Consulting Editor

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General Preface

“What can be said at all can be said clearly.”

“Σοφὸν τοί τό σωφὲς οὐ τό μὴ σωφὲς.”
Εὐριπ., Ὀρέστης

It is nowadays generally accepted that the theory of principal fiber bundles is the appropriate mathematical framework for describing one of the most beautiful, as well as important, physical theories, viz. the so-called gauge field theory, or gauge theories, being in effect to quote M. F. Atiyah, “physical theories of a geometrical character.”

Now, in this context, a principal fibration (or internal symmetry group) of the physical system (particle field) under consideration. Yet, the particular physical system at issue is carried by, or lives on, a “space” (vacuum) that in the classical case is usually a smooth (viz. $C^\infty$-) manifold. Within our abstract framework, instead, this is in general an arbitrary topological space, being also the base space of all the fiber spaces involved.

Accordingly, we do not use any notion of calculus (smoothness) in the classical sense, though we can apply, most of the powerful machinery of the standard differential geometry, in particular, the theory of connections, characteristic classes, and the like. However, all this is done abstractly, which constitutes an axiomatic treatment of differential geometry in terms of sheaf theory and sheaf cohomology (see A. Mallios [VS: Vols I, II]), while, as already noted, no calculus is used at all! So the present study can be construed as a further application of that abstract (i.e., axiomatic) point of view in the realm of gauge theories, given, as mentioned before, the intimate connection of the latter theories with (differential) geometry.

Thus, working within the aforementioned abstract set-up, we essentially replace all the previous fiber spaces (viz. principal and/or vector bundles) by the corresponding sheaves of sections, the former being, of course, just our model (motivation), while our study is otherwise, as has already explained above, quite abstract(!), that is, axiomatic. Of course, in the classical case the two perspectives are certainly mathematically speaking (categorically!) equivalent; however, the sheaf-theoretic language, to which we also stick throughout the present treatment, is even in the standard case, in common usage in the recent physics literature (cf., for instance, Yu. I. Manin [2] or even S. A. Selesnick [1]). Thus, it proves that the same language is at least physically more transparent, while finally being more practical. In addition,
wave functions are considered as sections (i.e., functions whose domain is varied as well as their range, along with the point of application) of appropriate bundles (this volume, Chapt. IV; Section 10). Furthermore, it is still very likely that the kind of common base space of the sheaves involved herewith can also be thought of as corresponding to recent aspects of the “vacuum,” for instance, “…the structure of such spaces is governed by topology, rather than geometry” (cf. P. J. Braam [1: p. 279]).

On the other hand, a significant advantage of the present abstract formulation of the classical gauge field theory (i.e., the smooth case) lies in the possibility of employing the standard conceptual machinery of the usual (smooth) differential geometry, even for base spaces (of the fiber spaces, as above) that (i) are not smooth enough, (ii) include a large amount of singularities in the classical sense, and (iii) are not smooth at all (!), but provide the appropriate framework for the exploitation of the axiomatic theory [VS], as this happens in certain important cases (see concrete examples throughout the sequel). Of course, this potential generalization of the classical theory might very likely be of a particular significance to (mathematical) physicists, who long ago were already aware of, as well as tantalized by, the aforesaid type of spaces. Furthermore, the same abstract approach, has certainly theoretical/pedagogical advantages, being namely, of greater perspective, clarity and unification. It is thus more akin to the nowadays generally accepted aspect that “the basic ideas of modern physics are quite simple” (see, for instance, H. Fritzesh [1: p. 211]), or even that “…the problems of quantum gravity are much more than purely technical ones; they touch upon very essential philosophical issues” (cf. G. ’t Hooft [1: p. 2]). So it is quite natural to try to manufacture a similar situation pertaining to the mathematics involved; thus, something like this would also be in concord with the apostrophes, as stated in the epigraph of this preface.

Further details about each of the two individual volumes are given by separate prefaces.
Preface to Volume II

This second volume of the present treatise continues our study of gauge theories, in the framework of abstract differential geometry, by referring now to Yang–Mills fields in general, the particular case of Maxwell fields being covered already by the relevant exposition in the first volume.

