I want to dedicate this work to my Russian scientific supervisor Garald Isidorovich Natanson (May 9, 1930–July 24, 2003), professor at the chair of mathematical analysis of St Petersburg State University, доктор of mathematical sciences.

Thank you, Garald Isidorovich, that I was allowed to meet you and listen to your vivid narrations about what mathematics in St Petersburg was about. I hope that I could save just a little bit of your spirit and so fill this work with life. R.I.P.

Consulting Editor

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis

Karl-Georg Steffens

The History of Approximation Theory

From Euler to Bernstein

Birkhäuser
Boston • Basel • Berlin

Karl-Georg Steffens
Auf der neuen Ahr 18
52372 Kreuzau
Germany
kgsteffens@web.de

Consulting Editor:
George A. Anastassiou
University of Memphis
Department of Mathematical Sciences
Memphis, TB 38152
USA

Cover design by Mary Burgess.

AMS Subject Classification: 11J68, 32E30, 33F05, 37Mxx, 40-XX, 41-XX, 41A10

Library of Congress Control Number:

ISBN 0-8176-4353-2 eISBN 0-8176-4353-9
ISBN-13 978-0-8176-

Printed on acid-free paper.

Printed in the United States of America. (TXQ/SB)

9 8 7 6 5 4 3 2 1

www.birkhauser.com

Preface

The aim of the present work is to describe the early development of approximation theory. We set as an endpoint the year 1919 when de la Vallée-Poussin published his lectures [Val19]. With these lectures all fundamental questions, that is, non-quantitative theorems, series expansions and quantitative problems, received their first summarized discussion.

The clear priority of the present investigations are the contributions of Pafnuti Lvovich Chebyshev and of the St Petersburg Mathematical School founded by him. Although some overviews and historical contributions have been published on this subject (e. g., [Gon45], [Gus61] and [But92]) we think that nevertheless it makes sense to go into this topic again for at least five reasons:

Firstly, you find contradictory statements about the exact efforts of Chebyshev and his pupils. So the statement that Chebyshev himself proved the alternation theorem is wrong and the claim that St Petersburg mathematicians had not been interested in the theory of functions is pure nonsense.

Secondly, the available material of Soviet origin is sometimes tendentious and exaggerated in its appreciations of the persons involved, both positively in the almost cultic adulation of Chebyshev and negatively in neglecting the scientific results of mathematicians like Sochocki and his students who did not stand in the limelight, or in belittling the work of Felix Klein.

Thirdly, nearly all historical comments are written in Russian or in one of the languages of the Soviet Union (except for some articles, for example the contributions of Butzer and Jongmans ([BuJo89], [BuJo91] and [BuJo99]) and some papers of Sheynin. So it was time to explain this era of enormous importance for the development of mathematics in Russia and the Soviet Union to those who are not able to read Russian and do not have the time or opportunity to dig in Russian archives and libraries.

In this regard, we feel that it would be disrespectful and unhelpful to refer to Russian contributions which nearly no-one could have access to. Therefore you will find many quotations from the works listed in the References.

Fourthly you will recognize that we did not want only to *describe* the results of the St Petersburg Mathematical School, but also to discuss its historico-philosophical background, and so its character and how it interacted with other European schools.

And lastly we present some interesting facts about the rôle played by Gôttingen in spreading Russian contributions and in their further development.

The breadth of this subject made some restrictions necessary. Definitely you will miss the problems of moments. But we think that also without them the basic tendency of the development described here would not have changed. Only the rôle of A. Markov, Sr. then would have been of even higher importance.

We did not analyse the work of Bernstein as carefully as the contributions of other authors because the German translation of Akhiezer's scientific biography of Bernstein [Akh55] was published in 2000 by R. Kovacheva and H. Gonska.

Expression of Thanks

I especially want to express my deep gratitude to George Anastassiou (Memphis, TN) and Heiner Gonska (Duisburg, Germany). Only by their initiative could this work be published in its present form. Many thanks to Birkhäuser that it followed their recommendation.

