skewness and one for tailweight, and then, at the second stage, to select an appropriate test for that classified type of distribution. It will be shown that under certain conditions such a two-staged adaptive test maintains the level. Meanwhile, there are a lot of proposals for adaptive tests in the literature in various statistical hypotheses settings. It turns out that all these adaptive tests are very efficient over a broad class of distributions, symmetric and asymmetric ones.

1 Introduction

In the parametric case of testing hypotheses the efficiency of a test statistic strongly depends on the assumption of the underlying distribution of the data, e.g. if we assume normality then optimal tests are available for the one- two- and c-sample location or scale problem such as t-tests, F-tests and Chi-square-tests. In the nonparametric case the distribution of the test statistic is not based on a special distribution of the data like the normal, only the assumption of continuity of the distribution is needed in general. It is well known, however, that the efficiency of nonparametric tests depends on the underlying distribution, too, e.g. the Kruskal–Wallis test in the c-sample location problem has high power for symmetric and medium- up to long-tailed distributions in comparison to its parametric and nonparametric competitors whereas the Kruskal–Wallis test can be poor for asymmetric distributions.

Herbert Büning
Freie Universität Berlin, D-14195 Berlin, Germany
Herbert.Buening@fu-berlin.de

But for the practising statistician it is more the rule rather than the exception that he has no clear idea of the underlying distribution of his data. Consequently, he should apply an adaptive test which takes into account the given data set.

At present, we register a lot of papers on adaptive tests in the literature, concerning one-, two- and c-sample location or scale problems with two-sided and one-sided ordered alternatives as well as umbrella alternatives.

Most of these adaptive tests are based on the concept of Hogg [21], that is, to classify, at first, the type of the underlying distribution with respect to some measures like tailweight and skewness and then to select an appropriate rank test for the classified type of distribution. It can be shown that this two-staged test procedure is distribution-free, i.e. it maintains the level over the class of all continuous distribution functions.

In our paper Hogg's concept of adaptive tests is presented and demonstrated by a real data set. Adaptive tests are generally not the best ones for a special distribution but mostly second best whereas the parametric competitors are poor in many cases. That is just the philosophy of an adaptive test to select the best one for a given data set. It works in the sense of "safety first" principle. For clarity of exposition we confine our attention to the c-sample location problem. A power comparison by means of Monte Carlo simulation shows that the adaptive test is very efficient over a broad class of distributions in contrary to its parametric and nonparametric competitors.

2 Model, Hypotheses and Data Example

We consider the following c-sample location model:

Let $X_{i1},\dots,X_{in_i},\ i=1,\dots,c$, be independent random variables with $X_{ij}\sim F_X(x-\theta_i),\ j=1,\dots,n_i,\ \theta_i\in \mathbb{R}$,
where the distribution function F_X is assumed to be continuous. We wish to test

$$\mathrm{H}_0:\theta_1=\dots=\theta_c.$$

As alternative hypotheses we consider

the two-sided alternative $\mathrm{H}_1^{(1)}:\theta_r\neq\theta_s$ for at least one pair $(r,s),\ r\neq s$,
the ordered alternative $\mathrm{H}_1^{(2)}:\theta_s\leq\dots\leq\theta_c$ with at least one strict inequality,
the umbrella alternative $\mathrm{H}_1^{(3)}:\theta_1\leq\dots\leq\theta_{l-1}\leq\theta_l\geq\theta_{l+1}\geq\dots\geq\theta_c$
with at least one strict inequality for peak $l,\ 2\leq l\leq c-1$.

Now, let us present a data example for $\mathrm{H}_1^{(1)}$, the example is given by Chatfield ([13] p. 101).

Example 1. A study was carried out at a major London hospital to compare the effects of different types of anaesthetic used in major operations. Eighty patients undergoing a variety of operations were randomly assigned to one of the four anaesthetics and a variety of observations were taken on each patient before and after the operation. This exercise concentrates on just one of the response variables, namely the time, in minutes, from the reversal of the anaesthetic until the patient opened his or her eyes.

The data are shown in Table 1.

Figure 1 shows the boxplots of the data.

Obviously, we cannot assume normality for that kind of data, the underlying distributions might be skewed to the right. Thus, what is an appropriate test for testing H_0? An answer will be given at the end of Sect. 3.3.

Data examples for testing H_0 against the alternatives $H_1^{(2)}$ and $H_1^{(3)}$ can be found in Hand et al. ([18], p. 212) and Simpson and Margolin [35], respectively.