We start with Chapter I, pertaining to the general theory of Yang–Mills fields, in what in particular concerns the corresponding (always, abstract) Yang–Mills equations, along with the material connected with the relevant Yang–Mills functional and its variation. To this end, we develop systematically all the necessary abstract differential-geometric machinery, as defined by a given $A$-metric, and the associated abstract Laplace–Beltrami operators and their consequences, such as the Dirac–Kähler operator and Green’s formula. We use the latter in the sequel, when studying the geometry of the moduli space of $A$-connections (cf. Chapt. III).

One of our main conclusions here, and in complete analogy with the classical (smooth) case, is that (cf. Section 8 of the present chapter)

\[ (*) \]

the solution of the Yang–Mills equations, which correspond to a given vector sheaf $E$, are exactly the critical points of the Yang–Mills functional that can be associated with $E$.

Yet, by analogy with our study in Chapter IV of Volume I, we also give a corresponding cohomological classification of Yang–Mills fields.

In Chapter II we deal with the space of Yang–Mills $A$-connections (viz., those $A$-connections (in point of fact, their curvatures) that satisfy the Yang–Mills equations corresponding to our abstract setting). Furthermore, due to the physical importance of considering gauge invariant $A$-connections, one finally considers the quotient of the previous space, through the gauge group (viz., in our case the group of ($A$-)automorphisms of the particular vector sheaf (locally free $A$-module of finite rank) $E$ under consideration)—in other words, the so-called moduli space (or orbit space) of $E$. We thus begin in the first section the rudiments of the “geometry of Yang–Mills $A$-connections,” which is further specialized in the subsequent Chapter. The corresponding moduli space of self dual $A$-connections is also considered in
The chapter ends with two more sections that are mainly connected with “second quantization” and its intrinsic commutative character; see e.g. “Morita equivalence”, or even Finkelstein’s aphorism, pertaining to “non-commutativity”, in connection with the quantum deep (see Section 5, (5.32)). Indeed, the former situation affects negatively, in effect, still the “relativistic aspect of quantization” (ibid.). In this context, one can further remark that,

the innate character of Nature is really “bosonic” (: symmetric): everything is light; see also, for instance, Chapter IV, Section 10, (10.21).

Within the same context, more recent developments, started from a topos-theoretic perspective of ADG, can be found in E. Zafiris [1], A. Mallios [14], [16], [17], A. Mallios – E. Zafiris [1], and A. Mallios – P.P. Ntumba [1], [2], [3], P.P. Ntumba [1], the latter items referring, in effect, to a symplectic aspect of ADG (yet, to what we may call, Abstract Symplectic Geometry). Yet, a topos-theoretic aspect of ADG can also be found in I. Raptis [3], [5].

As already mentioned, we continue in Chapter III our study of the geometry of the moduli space of a given vector sheaf, following, always within the advocated abstract setup, the corresponding classical pattern, mainly as indicated by I.M. Singer [1]. See, for instance, Section 7 of the chapter, referring to (the abstract form of) Gribov’s ambiguity à la Singer. In particular, we consider the notion of the tangent space at a point (A-connection) of the space of A-connections, appropriately expressed according to Singer as those tangent vectors, at the point at issue, of suitably defined curves through the same point in the space of A-connections; we have thus here a classical analogue of a Newtonian description of the notion of a tangent vector. The same aspect, suitably localized, is finally transferred to the orbit of the point concerned (viz., to the corresponding moduli space). A brief account of this has been presented in A. Mallios [6].

Finally, Chapter IV of the present volume is concerned with general relativity, when this is considered as a gauge theory, thus, to recall A. Einstein himself (see, for instance, the relevant apophthegm at the beginning of the chapter), as a field theory—that is, according to the terminology of the present treatise, a theory, pertaining to a Yang–Mills field; in particular, for the case at issue, to a Maxwell field (viz., to the massless 2-spin graviton (boson), which might be called an Einstein field to distinguish it from other bosons). One of our main results is the Einstein equations (in vacuo), which can be obtained within our abstract framework by following the classical pattern of the “variation of the Lagrangian density,” alias that of the so-called Einstein–Hilbert action (functional); all this, suitably formulated in terms of the abstract differential-geometric setup, has been applied throughout this study.