Also I want to thank all the people who helped me when I collected material, did not have the necessary ideas or asked for an English word. Special thanks to the wonderful support I enjoyed in St Petersburg and Kyiv, especially from Garald Isidorovich Natanson, Nataliya Sergeevna Ermolaeva and Aleksey Nikolaevich Bogolyubov.

And finally I will never forget Bruno Brosowski (Frankfurt/Main, Germany) who initially led me to the fantastic Chebyshev approximation theory.

Kreuzau, October 2004 *Karl-Georg Steffens*

Introduction

"All exact science is dominated by the idea of approximation."

This statement, attributed to Betrand Russell, shows the borders of exact science, but is also intended to point out how to describe nature by means of mathematics.

The strength of mathematics is abstraction concentrating on simple and clear structures which it aims to rule completely.

To make abstract theories useful it is necessary to adopt them to certain *a posteriori* assumptions coming from reality: Measured data cannot be more exact than the instrument which recorded them; numerical computations are not better than the exactness of the computer.

Whenever one computes, one *approximates.*

It is therefore not surprising that the problem of approximative determination of a given quantity is one of the oldest challenges of mathematics. At least since the discovery of irrationality, considerations of this kind had become necessary. The formula for approximating the square root of a number, usually attributed to the Babylonians, is a case in point.

However, approximation theory is a relatively young mathematical branch, for it needed a concept which describes the mutual dependence between quantities exactly, i.e., the concept of a function. As is well-known, the first approach to defining a function based on this dependence and to abstract from formulae was developed by Euler.

The first abstract definitions of this concept were followed by reflections on how to represent functions to render them pratically useful. Thus, formulae were developed to assist in approximating mainly transcendental functions. At first these representations relied on Taylor's formula and some interpolation formulae based on Newton's ideas.

Although these formulae gave good approximations in certain special cases, in general they failed to control the approximation error because the functions were not approximated uniformly; the error grew beyond the interpolation points or the expansion point. The least square method developed by Gauß provided some improvement, but points might still exist within the interval considered where the error of approximation becomes arbitrarily large.

It follows that new ideas had to be found to solve problems in those cases where it was important to control approximating errors over whole intervals. The present work starts with the first known problems which made such considerations indispensable.

Probably the first work on this subject is attributed to Leonhard Euler who tried to solve the problem of drawing a map of the Russian empire with exact latitudes. He gave a best possible approximation of the relationship between latitudes and altitudes considering all points of one meridian between given latitudes, i. e., over a whole interval [Eul77].

Because of the enormous size of the Russian empire all known projections had very large errors near the borders of the map, therefore Euler's approach proved helpful.

A problem encountered by Laplace was similar in character. One paragraph of his famous work [Lap43] (first published in 1799) dealt with the question of determining the ellipsoid best approximating the surface of the earth. Here, too, it was important to have the error held small for every point on the earth's surface.

Euler solved his problem for a whole interval, whereas Laplace assumed a finite number of data which was very much larger than the number of parametres in the problem. This fact alone prevented a solution of the problem by interpolation.

In 1820 Fourier generalized Laplace's results in his work 'Analyse des équations determinés' [Fou31], where he approximatively solved linear equational systems with more equations than parametres by minimizing the maximum error of every equation.

In 1853 Pafnuti Lvovich Chebyshev was the first to consolidate these considerations into the 'Theory of functions deviating the least possible from zero,' as he called it.

Starting out from the problem of determining the parameters of the driving mechanism of steam engines—also called Watts parallelogram—in such a manner that the conversion of straight into circular movement becomes as exact as possible everywhere, he was led to the general problem of the uniform approximation of a real analytic function by polynomials of a given degree. The first goal he achieved was the determination of the polynomial of nth degree with given first coefficient which deviates as little as possible from zero over the interval $[-1, 1]$. Today this polynomial is called a Chebyshevian polynomial of the first kind.