Table 1 Time in minutes, from reversal of anaesthetic until the eyes open for each of 20 patients treated by one of four anaesthetics A,B,C or D

A	3	2	1	4	3	2	10	12	12	3	19	1	4	5	1	1	7	5	1	12
B	6	4	1	1	6	2	1	10	1	1	1	2	10	2	2	2	2	1	3	7
C	3	5	2	4	2	1	6	13	1	1	1	4	1	1	1	8	1	2	4	0
D	4	8	2	3	2	3	6	2	3	4	8	5	10	2	0	10	2	3	9	1

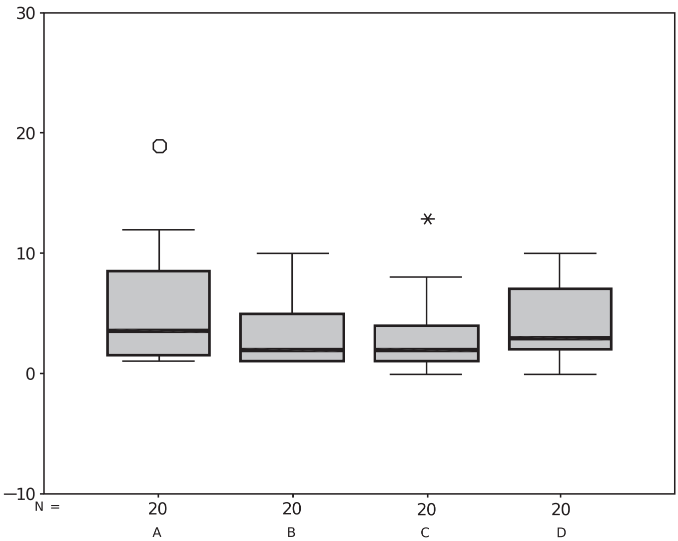


Fig. 1 Boxplots of the data of Example 1

3 Tests for Two-sided Alternatives

3.1 Parametric F-test

Let $X_{i1},\dots,X_{in_i},\ i=1,\dots,c$, be independent and normally distributed random variables, i.e.

$$X_{ij} \sim N(\mu_i,\sigma_i^2),\ j=1,\dots,n_i \text{ with } \sigma_1^2=\dots=\sigma_c^2=\sigma^2.$$

We wish to test

$$\mathrm{H}_0:\ \mu_1=\dots=\mu_c \quad \text{versus} \quad \mathrm{H}_1:\mu_r \neq \mu_s \text{ for at least one pair } (r,s),\ r\neq s.$$

Then the likelihood ratio F-test is based on the statistic

$$F=\frac{(N-c)\sum\limits_{i=1}^{c} n_i(\bar{X}_i-\bar{X})^2}{(c-1)\sum\limits_{i=1}^{c}\sum\limits_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2},\ \text{where } N=\sum_{i=1}^{c} n_i,\ \overline{X_i}=\frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \text{ and } \overline{X}=\frac{1}{N}\sum_{i=1}^{c} n_i\overline{X_i}.$$

Under H_0, the statistic F has an F-distribution with $c-1$ and $N-c$ degrees of freedom. If we assume non-normal distributions with at least finite second moments it can be shown that, under H_0, F has asymptotically a chi-square distribution with $c-1$ degrees of freedom, see, e.g. Tiku et al. [37].

3.2 Rank Tests

Let $X_{(1)},\dots,X_{(N)}$ be the combined ordered sample of $X_{11},\dots,X_{1n_1},\dots,X_{c1},\dots,X_{cn_c}$, $N=\sum_{i=1}^{c} n_i$.

We define indicator variables V_{ik} by

$$V_{ik}=\begin{cases}1 & \text{if } X_{(k)} \text{ belongs to the ith sample}\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, we have real valued scores $a(k),\ k=1,\dots,N$, with mean $\bar{a}=\frac{1}{N}\sum\limits_{k=1}^{N} a(k)$.

Now, we define for each sample a statistic A_i in the following way

$$A_i=\frac{1}{n_i}\sum_{k=1}^{N} a(k)V_{ik},\ 1\leq i\leq c.$$

A_i is the average of the scores for the ith sample. Then the linear rank statistic L_N is given by

$$L_N = \frac{(N-1)\sum_{i=1}^{c} n_i(A_i - \bar{a})^2}{\sum_{k=1}^{N}(a(k)-\bar{a})^2}.$$

Under H_0, L_N is distribution-free and has asymptotically a chi-square distribution with $c-1$ degrees of freedom, that means, H_0 has to be rejected in favour of $H_1^{(1)}$ if $L_N \geq \chi^2_{1-\alpha}(c-1)$.

In the following, some examples of rank tests are given; for references, see, e.g. Gastwirth [14], Randles and Wolfe [32], Büning [3, 5], Gibbons and Chakraborti [15] as well as Büning and Trenkler [12]. In parenthesis that type of distribution is indicated for which the test has high power.

Example 2 (Gastwirth test G (short tails)).

$$a_G(k) = \begin{cases} k - \frac{N+1}{4} & \text{if } k \leq \frac{N+1}{4} \\ 0 & \text{if } \frac{N+1}{4} < k < \frac{3(N+1)}{4} \\ k - \frac{3(N+1)}{4} & \text{if } k \geq \frac{3(N+1)}{4}. \end{cases}$$

Example 3 (Kruskal–Wallis test KW (medium tails)).

$$a_{KW}(k) = k.$$

As an efficient test for long tails Büning [5] proposed the so called LT-test with scores chosen analogously to Huber‘s Ψ-function referring to M-estimates, see Huber [25].