What is of particular interest here, and is also very likely to have potential applications in problems related, for instance, with quantum gravity, is the possibility of using as sheaf of coefficients (our generalized arithmetic) the sheaf of (differential) algebras of generalized functions, à la E.E. Rosinger, functions that contain by definition a large number of singularities, in effect the largest one dealt with so far. So it is a natural hunch that one can apply
a gauge theory (field theory/graviton) to understand the “atomistic and quantum structure” of reality

cf. A. Einstein [1: p. 165]) as a result of the abstract differential-geometric machinery developed so far, this being independent, as already pointed out, of any set of singularities (cf., for instance, Rosinger’s algebras), in the classical sense. A more detailed discussion is given in the Section 9 of this chapter. Moreover, for the sake of completeness, a brief account concerning Rosinger’s algebra (as well as that of multifoam algebra) sheaf is provided in Section 5 of the same chapter.

Finally, the fact that general relativity, although referred to a Maxwell field (graviton/boson), is treated in this second volume is due just to technical reasons, having to do with the way of describing it by analogy with the classical case. So this description is still afforded by means of a Lorentzian $\mathcal{A}$-metric (cf. Section 2 of Chapter IV, along with Section 2 of Chapter I); again, no “manifold” concept is needed, this sort of $\mathcal{A}$-metrics, together with necessary relevant notions, being studied in the same volume of our treatise.
The following lines represent only a small part of my indebtedness to all those people who in several ways contributed, by their contact or personal communication, the present material as well as all of the present consideration of ADG \([\text{abstract (≡ modern) differential geometry}]\), together with its potential physical applications thus providing an indispensable and corroborative factor of the whole project at issue: Thus it was Elemér Rosinger who some years ago, during one of my visits to the University of Pretoria in South Africa, heard about my intention to present general relativity, the mathematical part of course [e.g., Einstein’s equation \((in \text{ vacuo})\) in terms of ADG, and in particular using his (sheaf of) algebras of generalized functions]; the reaction then was more than enthusiastic, so that project was finally realized in A. Mallios \([8]\). Somewhat earlier, I had already started to think of the possibility of presenting Yang–Mills theory in terms of ADG, motivated here by the relevant remark (M.F. Atiyah) that the same, being a gauge theory, is, in effect, of a geometrical character (hence, ADG), yet supported by the common aspect that “basic ideas of modern physics are quite simple” (H. Fritzsch) [ADG is, in principle, a \textit{naive} theory; \textit{viz.}, \textit{axiomatic} (S. Mac Lane)]. So the first relevant ideas were already presented in A. Mallios \([6]\), in full details in the same Volume II of this work, Chapters I–II, thanks, concerning the latter reference, to kind and lively interest in my whole work of K. Iséki and the late T. Ishihara. So Elemér Rosinger, in that context, post-anticipated me, in point of fact, while supporting me too, at the same time, concerning the idea of Yang–Mills theory, when he asked for an analogous abstract formulation of the Yang–Mills equations, yet this in his characteristic, for the whole enterprise enthusiastic, stimulating, and always lively manner.

On the other hand, the continued moral and quite definitive support of Steve Selesnick was certainly alive always and perceptible. What I call in this exposition \textit{Selesnick’s correspondence} (Vol. I; Chapter II) was the guiding principle, throughout the text, pertaining to its connection with physics, in spite of his usual reservations, referring to the usefulness of that otherwise extremely nice, very convenient, and workable(!) idea; later I met an analogous point of view, related with the electromagnetic field, in Yu. Manin’s Springer book on \textit{Gauge Field Theory} while quite recently, by that same author, concerning now any other field, in his article in [3]
(I owe this last quotation to Yannis Raptis). It was actually also Steve Selesnick who
was responsible for a delightful collaboration in the last few years with Raptis, someth-
ing that has led to an especially fruitful and substantial result, referring in particular
to potential physical consequences of ADG for quantum relativity and the problem
of the so-called singularities in general.