Further results were presented by Chebyshev in his work 'Sur les questions des minimas' [Cheb59] (written in 1857), where he stated a very general problem: that of determining parameters $p_1, \ldots, p_n$ of a real-valued function $F(x, p_1, \ldots, p_n)$ so that over a given interval $[a, b]$ the maximum error

$$\max_{x \in [a,b]} |F(x, p_1, \ldots, p_n)|$$

is minimized. Under certain assumptions for the partial derivatives

$$\frac{\partial F}{\partial p_1 \dots \partial p_n}(x, p_1, \dots, p_n)$$

he was able to prove a generally necessary condition for the solution of the problem.

Using this condition he showed that in special cases (polynomial, weighted polynomial and rational approximation) it led to the necessary condition that F has a fixed number of points where it assumes the maximum value. These points are now known as deviation points. However, the alternation theorem which clearly follows from this result has never been proved by Chebyshev himself.

The aim he sought to achieve with this contribution is to find the polynomial uniformly deviating as little as possible from zero for any number of given coefficients. Later his pupils would work on several problems arising from this general challenge. This remained the determining element of all contributions of the early St Petersburg Mathematical School on the subject of approximation theory.

[Cheb59] was the only work by Chebyshev devoted to a general problem of uniform approximation theory. But it was followed by a series of more than 40 publications in which he dealt with the solution of special uniform approximation problems, mainly from the theory of mechanisms.

Another major part of his work was devoted to least squares approximation theory with respect to a positive weight function θ. In his contribution 'About continuous fractions' [Cheb55/2] he proved that (as we say now) the orthogonal projection of a function is its best approximation in the space $L_2(\theta)$. In a number of subsequent papers he discussed this fact for certain orthonormal systems and defined general Fourier expansions.

He merged the theoretical results in the publication 'On functions deviating the least possible from zero' [Cheb73], in which he determined the monotone polynomial of given degree and the first coefficient which deviates as little as possible from zero. This was the first contribution to what we know now as shape preserving approximation theory.

All of Chebyshev's work was aimed at delivering useful solutions to practical problems. The above-mentioned contributions all arose from practical problems, e. g., from the theory of mechanisms or ballistics. A small part of his work was devoted to problems from geodesy, cartography or other subjects. This ambition pervaded all of Chebyshevs work: to him practice was the 'leader' of mathematics, and he has always demanded from mathematics that it should be applicable to practical problems. Apparently he did not view this as being in contradiction to his early theoretical work, which had been devoted mainly to number theory. So we can speculate about his concept of application.

On the other hand, he clearly disassociated himself from contemporary attempts, mainly on the part of French and German mathematicians, to define the basic concepts of mathematics clearly and without contradictions. Chebyshev qualified the discussion about infinitely small quantities as ‘philosophizing’. It follows that his methods, without exceptions, were of an algebraic nature; he did not mention limits except where absolutely necessary. A characteristic feature of his work is the fact that, if convergence was intuitively possible, to him it was self-evident. Thus, he often omitted to point out that an argument was valid if a function converged (uniformly or pointwise).

Besides his scientific achievements which also extended into probability theory, Chebyshev distinguished himself as a founder of a mathematical school. The first generation of the generally so-called Saint Petersburg Mathematical School only consisted of mathematicians who began their studies during Chebyshev’s lifetime and were completely influenced by his work, but even more by his opinion about what mathematics should be. In Aleksandr Nikolaevich Korkin, the eldest of the schools’ members, we have a truly orthodox representative of the algebraic orientation. For example he referred to modern analytic methods of treating partial differential equations as ‘decadency’ because they did not explicitly solve explicit equations. Other members also disassociated themselves from new mathematical directions, most notably Aleksandr Mikhajlovich Lyapunov, who sweepingly disqualified Riemannian function theory as ‘pseudogeometrical’.