Example 4 (LT-test (long tails)).

$$a_{LT}(k) = \begin{cases} -\left(\left[\frac{N}{4}\right]+1\right) & \text{if } k < \left[\frac{N}{4}\right]+1 \\ k - \frac{N+1}{2} & \text{if } \left[\frac{N}{4}\right]+1 \leq k \leq \left[\frac{3(N+1)}{4}\right] \\ \left[\frac{N}{4}\right]+1 & \text{if } k > \left[\frac{3(N+1)}{4}\right]. \end{cases}$$

[x] denotes the greatest integer less than or equal to x.

Example 5 (Hogg–Fisher–Randles test HFR (right-skewed)).

$$a_{HFR}(k) = \begin{cases} k - \frac{N+1}{2} & \text{if } k \leq \frac{N+1}{2} \\ 0 & \text{if } k > \frac{N+1}{2}. \end{cases}$$

For left-skewed distributions interchange the terms $k-(N+1)/2$ and 0 in the above definition.

All these four rank tests are included in our simulation study in Sect. 4. They are “bricks” of the adaptive tests proposed in the next section.

For the case of ordered alternatives $H_1^{(2)}$ and umbrella alternatives $H_1^{(3)}$ the most familiar rank tests are the tests of Jonckheere [27] and Mack and Wolfe [28], respectively. They are based on pairwise two-sample Wilcoxon statistics computed on the ith sample vs. the combined data in the first $i-1$ samples, $2 \leq i \leq c$. It is well known that both tests have high power for symmetric and medium-tailed distributions. Büning [6], Büning and Kössler [9] modifies these tests by using two-sample statistics of Gastwirth and Hogg–Fisher–Randles rather than the Wilcoxon statistic. These so called Jonckheere-type- and Mack–Wolfe-type tests are very efficient for short-tailed and asymmetric distributions.

3.3 Adaptive Tests

Husková [26] and Hájek et al. [16] distinguishes between two different concepts of adaptive procedures, nonrestrictive and restrictive ones. In the case of nonrestrictive procedures the optimal scores $a_{\text{opt}}(k)$ for the locally most powerful rank test, which depend on the (unknown) underlying distribution function F and its density f, are estimated directly from the data. This approach is applied, e.g. by Behnen and Neuhaus [1] in many testing situations. We will apply the adaptive procedure of Hogg [21] which belongs to the class of restrictive procedures, i.e. a "reasonable" family of distributions and a corresponding class of "suitable" tests are chosen. The adaptive test of Hogg is a two-staged one. At the first stage, the unknown distribution function is classified with respect to some measures like tailweight and skewness. At the second stage, an appropriate test for that classified type of distribution is selected and then carried out. Hogg [22] states: *"So adapting the test to the data provides a new dimension to nonparametric tests which usually improves the power of the overall test."*

This two-staged adaptive test maintains the level α for all continuous distribution functions as shown by the following

Lemma 1. *(1) Let $\mathscr{F}$ denote the class of distribution functions under consideration. Suppose that each of m tests based on the statistics $T_1, \ldots, T_m$ is* distribution-free *over the class $\mathscr{F}$; i.e. $P_{H_0}(T_h \in C_h | F) = \alpha$ for each $F \in \mathscr{F}$, $h = 1, \ldots, m$.*

(2) Let S be some statistic that is independent *of $T_1, \ldots, T_m$ under $\mathbf{H}_0$ for each $F \in \mathscr{F}$. Suppose we use S to decide which test T_h to conduct. (S is called a selector statistic.). Specially, let U_S denote the set of all values of S with the following decomposition:*

$$U_S = D_1 \cup D_2 \cup \cdots \cup D_m, \; D_h \cap D_k = \varnothing \text{ for } h \neq k,$$

so that $S \in D_h$ corresponds to the decision to use the test T_h. The overall testing procedure is then defined by:

If $S \in D_h$ then reject H_0 *if $T_h \in C_h$.*

This two-staged adaptive test is, under H_0, *distribution-free over the class $\mathscr{F}$, i.e. it maintains the level α for each $F \in \mathscr{F}$.*

Proof.
$$\begin{aligned}P_{\mathrm{H}_0}(\text{reject } \mathrm{H}_0 \mid F) &= P_{\mathrm{H}_0}\left(\bigcup_{h=1}^{m} (S \in D_h \wedge T_h \in C_h \mid F)\right)\\ &= \sum_{h=1}^{m} P_{\mathrm{H}_0}(S \in D_h \wedge T_h \in C_h | F)\\ &= \sum_{h=1}^{m} P_{\mathrm{H}_0}(S \in D_h | F) \cdot P_{\mathrm{H}_0}(T_h \in C_h | F)\\ &= \alpha \cdot \sum_{h=1}^{m} P_{\mathrm{H}_0}(S \in D_h | F) = \alpha \cdot 1 = \alpha. \quad \square\end{aligned}$$

Let us apply this Lemma on our special problem:

1. $\mathscr{F}$ is the class of all continuous distribution functions F and $T_1, \ldots, T_m$ are linear rank statistics. Then T_h is distribution-free over $\mathscr{F}$, $h = 1, \ldots, m$.