The beautiful and very informative recent work of Stathis Vassiliou on the
Geometry of Principal Sheaves, to appear in the MIA series of Kluwer, came at
the right time to vindicate and further extend the scope as well as the applicabi-
licity of ADG. The ongoing work of Maria Papatriantafillou comes to cover the quite
natural formally categorical treatise of ADG, both of the aforesaid recent two aspects
of ADG being altogether definitive and necessary complements of the whole, thus
far, enterprise on the matter. Within an analogous vein of ideas the recent work
of Elias Zafiris comes already to test the ADG point of view in a topos-theoretic
environment for the subject, yet with possible applications to quantum gravity as
well.

During the time of several visits in the last few years to Rabat (Fès, included),
Morocco, I had the opportunity to talk about ADG and its potential physical con-
sequences mainly with Mohamed Oudadess and, in effect, with the whole “équipe
de’ analyse fonctionnelle” that thrives there, in particular, as it concerns topological
algebras theory, thus having always an eager and also critical audience, being a test,
of my own perceptions on the subject. Indeed, a very pleasant atmosphere, still inspir-
ing too, Mohamed Oudadess, at least, being steadily a prompt and critical listener (!)
providing me thus with a precious experience of having first reactions of a thoughtful
“amateur” (the last denomination is, of course, his own) to the matter that often led
me to greater elaborations of the ideas discussed and to increase understanding.

I have had in similar supporting and inspiring reactions in the past from contact
with Nelu Colojoara, the late Gerd Lassner, Konrad Schmudgen, Susanne Dierolf,
the late Klaus Floret, Franek Szafraniec, Jan de Graaf, Fredy van Oystaeyen, Roman
Zapatrin, and last, but not least, with Chris Isham for his incisive corroborative
critique, especially concerning our relevant joint work on the subject with Yannis
Raptis. The reaction of my Russian editors Vassia Lyubetsky and Sasha Zarelua was
supportive, vindicative, and much enlightening, as well.

My special thanks here are due too, for partial financial support during the recent
few years, to the office of the Special Research Program conducted by the University
of Athens and, in particular, to the Vice-Rector, at that time, Prof. Michael Dermitz-
akis for his lively and very kind support to my own research work.

The realization and appearance of the material contained in the present two
volumes would have not been accomplished without the skilful and, really wonder-
ful, typing (LATEX) talent of our secretary in the Section of Algebra and Geometry of
our Department, the late Popi Bolioti. It is to her fond memory to record here too the
excellent job she has done.

The present two-volume work owes its appearance to the enthusiasm, eager
interest, and prompt reaction of Prof. George A. Anastassiou (University of Memphis,
USA), as well as to the editorial help and extremely kind attention of the Executive
Editor of Birkhäuser, Boston, Ms. Ann Kostant, and her so efficient and competent
editorial staff at Birkhäuser production. It is a particular pleasure to express at this place my heartfelt thanks and deep appreciation as well to all of the above people for their kindness and the warm attitude they showed toward my work.

[It is really amazing that the whole story began simply from one source: the Math. Z. (146 (1976)) article of Stephen Allan Selesnick (!); see also the Acknowledgments of the first two volumes on ADG. Then the enterprise has been continued by pointing out the quite instrumental role the notion of connection has had in the whole development of CDG, along with its physical applications.]
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Yang–Mills Theory: General Theory
Abstract Yang–Mills Theory

“Today there is an amazing confluence of the gauge theories in physics (for the Yang–Mills equations) and the geometrical theory of connections on fiber bundles.”