However, this radical position was not typical of all students of Chebyshev. Egor Ivanovich Zolotarev showed an interest in basic mathematical questions, both in his written work, where we see his deep knowledge of function theory, and in his lectures, where he endeavoured to define concepts like the continuous function as early as in the 1870s. Julian Sochocki’s work was even exclusively devoted to the theory of functions in the manner of Cauchy and later these results were used by others (e.g., by Posse) to prove Chebyshev’s results in a new way.

Nevertheless the Saint Petersburg Mathematical School was characterized by the orientation towards solving concretely posed problems to get an explicit formula or at least a good algorithm which is suitable for practical purposes.

It is not surprising, then, that the contributions of the members of the Saint Petersburg Mathematical School were predominantly of a classical nature and employed almost exclusively algebraic methods.

This is particularly true for the schools’ work on approximation theory. Outstanding examples include the papers by Zolotarev and the brothers Andrej Andreevich Markov, Sr. and Vladimir Andreevich Markov, which were devoted to special problems from the field of uniform approximation theory.

Zolotarev investigated the problem of determining the polynomial of given degree which deviates as little as possible from zero while having its two highest coefficients fixed. Thus he directly followed the objective set by Chebyshev himself. Andrej Markov’s most important contribution on this subject was the determination of a polynomial least deviating from zero with respect to a special linear condition of its coefficients. Vladimir Markov generalized this problem and solved his brother's problem for any linear side condition.

The above-mentioned three contributions were all distinguished by the fact that they presented a complete theory of their problems. This is most distinctly illustrated by the work of Vladimir Markov, who proved a special alternation theorem in this context, as well as another theorem which in fact can be called a first version of the Kolmogorov criterion of 1948. As the most important result of Vladimir Markov's paper we today acknowledge the inequality estimating the norm of the kth derivative of a polynomial by the norm of the polynomial itself. Later Sergej Natanovich Bernstein used this result to prove one of his quantitative theorems. However, consistent with the nature of the task, these investigations remained basically algebraic.

The last contribution to early uniform approximation theory coming from Saint Petersburg were Andrej Markov's 1906 lectures 'On Functions Deviating the Least Possible from Zero', [MarA06] where he summarized and clearly surveyed the respective results of Saint Petersburg mathematicians. For the first time a Petersburg mathematician presented the uniform approximation problem as a problem of approximating a continuous function by means of polynomials and proved a more general alternation theorem. Conspicuously, however, he soon returned to problems of the Chebyshev type. It is somewhat amazing that he never referred to any of the results achieved by Western European mathematicians in this context. Even the basic Weierstrassian theorem of 1885, which states the arbitrarily good approximability of any continuous function by polynomials, was not cited in these lectures.

It thus emerges with particular clarity that, because of its narrow setting of problems and its rejection of analytical methods, the uniform approximation theory of functions as developed by the Saint Petersburg Mathematical School ended in an impasse at the beginning of the twentieth century.

Outside Russia, approximation theory had other roots. A more theoretical approach had been preferred abroad because of the strong interest in basic questions of mathematics generated since the end of the 18th century by the problem of the 'oscillating string'.

The clarification of what is most likely the most important concept in modern analysis—the continuous function—generated intense interest in the consequences resulting from Weierstrass' approximation theorem. The latter had defined the aim; one now tried to make use of it, i.e., to find explicit and simple sequences of algebraic or trigonometric polynomials which converge to a given polynomial. Secondly it had to be determined how fast these sequences could converge, how fast the approximation error decreases. Such were the objectives of the long series of alternative proofs which emerged early after the original work of Weierstrass.

Natural candidates for such polynomial sequences were the Lagrange interpolation polynomials and the Fourier series.

It is surprising that a first positive result was found for the Fourier series, although the existence of continuous functions with a divergent Fourier series had been known since 1876. Lipót Fejér showed in 1900 that every function could be approximated by a version of its Fourier series, where the sum was taken from certain mean values of the classical Fourier summands.