2. S is a function of the order statistics of the combined sample. Under $\mathbf{H}_0$, the order statistics are the complete sufficient statistics for the common, but unknown F, and therefore independent of every statistic whose distribution is free of F (theorem of Basu, see, e.g. Roussas [33], p. 215). Thus, under $\mathbf{H}_0$, S is independent of the linear rank statistics $T_1, \ldots, T_m$.

As a selector statistic S we choose $S = (\hat{M}_S, \hat{M}_T)$, where $\hat{M}_S$ and $\hat{M}_T$ are measures of skewness and tailweight, respectively, defined by
$\hat{M}_S = \dfrac{\hat{x}_{0.975} - \hat{x}_{0.5}}{\hat{x}_{0.5} - \hat{x}_{0.025}}$ and
$\hat{M}_T = \dfrac{\hat{x}_{0.975} - \hat{x}_{0.025}}{\hat{x}_{0.875} - \hat{x}_{0.125}}$ with the p-quantile $\hat{x}_p$ given by

$$\hat{x}_p = \begin{cases} X_{(1)} & \text{if } p \le 0.5/N \\ (1-\lambda)X_{(j)} + \lambda X_{(j+1)} & \text{if } 0.5/N < p \le 1 - 0.5/N \\ X_{(N)} & \text{if } p > 1 - 0.5/N \end{cases}$$

where $X_{(1)}, \ldots, X_{(N)}$ again are the order statistics of the combined c samples and $j = [np + 0.5]$, $\lambda = np + 0.5 - j$. Obviously, $\hat{M}_S < 1$, if F is skewed to the left, $\hat{M}_S = 1$, if F is symmetric and $\hat{M}_S > 1$, if F is skewed to the right. $\hat{M}_T \ge 1$, the longer the tails the greater $\hat{M}_T$. The measures $\hat{M}_S$ and $\hat{M}_T$ are location and scale invariant.

In Table 2 values of the corresponding theoretical measures M_S and M_T are presented for some selected distributions where CN1, CN2 and CN3 are contaminated normal distributions:

$CN_1 = 0.95N(0,1) + 0.05N(0,3^2)$, $CN_2 = 0.9N(0,1) + 0.1N(0,5^2)$, both symmetric, and $CN_3 = 0.5N(1,4) + 0.5N(-1,1)$, a distribution skewed to the right.

We see, the exponential distribution is extremely right-skewed and the Cauchy has very long tails. Now, two questions arise:

First, what is an appropriate number m of categories $D_1, \ldots, D_m$?

Such a number m may be three, four or five, in most proposals four categories are preferred, three for symmetric distributions (short, medium, long tails) and one for distributions skewed to the right. A fifth category can be defined for left-skewed distributions.

Table 2 Theoretical values of M_S and M_6

Distributions	M_S	M_T
Uniform	1.000	1.267
Normal	1.000	1.704
CN1	1.000	1.814
Logistic	1.000	1.883
Double exp.	1.000	2.161
CN2	1.000	2.606
Cauchy	1.000	5.263
CN3	1.765	1.691
Exponential	4.486	1.883

Second, how do we fix the bounds of the categories?

The bounds depend on the theoretical values of M_S and M_T (see Table 2) in order to consider different strength of skewness and tailweight. Simulations by trial and error may improve the bounds in the adaptive scheme. To our experience, however, the very special choice of the bounds is not the crucial point, it is much more important to include efficient rank tests in the adaptive scheme, an efficient rank test not only for the corresponding category but also in the neighbourhood of that category because of possible misclassifications, see Table 3.

Now, for our special c-sample location problem we propose the following four categories which are based on S:

$$D_1 = \{S|0 \leq \hat{M}_S \leq 2;\ 1 \leq \hat{M}_T \leq 1.5\}$$
$$D_2 = \{S|0 \leq \hat{M}_S \leq 2;\ 1.5 < \hat{M}_T \leq 2\}$$
$$D_3 = \{S|\hat{M}_S \geq 0;\ \hat{M}_T > 2\}$$
$$D_4 = \{S|\hat{M}_S > 2;\ 1 \leq \hat{M}_T \leq 2\}.$$

This means, the distribution is classified as symmetric with short- or medium tails, if S falls in the category D_1 or D_2, respectively, as long-tailed if S belongs to D_3 and as skewed to the right with short- or medium tails if S falls in D_4.

We now propose the following adaptive test A:

$$A = \begin{cases} G & \text{if } S \in D_1 \\ KW & \text{if } S \in D_2 \\ LT & \text{if } S \in D_3 \\ HFR & \text{if } S \in D_4. \end{cases}$$

Figure 2 shows the adaptive scheme of test A.