“The objects of our study in this chapter belong to what we may call the Yang–Mills category (see Section 4.2 for concrete definitions), while the corresponding morphisms are suitable connection-preserving sheaf morphisms (ibid. (4.16)). Now, since the necessary background material for the subject matter at issue has not been systematically developed so far, within the framework of abstract differential geometry (see A. Mallios [VS: Vols. I and II]), which is employed by the present treatise, we give below a detailed exposition of all the relevant issues that will be needed in the sequel. In this context, see also, however, A. Mallios [6: p. 164, Appendix II] for a brief account on the same material. Among the various standard presentations of this subject, see, for instance, T. Petrie–J. Randall [1]. So we start with the ensuing fundamental notions for all the subsequent discussion.

1 The Differential Setting

As was the case in Volume I of the present treatise, here too, the adjective “differential” has, of course, only a formal meaning, referring to a particular type of (“differential”) operator connected with the subject matter under consideration, given that no “smooth structure” at all (!) is assumed on our base space $X$. Thus, the ensuing discussion of this section aims, in effect, to collect together all the “differential operators” that have been employed in the preceding (see [VS: Vols. I and II], yet, Volume I of this treatise), to the extent that they provide the corresponding to our case (generalized, alias abstract) de Rham complex, along with the respective “connection operators” (see below). All of this will be necessary for our subsequent considerations in Section 2, where we shall define the corresponding “dual (alias, adjoint) differential setup.”
So, as usual, we start with a $\mathbb{C}$-algebraized space

\[(X, \mathcal{A}),\]

which is further assumed to be endowed with a given differential triad (see also Chapt. I, Section 1),

\[(\mathcal{A}, \partial, \Omega^1).\]

Indeed, one can assume, depending on the particular case under consideration, the existence of a sequence of exterior differentials (or also differential operators, or else differentials of the first kind)

\[(d^n)_{n\in\mathbb{Z}_+},\]

where we still set

\[(1.3') \quad d^0 \equiv \partial,
\]

as in (1.2) (see also (1.4) and (1.7) below). In this context, see, for instance, A. Mallios [5: pp. 17ff], or even [VS: Chapt. VIII, pp. 226ff] as it concerns the above sequence of differentials. So one defines

\[(1.4) \quad d^n : \Omega^n \longrightarrow \Omega^{n+1}, \quad n \in \mathbb{Z}_+,
\]

where we set

\[(1.5) \quad \Omega^n := \bigwedge_{i=1}^{n} (\Omega^1)^i \equiv \Omega^1 \wedge \cdots \wedge \Omega^1, \quad n \in \mathbb{N},
\]

and we also set

\[(1.5') \quad \Omega^0 := \mathcal{A},
\]

as in (1.2). Here, each one of the $d^n$’s, as above, is, by definition, a $\mathbb{C}$-linear morphism of the $\mathcal{A}$-modules concerned, such that the following (defining) relation is further assumed to be valid; namely one has

\[(1.6) \quad d^{p+q}(s \wedge t) = d^p(s) \wedge t + (-1)^p s \wedge d^q(t),
\]

for any $s \in \Omega^p(U)$ and $t \in \Omega^q(U)$, and any open $U \subseteq X$, with $p$ and $q$ in $\mathbb{Z}_+$. On the other hand, supposing that the relations

\[(1.7) \quad d^1 \circ d^0 \equiv d^1 \circ \partial = 0
\]

as well as

\[(1.7') \quad d^2 \circ d^1 = 0
\]
are valid, one further proves that

(1.8) \[ d^{p+1} \circ d^p = 0, \text{ for any } p \geq 2. \]

Yet, concerning the equations (1.7), (1.7′), and (1.8), we also set, for convenience, as already done in [VS] and Volume I too, simply

(1.9) \[ d \circ d \equiv dd \equiv d^2 = 0. \]

Thus, one arrives at the following generalized (alias, abstract) de Rham complex—in point of fact, a cochain complex of \( \mathbb{C} \)-vector space sheaves on the space \( X \) (see (1.1)):

(1.10) \[
0 \rightarrow \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \equiv \Omega^0 \xrightarrow{d^0 \equiv \partial} \Omega^1 \xrightarrow{d^1} \Omega^2 \rightarrow \cdots \rightarrow \Omega^n \xrightarrow{d^n} \Omega^{n+1} \rightarrow \cdots.
\]