For the case of interpolating polynomials the question likewise seemed to be negatively answered by the results of Runge [Run04] and Faber [Fab14]. However, it was again Fejér who showed that for every continuous function the sequence of the 'Hermite–Fejér Interpolants' (as we now call them) converges to the function itself.

Chebyshev's results became more commonly known in Western Europe only after the first edition of his collected works in 1899. With the introduction of analytical methods his findings could be theoretically expanded by the work of Hilbert's pupil Paul Kirchberger in 1902 [Kir02], and Émile Borel in 1905 [Bor05].

We call Dunham Jackson the founder of the quantitative approximation theory which is designed, inter alia, to determine the degree of the approximation error subject to specific requirements on the approximating function. Jackson proved a series of direct theorems in his doctoral dissertation of 1911 [Jack11]. Actually it was Hilbert's pupil Sergej Natanovich Bernstein who, a little earlier, had proved theorems of this kind—he is today considered the author of the inverse theorems which laid the foundation of the constructive function theory that characterizes functions by the order of their approximation error.

The roots of the constructive function theory lay in a very special-looking problem to which Lebesgue's proof of Weierstrass' theorem had already attracted attention: the approximation of the function $|x|$. In 1898 Lebesgue proved Weierstrass' theorem by initially approximating a continuous function by polygon lines and subsequently proving that a polygon line can be arbitrarily well approximated by polynomials [Leb98].

In the years that followed, the question of how fast $|x|$ can be approximated and the answer given by Bernstein gave rise to investigations which connected the approaches of Chebyshev and Weierstrass, that is, algebraic and analytic ideas. Especially the way he used the results of Vladimir Markov led to the insight that the degree of approximation reveals certain properties of a function. Thus, approximation theory, born from practical mechanics, helped to solve important basic mathematical problems.

Bernstein's results completed the frame of modern approximation theory, as first described in de la Vallée-Poussin's lectures from 1919, within which the theory has remained to this day [Val19].

The following key theses are first presented and substantiated in the present book:

1. The chief aim of the activities of the Saint Petersburg Mathematical School around P. L. Chebyshev on the subject of approximation was to

determine the polynomial of nth degree which deviates as little as possible from zero while having an arbitrary but fixed number of given leading coefficients.
2. This aim prevented the application of modern analytical methods to approximation theory during the early period in Saint Petersburg.
3. A merger of Weierstrass' and Chebshev's approaches was first achieved by Bernstein. Thus, we see that the Göttingen School around David Hilbert and Felix Klein had a decisive influence on the early development of approximation theory.

The present book is structured as follows:

1. In the first chapter we discuss the two works that may be considered forerunners of uniform approximation theory: Euler's cartographic investigations and Laplace's geodetic problem.
2. The second chapter is dedicated to the work of P. L. Chebyshev: His most important contributions to the uniform approximation problem are analysed and arranged in historical context. In addition, Chebyshev's philosophy of mathematics is discussed.
3. The work of Chebyshev's students founding the Saint Petersburg Mathematical School is reviewed in chapter three. We analyse in what manner they continue Chebyshev's work and adopt his aims. It becomes clear that the ideas of the mathematicians of the Saint Petersburg Mathematical School are not perfectly homogeneous. We examine their opinion about basic principles of mathematics, especially the concept of a function.
4. The absolutely different development of approximation theory in Western Europe is summarized in the fourth chapter. Starting from the problems connected with Weierstrass' approximation theorem, I have focused on questions of approximative representations of functions and their (uniform) convergence. The role of the undoubted centre of mathematics of that time, the Göttingen School around David Hilbert and Felix Klein, in the development of approximation theory is outlined on the basis of material from several archives.
5. The fifth chapter addresses the thesis that the framework of the foundation of modern approximation theory was shaped by the contributions of Bernstein. The content of what he called 'Constructive Function Theory' is described. We shall see that he achieved the link between Chebyshev's and Weierstrass' approaches.