The adaptive test above is based on the measures $\hat{M}_S$ and $\hat{M}_T$ calculated from the combined ordered sample $X_{(1)},\ldots,X_{(N)}$ in order to guarantee that the resulting test is distribution-free in the sense of the Lemma. Another way is to calculate the measures $\hat{M}_S$ and $\hat{M}_T$ from each of the c samples separately and then to consider the weighted sum of these measures, that is

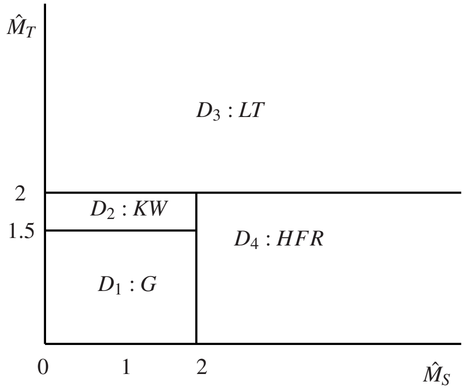


Fig. 2 Adaptive scheme

$$\bar{M}_S = \frac{n_1\hat{M}_{S1} + \cdots + n_c\hat{M}_{Sc}}{N} \text{ and } \bar{M}_T = \frac{n_1\hat{M}_{T1} + \cdots + n_c\hat{M}_{Tc}}{N},$$

where $\hat{M}_{Si}$ and $\hat{M}_{Ti}$ are the measures for skewness and tailweight of the ith sample, $i = 1, \ldots, c$.

The adaptive test based on the measures from the *combined* sample is denoted by *AC* and that based on the measures from the *single* samples by *AS*. The adaptive test *AC* is distribution-free, the measures $\hat{M}_S$ and $\hat{M}_T$, however, are affected by the amount of the shift under H_1, whereas the adaptive test *AS* is *not* distribution-free, but $\bar{M}_S$ and $\bar{M}_T$ are not affected by the shift.

Table 3 shows the classification performance of $(\hat{M}_S, \hat{M}_T)$ and $(\bar{M}_S, \bar{M}_T)$ for the case of $c = 4$, $n_1 = n_2 = n_3 = n_4 = 20$. The data were generated by simulation (10,000 replications) from the uniform (Uni), normal (Norm), logistic (Log), double exponential (Dexp), Cauchy (Cau), the contaminated normal CN3 and the exponential (Exp) distribution.

The amount of shift is determined by the parameters $\theta_i = k_i\sigma_F$, $i = 1, \ldots, 4$, where σ_F is the standard deviation of the underlying distribution function F. For the Cauchy we choose $\sigma_{\text{Cau}} = F_{\text{Cau}}^{-1}(0.8413) = 1.8373$ because of $\Phi(1) = 0.8413$ where Φ is the standard normal distribution function.

Let us consider, as an example, the *AC*-test with data from the uniform distribution and $k_i = 0$, $i = 1, \ldots, 4$. Then in 9,911 of 10,000 cases these data were (correctly) classified as symmetric and short-tailed (D_1), in 72 cases as symmetric and medium-tailed (D_2), in 0 cases as long-tailed (D_3) and in 17 cases as skewed to the right (D_4). Under the null hypothesis the classification schemes based on $(\hat{M}_S, \hat{M}_T)$ and $(\bar{M}_S, \bar{M}_T)$ are quite effective for all distributions considered.

In contrary to the *AS*-test the *AC*-test – based on the classification performance of $(\hat{M}_S, \hat{M}_T)$ – is strongly affected by the amount of shift for the uniform, double exponential and the two distributions skewed to the right, CN3 and Exp. As

Table 3 Skewness and tailweight classification of the adaptive tests *AC* and *AS*, $c = 4$, $n_1 = n_2 = n_3 = n_4 = 20$

k_1, k_2, k_3, k_4	Uni	Norm	Log	Dexp	Cau	CN3	Exp
0,0,0,0							
D_1							
AC	9,911	1,171	247	47	0	1041	5
AS	9,786	1,484	410	66	1	1083	4
D_2							
AC	72	8,131	6,538	3,082	13	5,711	4
AS	151	7,901	6,704	3,544	14	4,977	4
D_3							
AC	0	690	3,203	6,855	9,987	822	3,409
AS	0	577	2,816	6,313	9,983	901	3,319
D_4							
AC	17	8	12	16	0	2,426	6,582
AS	63	38	70	77	2	3,039	6,673
0,0.2,0.4,0.6							
D_1							
AC	9,407	1,101	308	52	0	1,088	70
AS	9,799	1,416	423	83	0	1,092	13
D_2							
AC	588	8,154	6,807	3,571	9	6,393	166
AS	140	7,915	6,726	3,585	15	4,952	5
D_3							
AC	0	735	2,826	6,366	9,990	801	3,852
AS	0	634	2,797	6,268	9,984	900	3,350
D_4							
AC	5	10	17	11	1	1,718	5,912
AC	61	35	54	64	1	3,056	6,632
0,0.4,0.8,1.2							
D_1							
AC	6,128	1,191	450	95	0	1,161	315
AS	9,765	1,442	376	60	0	1,072	9
D_2							
AC	3,868	8,171	7,285	4,830	20	7,366	2,123
AS	181	7,922	6,656	3,505	8	5,034	3
D_3							
AC	1	634	2,248	5,058	9,980	774	3,132
AS	0	598	2,898	6,371	9,989	949	3,322
D_4							
AC	3	4	17	17	0	699	4,430
AS	54	38	70	64	3	2,945	6,666