Of course, in the previous sequence the complexes \( \mathbb{C} \) stand, in effect, for the constant (\( \mathbb{C} \)-vector space) sheaf of the complexes, while one still obtains the relation

(1.11) \[ \partial \circ \varepsilon = 0, \]

according to our hypothesis for \( \mathcal{A} \) and (1.6), along with (1.3′) (see also Chapt. I, (1.16)). However, the above complex (1.10) is not, in general, exact! (In other words, the “abstract Poincaré lemma” is lacking, in general, but on the other hand, see also, for instance, Chapter IV, Section 5 in the sequel.)

So, to summarize, we have so far considered a \( \mathbb{C}_X \)-complex on \( X \) associated with the given \( \mathbb{C} \)-algebraized space, as in (1.1)—that is, the cochain complex of \( \mathbb{C} \)-vector space sheaves on \( X \), of positive degree,

(1.12) \[
(\Omega^*, d) \equiv \{(\Omega^n, d^n)\}_{n \in \mathbb{Z}_+}
\]

(see (1.5) and (1.5′), along with (1.3′) and (1.4), as well as (1.6)). Furthermore, we also call (1.12) the abstract de Rham complex of \( X \), yet, occasionally, apart from (1.10), as the case may be. In this connection, we still note that in the particular case that (1.10) is exact, while further suitable assumptions are imposed on the pair \((X, \mathcal{A})\), as in (1.1), it is, in point of fact, through (1.12) that one gets at the (abstract de Rham) cohomology of \( X \), with complex coefficients that can still be expressed by means of our “structure sheaf” \( \mathcal{A} \), a fact that might be of fundamental importance in the applications (loc. cit.). In this regard, see also [VS: Chapt. III] for a detailed account of the relevant terminology employed herewith.

1.1 Vectorization of the Abstract de Rham Complex (Prolongations)

We come now to what one might call a “vectorization” of the preceding, with respect to a given \( \mathcal{A} \)-module \( \mathcal{E} \) on \( X \), by considering the following sequence of “differentials of the second kind”:

(1.13) \[
(D^n)_{n \in \mathbb{Z}_+},
\]
such that one has

\[(1.14) \quad D^n : \Omega^n(\mathcal{E}) \longrightarrow \Omega^{n+1}(\mathcal{E}), \quad n \in \mathbb{Z}_+, \]

where we set

\[(1.15) \quad \Omega^n(\mathcal{E}) := \Omega^n \otimes_A \mathcal{E} \cong \mathcal{E} \otimes_A \Omega^n \]

for any \( n \in \mathbb{Z}_+ \) (see also Vol. I; Chapt. I, (2.1), concerning the last \( \mathcal{A}\)-isomorphism in (1.15) of the \( \mathcal{A}\)-module involved). On the other hand, the same “differential operators” as in (1.14) are actually given, through the following relation (definition):

\[(1.16) \quad D^n := 1_{\mathcal{E}} \otimes d^n + (-1)^n \Omega^n \wedge D, \quad n \in \mathbb{Z}_+, \]

where we have still set

\[(1.17) \quad D^0 \equiv D \]

(see also (1.22) in the sequel).

Thus, by looking at (1.16), in terms of (local) sections of the \( \mathcal{A}\)-modules involved therein, one obtains (see also (1.14) and (1.15))

\[(1.18) \quad D^n(s \otimes t) = s \otimes d^n(t) + (-1)^n t \wedge D(s), \quad n \in \mathbb{Z}_+, \]

for any \( s \in \mathcal{E}(U) \) and \( t \in \Omega^n(U) \), with \( U \) open in \( X \). Yet, we also take into account here that (\( \text{viz.}, \) (1.15), for \( n = 0 \))

\[(1.19) \quad \Omega^0(\mathcal{E}) \equiv \Omega^0 \otimes_A \mathcal{E} \equiv \mathcal{A} \otimes_A \mathcal{E} = \mathcal{E} \]