Contents

1

Forerunners

1.1 Euler's Analysis of Delisle's Map

Cartography arose in the beginning of the 18th century in Russia with the first map covering the whole Russian empire; it was made by I. K. Kirillov in 1734 and held the scale 1:11,7 Mill. In 1745 it was followed by the "Mappa Generalis Totius Imperii Russici" (1:8,9 Mill.) developed by the St Petersburg Academy of Sciences under the main supervision of the astronomer Joseph Nicolas Delisle (1688-1768), and with co-operation of Leonhard Euler[1] (see [KayoJ]).

1.1.1 The Delislian Projection

S. E. Fel' [Fel60, p. 187] describes the construction of the maps of this atlas: "The maps of this atlas are drawn in the cone projection which preserves distances and is attributed to J. Delisle [...] The main scale is preserved in the two cutting parallels and all meridians. The map of Russia covers the region between 40^o and 70^o of northern latitude, thus the middle parallel lies at 55^o, and the cutting parallels are chosen at $47^o30'$ and $62^o30'$. So they have equal distance to the middle and the outer parallels. The meridians are divided with preservation of distances."

This kind of projection is often used when it is necessary to project a big map like the Russian empire.

Stereographic projection often used to map polar regions have the following advantages and disadvantages:

[1] Leonhard Euler (*Basel 1707, †St Petersburg 1783), 1720–1724 studies of mathematics and physics at Basel university, 1727 move to St Petersburg, 1730 professorship of physics at the Academy of Sciences, 1733 professorship of mathematics as successor of Daniel Bernoulli, 1735 co-operation with the department of geography, 1741 move to Berlin, 1744 director of the department of mathematics ("mathematische Klasse") of the Berlin Academy of Sciences, 1766 again professorship at the St Petersburg Academy of Sciences.

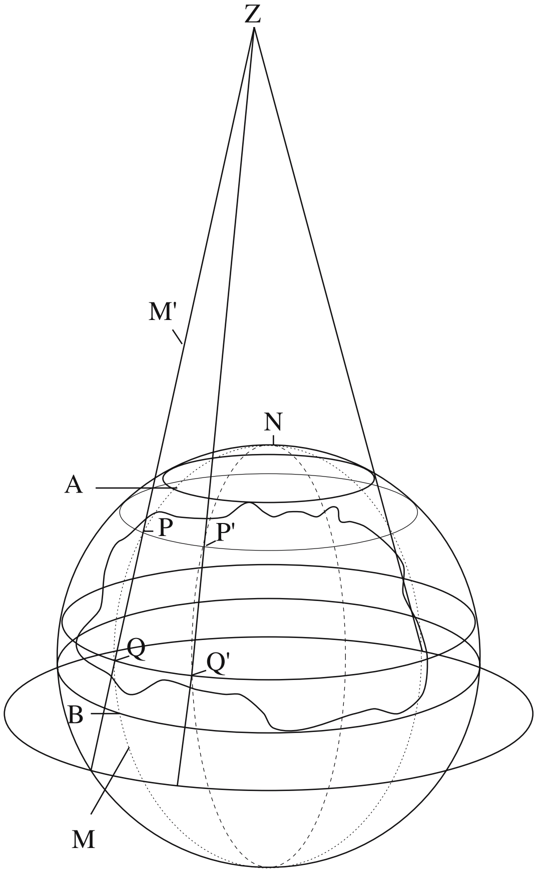

Figure 1.1. *Scheme of the Delislian cone projection with the cutting parallels PP' and QQ'*

1. Advantages:
   a) Parallels and meridians intersect perpendicularly
   b) It gives locally good approximation
2. Disadvantages:
   a) The latitudes are not equally long because the scale grows from the center to the border of the map
   b) In the case of an equatorial projection (or another non-polar projection) the meridians curve to the borders of the map. Thus taking details from such a map does not make much sense

Because of the latter disadvantage one would have to choose polar projections for an overall map of Russia. But because of the growing scale one would get a global inaccuracy of the map.