the differences of $\theta_1, \ldots, \theta_4$ increase, all these four distributions tend to be classified more as having medium tails. But for large differences of the location parameters each of the tests in the adaptive scheme should reveal these differences. For the normal and the Cauchy distribution the classification performance of $(\hat{M}_S, \hat{M}_T)$ is hardly affected by the shift. Similar results hold for $c = 3$ samples and other sizes.

Now, let us analyze the data Example 1 from Sect. 2. What is an appropriate test for these data? First, we calculate the measures $\hat{M}_S$ and $\hat{M}_T$ of the combined ordered sample $X_{(1)},\ldots,X_{(N)}$ of the four samples in order to guarantee that the resulting adaptive test *AC* maintains the level. For the data we get $\hat{M}_S = 3.80$ and $\hat{M}_T = 1.41$, i.e. the distribution of the data is extremely skewed to the right, see Table 2. The selector statistic $S = (3.80, 1.41)$ belongs to D_4 and we have to apply the *HFR*-test. Because of $HFR = 5.636 < \chi^2_{0.95}(3) = 7.815$, $\mathbf{H}_0$ is not rejected at level $\alpha = 5\,\%$. It should be noted that the adaptive test *AC* is only asymptotically distribution-free because an asymptotical critical value of *HFR* is used.

If we calculate the measures $\bar{M}_S$ and $\bar{M}_T$ from each of the four samples separately, we get $\bar{M}_S = 5.51$ and $\bar{M}_T = 1.79$. Thus, we have to apply the *HFR*-test, too, and we get the same test decision. But notice, the adaptive test *AS* based on the selector statistic $S = (5.51, 1.79)$ is not distribution-free.

In the same sense as described above adaptive tests may be constructed for ordered alternatives $\mathbf{H}_1^{(2)}$ and umbrella alternatives $\mathbf{H}_1^{(3)}$ by including Jonckheere-type or Mack–Wolfe-type tests in the adaptive scheme, see Büning [6] and Büning and Kössler [9].

4 Power Study

We investigate via Monte Carlo methods (10,000 replications) the power of all the tests from Sect. 3. The selected distributions are the same as in Table 2 where each of them has mean or median (Cauchy) equal to zero. Here, we again consider only the case of $c = 4$ samples with equal sizes $n_i = 20,\ i = 1,\ldots,4$. The location parameters θ_i are defined by $\theta_i = k_i\sigma_F$ as in Sect. 3.3. The nominal level of the tests is $\alpha = 5\,\%$. Table 4 presents the power values.

We can state:

The F-test maintains the level α quite well for all distributions considered with the exception of the Cauchy for which finite moments do not exist. In this sense, the approximation of the distribution of F by the chi-square distribution does not work, see Sect. 3.1. Thus, for the Cauchy a power comparison of the F-test with the other tests becomes meaningless.

For each of the distributions (with exception of the normal) there is a linear rank test which has higher power than the F-test, e.g. the Gastwirth test for the uniform, the Kruskal–Wallis test for CN1 and the logistic, the *LT*-test for the double exponential and CN2 and the Hogg–Fisher–Randles test for both distributions skewed to the right, CN3 and Exp.

The adaptive tests, *AC* and *AS*, are the best ones over this broad class of distributions. The *AS*-test has (slightly) higher power than the *AC*-test, but since in all cases the actual level of the *AS*-test starts higher than the level of the *AC*-test, it is difficult to assess the higher power values of the *AS*-test in comparison to the *AC*-test. Except for the normal distribution the *AC*-test is more powerful than the F-test for all symmetric and asymmetric distributions.

Table 4 Power of some tests (in percent) under selected distributions $\alpha = 5\%$, $c = 4$, $(n_1, n_2, n_3, n_4) = (20, 20, 20, 20)$