(see (1.5’), along with [VS: Chapt. II, (5.15)]). In particular, by further applying (1.18), for \( n = 0 \), one obtains, in view also of (1.3’), (1.17), (1.19), and (1.14), the following \( \mathbb{C}\)-linear morphism:

\[(1.20) \quad D : \mathcal{E} \longrightarrow \Omega^1(\mathcal{E}), \]

such that one has (a result, in effect, of (1.18), in view of (1.19))

\[(1.21) \quad D(\alpha \cdot s) = \alpha \cdot D(s) + s \otimes \partial(\alpha) \]

for any \( \alpha \in \mathcal{A}(U), s \in \mathcal{E}(U) \), and open \( U \subseteq X \). Accordingly, one thus concludes that

the above map, as in (1.20) (see also (1.17)),

\[(1.22.1) \quad D \equiv D^0, \]

(1.22)

is, in effect, an \( \mathcal{A}\)-connection of \( \mathcal{E} \). Thus, by still extending the classical terminology, we can consider the rest of the “differential operators”, as in (1.13) (\( \text{viz.}, \) for \( n \geq 1 \)), as the prolongations of \( D(\equiv D^0) \). (See also (1.25) in the sequel.)
Consequently, based on the preceding, we can say that the vectorization of (1.12), with respect to a given $\mathcal{A}$-module $\mathcal{E}$ on $X$, as above, is given now by the relation (see (1.14) and (1.15))

\[(Q^*(\mathcal{E}), D) \equiv \{(\Omega^n(\mathcal{E}), D^n)\}_{n\in\mathbb{Z}_+}.\]

Yet, the relation of (1.12) with (1.23) is further explained by the subsequent discussion; indeed, one can say that

\[(1.12)\] is obtained from (1.23), by simply putting $\mathcal{E} = \mathcal{A}$ in the latter;

namely, based on (1.18), in conjunction also with (1.15) and (1.19), one first obtains that

\[(1.25)\]

\[D^n|_{\Omega^n(\mathcal{A})=\mathcal{A}\otimes\mathcal{A}}=d^n, \quad n \in \mathbb{N}.\]

On the other hand, by looking at (1.6), for $p = 0$ and $q = 1$, one has

\[(1.26)\]

\[d^1(\alpha \cdot s) = \alpha \cdot d^1(s) - s \land \partial(\alpha)\]

for any $\alpha \in \mathcal{A}(U)$, $s \in \Omega^1(U)$, and open $U \subseteq X$, which thus, in view of (1.18), when the latter is applied for $n = 1$, can be construed as yielding that

\[(1.27)\]

\[d^1 \text{ is the first prolongation of } d^0 \equiv \partial.\]

(In this concern, see also, for instance, [VS: Chapt. VIII, p. 192, Remark 2.1]). So the above relation (1.25), along with (1.27), justifies our assertion in (1.24).

However, the same relations, as before, prove also that

\[(1.28)\]

\[d^n, \quad n \in \mathbb{Z}_+, \text{ as in (1.3) (see (1.3')), or else “differentials of the first kind” can be viewed simply as the successive prolongations of the given (see (1.2)) standard (flat) $\mathcal{A}$-connection }\partial \equiv d^0 \text{ on } \mathcal{A}.

In this regard, see also A. Mallios [5: p. 19, (1.19)].

So the preceding, as given by (1.3) and (1.13), constitute our abstract differential setup, in terms of which one can recast, while at the same time extend, within the present axiomatic framework, several fundamental notions and results of the classical theory (e.g., differential geometry of smooth manifolds). We proceed in Section 2 to the dual framework of the above under the only proviso that one is supplied with an appropriate $\mathcal{A}$-metric on a given $\mathcal{A}$-module $\mathcal{E}$ on $X$, as before. Indeed, the problem is actually reduced, for suitable $(X, \mathcal{A})$, to a similar one for $\mathcal{A}$! See, thus, for example, (2.19) below.