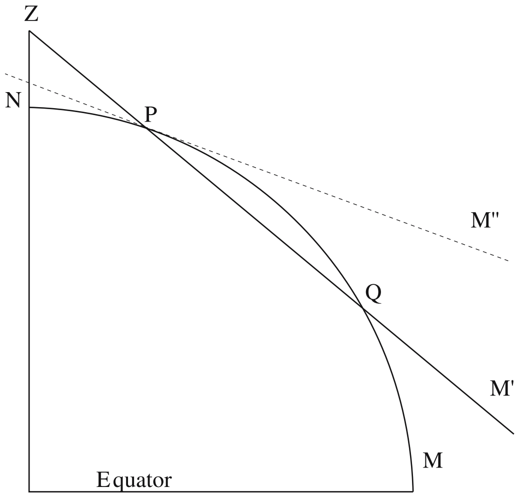

Figure 1.2. *Distance preserving division of meridians*

The map of the entire Russian Empire drawn by J. M. Hasius[2] is a polar projection. Euler had it in mind as he compared the different projections.

On the other hand, the Delislian projection meets the following claims:

- All meridians are represented by straight lines
- All parallels have the same size
- Meridians and parallels intersect perpendicularly

In 1777 Euler analysed the local and global accuracy of the Delislian conic projection in the contribution *De proiectione geographica De Lisliana in mappa generali imperii Russici usitata* [Eul77], where he tried to approximate the proportion of longitudes and latitudes of the map to the real proportion of the terrestrial globe.

The Delislian conic projection usually has cutting parallels with equal distance to the center and the borders of the map, as described above. Then the error of the proportions within the section between the cutting parallels is smaller then between the borders and the cutting parallels (see [Fel60, p. 187]).

1.1.2 Euler's Method

Now we want to get into a more exact analysis of Delisle's conic projection.

Consider a cone with the following properties (see figure 1.1):

[2] In 1739 Johann Matthias Hasius (1684–1742) published in Nuremberg the "Imperii Russici et Tatariae universae tam majoris et Asiaticam quam minoris et Europaeae tabula" (cited after [Eul75, p. 583, entry 195]).

1. It has a common axis with the earth.
2. Its top Z lies above the north pole.
3. It has two common parallels with the earth.

Now take a meridian M of the globe with points P and Q intersecting the cone.

The map M' of M on the cone is now divided preserving distances, that is, every distance of latitudes on the cone is the same as of the globe with respect to a constant factor. To illustrate this we can turn the cone into position M'' (see figure 1.2) and then unwind the meridian of the globe onto the cone.[3]

Using this construction it is possible to define a "latitude" for the top of the cone Z, since the position of Z and the distance $|\overline{ZP}|$ is determined by P and Q.

Thus, the projection is exact according to the latitudes.

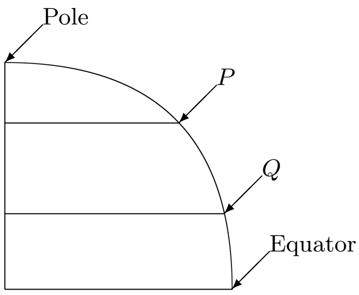

Figure 1.3. *The proportion between longitudes and latitudes from the equator to the pole is the cosine of the latitude to 1.*

To analyse the error of the longitudes on the cone, which is especially obvious regarding the pole, we will compute the length of one degree of longitude on the map.

On the surface of the globe the proportion between longitudes is the cosine of the latitude to 1. (see figure 1.3). One degree of longitude on the parallel PP' has therefore the length $\delta \cos p$, if δ is the length of one degree of latitude on the surface of the globe. On the map the length of one degree of longitude on the parallel PP' is $\omega|\overline{ZP}|$, if ω is the angle corresponding with one degree of longitude on the map (see figure 1.4).