Tests	k_1, k_2, k_3, k_4	Uni	Norm	CN1	Log	Dexp	CN2	Cau	CN3	Exp
F	0, 0, 0, 0	4.8	4.9	4.9	4.8	4.8	4.3	1.8	5.2	4.3
	0, 0.2, 0.4, 0.6	3.5	33.7	36.4	35.4	34.7	40.3		34.7	36.9
	0, 0.3, 0.6, 0.9	68.5	68.8	69.3	68.9	69.2	71.3		69.3	69.9
G	0, 0, 0, 0	4.5	4.6	4.9	4.6	4.9	4.8	5.1	4.7	4.2
	0, 0.2, 0.4, 0.6	50.4	27.6	33.4	28.2	25.0	51.9	12.1	32.7	70.7
	0, 0.3, 0.6, 0.9	85.1	59.5	65.2	57.2	52.6	84.1	21.0	65.6	90.7
KW	0, 0, 0, 0	4.6	4.7	4.7	4.4	4.9	5.0	5.0	4.8	4.3
	0, 0.2, 0.4, 0.6	29.3	31.7	39.3	36.7	45.4	70.5	31.2	37.6	64.2
	0, 0.3, 0.6, 0.9	61.5	65.8	75.2	71.4	81.0	96.7	60.8	72.3	92.1
LT	0, 0, 0, 0	4.6	4.9	4.9	4.8	4.7	5.1	5.3	4.8	4.5
	0, 0.2, 0.4, 0.6	18.7	28.9	36.0	35.1	49.2	70.0	39.9	43.2	53.4
	0, 0.3, 0.6, 0.9	40.8	60.2	71.4	69.2	84.2	97.0	72.7	66.8	87.7
HFR	0, 0, 0, 0	4.5	4.7	4.9	4.9	4.6	5.0	5.1	4.7	4.8
	0, 0.2, 0.4, 0.6	23.3	24.8	31.3	29.6	36.5	59.0	26.8	43.5	86.1
	0, 0.3, 0.6, 0.9	50.0	54.2	63.9	60.0	70.6	90.0	50.7	78.6	99.2
AC	0, 0, 0, 0	4.5	4.8	4.8	4.4	4.7	5.1	5.3	4.7	4.7
	0, 0.2, 0.4, 0.6	49.1	30.8	37.9	35.9	47.6	70.5	39.9	37.6	72.9
	0, 0.3, 0.6, 0.9	78.9	64.1	73.6	70.1	82.5	97.0	72.7	71.3	94.1
AS	0, 0, 0, 0	5.1	5.3	5.2	4.9	5.0	5.2	5.4	6.0	4.8
	0, 0.2, 0.4, 0.6	50.5	32.5	38.7	37.1	48.7	70.4	39.9	41.8	75.8
	0, 0.3, 0.6, 0.9	84.9	66.0	74.1	71.2	83.2	97.1	72.7	75.9	96.2

The adaptive test *AC* is not the best one for a special distribution but mostly second or third best. That is just the philosophy of an adaptive test, to select the best one for a given data set.

5 Outlook

In our paper we studied an adaptive c-sample location test which behaves well over a broad class of distributions, symmetric ones with different tailweight and right-skewed distributions with different strength of skewness. Further adaptive tests for the two- and c-sample location problem can be found in Hogg et al. [23], Ruberg [34], Hill et al. [20], Hothorn and Liese [24], Büning [4, 5, 6], Beier and Büning [2], Sun [36], O'Gorman [30], Büning and Kössler [9], Büning and Rietz [10] and Neuhäuser et al. [29]. For an adaptive two-sample scale test, see Hall and Padmanabhan [17] and Büning [8] and for an adaptive two-sample location-scale test of Lepage-type, see Büning and Thadewald [11]. An adaptive test for the general two sample problem based on Kolmogorov–Smirnov- and Cramér- von Mises-type tests has been proposed by Büning [7]. A very comprehensive survey of adaptive procedures is given by O'Gorman [31].

In our proposal for an adaptive test in Sect. 3.3 we restrict our attention to two measures for skewness and tailweight, $\hat{M}_S$ and $\hat{M}_T$. Other measures for skewness and tailweight are discussed in the literature, see, e.g. the measures $\hat{Q}_1$ and $\hat{Q}_2$ of Hogg [21]. Of course, we may add other types of measures in order to classify the unknown distribution function possibly more correctly, e.g. we can include an additional measure for peakedness, see Büning [3] and Hogg [21]. In this case we have a three dimensional selector statistic S defining our adaptive scheme. To our experience, there is, however, no remarkable gain in power of the adaptive test by adding the peakedness measure, see Handl [19]. Thus, we propose to use only two measures, one for skewness and one for tailweight.

As a result of all our studies on adaptive tests we can state without any doubt, that adaptive testing is an important tool for any practising statistician and it would be a profitable task to add adaptive procedures to statistical software packages.

References

[1] Behnen, K., Neuhaus, G.: Rank tests with estimated scores and their application. Teubner, Stuttgart (1989)

[2] Beier, F., Büning, H.: An adaptive test against ordered alternatives. Comput. Stat. Data Anal. **25**, 441–452 (1997)

[3] Büning, H.: Robuste und adaptive tests. De Gruyter, Berlin (1991)

[4] Büning, H.: Robust and adaptive tests for the two-sample location problem. OR Spektrum **16**, 33–39 (1994)

[5] Büning, H.: Adaptive tests for the c-sample location problem - the case of two-sided alternatives. Commun. Stat. Theor. Meth. **25**, 1569–1582 (1996)

[6] Büning, H.: Adaptive Jonckheere-type tests for ordered alternatives. J. Appl. Stat. **26**, 541–551 (1999)

[7] Büning, H.: An adaptive distribution-free test for the general two-sample problem. Comput. Stat. **17**, 297–313 (2002)

[8] Büning, H.: An adaptive test for the two-sample scale problem. Diskussionsbeiträge des Fachbereichs Wirtschaftswissenschaft der Freien Universität Berlin, Nr. 2003/10 (2003)

[9] Büning, H., Kössler, W.: Adaptive tests for umbrella alternatives. Biom. J. **40**, 573–587 (1998)

[10] Büning, H., Rietz, M.: Adaptive bootstrap tests and its competitors in the c-sample location problem. J. Stat. Comput. Sim. **73**, 361–375 (2003)

[11] Büning, H., Thadewald, T.: An adaptive two-sample location-scale test of Lepage-type for symmetric distributions. J. Stat. Comput. Sim. **65**, 287–310 (2000)

[12] Büning, H., Trenkler, G.: Nichtparametrische statistische Methoden. De Gruyter, Berlin (1994)

[13] Chatfield, C.: Problem-solving – A statistician's guide. Chapman & Hall, London (1988)

[14] Gastwirth, J.L.: Percentile modifications of two-sample rank tests. J. Am. Stat. Assoc. **60**, 1127–1140 (1965)

[15] Gibbons, J.D., Chakraborti, S.: Nonparametric statistical inference, 3rd ed. Dekker, New York (1992)

[16] Hájek, J., Sidák, Z.S., Sen, P.K.: Theory of rank tests. Academic, New York (1999)

[17] Hall, P., Padmanabhan, A.R.: Adaptive inference for the two-sample scale problem. Technometrics **39**, 412–422 (1997)

[18] Hand, D.J., Daly, F., Lunn, A.D., McConway, K.J., Ostrowski, E.: A handbook of small data sets. Chapman & Hall, London (1994)

[19] Handl, A.: Maßzahlen zur Klassifizierung von Verteilungen bei der Konstruktion adaptiver verteilungsfreier Tests im unverbundenen Zweistichproben-Problem. Unpublished Dissertation, Freie Universität Berlin (1986)

[20] Hill, N.J., Padmanabham, A.R., Puri, M.L.: Adaptive nonparametric procedures and applications. Appl. Stat. **37**, 205–218 (1988)

[21] Hogg, R.V.: Adaptive robust procedures. A partial review and some suggestions for future applications and theory. J. Am. Stat. Assoc. **69**, 909–927 (1974)

[22] Hogg, R.V.: A new dimension to nonparametric tests. Commun. Stat. Theor. Meth. **5**, 1313–1325 (1976)

[23] Hogg, R.V., Fisher, D.M., Randles, R.H.: A two-sample adaptive distribution-free test. J. Am. Stat. Assoc. **70**, 656–661 (1975)

[24] Hothorn, L., Liese, F.: Adaptive Umbrellatests - Simulationsuntersuchungen. Rostocker Mathematisches Kolloquium **45**, 57–74 (1991)

[25] Huber, P.J.: Robust estimation of a location parameter. Ann. Math. Stat. **35**, 73–101 (1964)

[26] Husková, M.: Partial review of adaptive procedures. In: Sequential Methods in Statistics **16**, Banach Center Publications, Warschau (1985)

[27] Jonckheere, A.R.: A distribution-free k-sample test against ordered alternatives. Biometrika **41**, 133–145 (1954)

[28] Mack, G.A., Wolfe, D.A.: K-sample rank tests for umbrella alternatives. J. Am. Stat. Assoc. **76**, 175–181 (1981)

[29] Neuhäuser, M., Büning, H., Hothorn, L.: Maximum test versus adaptive tests for the two-sample location problem. J. Appl. Stat. **31**, 215–227 (2004)

[30] O'Gorman, T.W.: A comparison of an adaptive two-sample test to the t-test, rank-sum, and log-rank tests. Commun. Stat. Simul. Comp. **26**, 1393-1411 (1997)

[31] O'Gorman, T.W.: Applied adaptive statistical methods – tests of significance and confidence intervals. ASA-SIAM Ser. Stat. Appl. Prob., Philadelphia (2004)

[32] Randles, R.H., Wolfe, D.A.: Introduction to the theory of nonparametric statistics. Wiley, New York (1979)

[33] Roussas, G.G.: A first course in mathematical statistics. Addison-Wesley, Reading, MA (1973)

[34] Ruberg, S.J.: A continuously adaptive nonparametric two-sample test. Commun. Stat. Theor. Meth. **15**, 2899–2920 (1986)
[35] Simpson, D.G., Margolin, B.H.: Recursive nonparametric testing for dose-response relationships subject to downturns at high doses. Biometrika **73**, 589–596 (1986)
[36] Sun, S.: A class of adaptive distribution-free procedures. J. Stat. Plan. Infer. **59**, 191–211 (1997)
[37] Tiku, M.L., Tan, W.Y., Balakrishnan, N.: Robust inference. Dekker, New York (1986)