Euler constructed a map where the maximal error of longitudes was minimized by a suitable selection of intersecting parallels P and Q.

[3] Then the cone's meridian is stretched by a factor which is equal to the quotient of the geodetic and the Euclidean distance between P and Q.

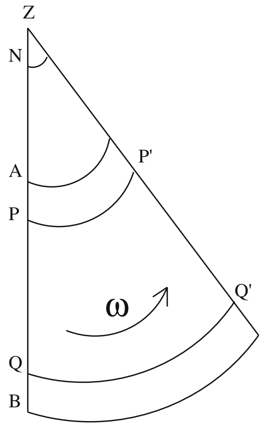

Figure 1.4. *Construction of one degree of longitude*

1.1.3 Determining the Intersecting Parallels *P* and *Q*

To construct with the above defined least maximal error we have to derive assumptions for the positions of the points P and Q.

With this we define p and q as the latitudes of P and Q on the globe. The length of the distance $\overline{ZP}$ and thus the position of the conic pole Z is then

$$\frac{|\widehat{QQ'}| - |\widehat{PP'}|}{|\overline{PQ}|} = \frac{|\widehat{PP'}|}{|\overline{ZP}|},$$

that is,

$$|\overline{ZP}| = \frac{|q-p|\cos p}{\cos q - \cos p}. \tag{1.1}$$

Now we determine the angle ω which corresponds to a degree of longitude on the map. It is

$$\omega = \frac{|\widehat{PP'}|}{|\overline{ZP}|}$$

and with (1.1)

$$\omega = \frac{\delta(\cos q - \cos p)}{|q-p|}, \tag{1.2}$$

δ being the length of a latitude of the globe.
Let z be the distance between Z and the Earth's pole on the globe.

The assumption that our projection preserves the latitude allows us to compute

$$|\overline{ZP}| = \frac{\pi}{2} - p + z.$$

Using (1.1) we get

$$z = \frac{|q-p|\cos p}{\cos q - \cos p} - \frac{\pi}{2} + p. \tag{1.3}$$

With the help of the equations (1.1) and (1.3) we will determine the positions of P and Q. Additionally we will assume that the errors of the projection at the upper border of the map A and the lower border B will be equal in value.

1.1.4 Minimization of the Error of the Projection

Firstly we will compute the error in the border parallels A and B. We set a and b as their latitudes.

Their distances from the Earth's pole can be computed as above: $\frac{\pi}{2} - a$ and $\frac{\pi}{2} - b$, respectively, similarly we have $|\overline{ZA}| = \frac{\pi}{2} - a + z$ and $|\overline{ZB}| = \frac{\pi}{2} - b + z$. To get the arc length of a degree of longitude δ_a and δ_b in A and B respectively, we must multiply these values with ω.

Hence we have:

$$\begin{aligned} \delta_a &= \omega|\overline{ZA}| \\ &= \omega(\frac{\pi}{2} - a + z) \\ &= \frac{\delta(\cos q - \cos p)(\frac{\pi}{2} - a + z)}{|q-p|}, \end{aligned} \tag{1.4}$$

$$\begin{aligned} \delta_b &= \omega|\overline{ZB}| \\ &= \omega(\frac{\pi}{2} - b + z) \\ &= \frac{\delta(\cos q - \cos p)(\frac{\pi}{2} - b + z)}{|q-p|}. \end{aligned} \tag{1.5}$$

But the exact values would be (see fig. 1.3) $\delta \cos a$ and $\delta \cos b$.

We remember that we wanted to determine P and Q under the assumption that the errors reach their maximal value in both border parallels A and B.

Therefore we can set

$$\omega(\frac{\pi}{2} - a + z) - \delta \cos a = \omega(\frac{\pi}{2} - b + z) - \delta \cos b$$

or

$$\omega = \delta \frac{\cos a - \cos b}{b - a}. \tag{1.6}$$

Thus we have a representation of ω by the known parameters a and b.