### 2 The Dual Differential Setting

Our purpose in the present section is to define, within the abstract framework of this treatise, the so-called *dual differential operators* of those already considered
in Section 1, something that one can really achieve, provided we put up the necessary setting there. So, what one actually needs here is the notion of an \( A \)-metric on a given \( A \)-module \( E \) on \( X \); we continue to assume herewith that we are given the context, as in (1.1) of Section 1. Thus, motivated by the standard situation, the latter notion concerns, in effect, a sheaf morphism, say,

\[
\rho : E \oplus E \to A,
\]

such that the following conditions are fulfilled:

(i) \( \rho \) is an \( A \)-bilinear morphism between the \( A \)-modules concerned, as in (2.1).

(ii) \( \rho \) is symmetric; that is, one has

\[
\rho(s, t) = \rho(t, s)
\]

for any \( s \) and \( t \) in \( E(U) \), with \( U \) open in \( X \).

There is another condition that we impose on \( \rho \) (viz., its positive definiteness). However, to formulate the latter notion, we need to have on our structure sheaf \( A \), as in (1.1), a richer structure; so we further suppose that

the underlying \( \mathbb{R} \)-algebra sheaf \( A \), as in (1.1), is a (partially) ordered \( (\mathbb{R}-) \)algebra sheaf on \( X \); that is, we assume the existence of a subsheaf

\[
P \subseteq A,
\]

(2.3) (2.3.1)

defining (sectionwise) the preorder in \( A \).

For details on the terminology employed in (2.3), see [VS: Chapt. IV, pp. 316ff].

In this connection, whenever we have the situation described by (2.3), as above, we then also speak of our previous pair, as in (1.1),

\[
(X, A),
\]

(2.4)
as a (partially) ordered algebraized space. Thus, under the hypothesis that (2.3) holds true, we further assume, concerning the conditions we impose on \( \rho \), that

(iii) \( \rho \) is positive definite, in the sense that one has

\[
\rho(s, s) \in P(U) \quad \text{(see (2.3.1)), such that}
\]

\[
\rho(s, s) = 0 \quad \text{(if, and) only if} \quad s = 0,
\]

(2.5) (2.6)

where \( s \in E(U) \) (see (2.1)) and \( U \) is open in \( X \).

**Note 2.1** The relation \( \rho(s, s) \in P(U) \), as applied in (2.5), will also be denoted in the sequel by

\[
\rho(s, s) \geq 0
\]

(2.5')

for any \( s \) and \( U \), as in (2.5).
Now, before we come to the next property of the map $\rho$ that we are going to employ, we first consider another map, say $\tilde{\rho}$, deduced from $\rho$ by virtue of property (i); indeed, one gets the existence of a map

$$\tilde{\rho} : \mathcal{E} \longrightarrow \mathcal{E}^* := \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$$

in fact, an $\mathcal{A}$-morphism of the $\mathcal{A}$-modules in (2.7), such that one actually defines

$$\tilde{\rho}(s) \equiv \rho_s : \mathcal{E} \longrightarrow \mathcal{A} : t \longmapsto \tilde{\rho}(s)(t) \equiv \rho(s, t)$$

for any $s$ and $t$ in $\mathcal{E}(U)$ and any open $U \subseteq X$.

Thus, assuming now that we are given the framework of (2.3), as above, we first remark that

property (iii) of $\rho$ (viz., the positive definiteness of $\rho$, see (2.5)) entails that $\tilde{\rho}$, as defined by (2.7), becomes an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules involved in (2.7); namely, at the following (canonical) imbedding

$$\mathcal{E} \subset \tilde{\rho}^\mathcal{E}$$. 

Now, the previous imbedding is not, in general, onto, as is virtually the case in the classical theory (i.e., when considering finite-dimensional $\mathbb{C}$ (or even $\mathbb{R}$)-vector spaces, hence, corresponding vector bundles, or their associated sheaves of sections).

Thus, whenever we have the relation

$$\mathcal{E} \cong \tilde{\rho}^\mathcal{E}$$

within an $\mathcal{A}$-isomorphism of the $\mathcal{A}$-modules concerned, we say that $\rho$ (see (2.1)) is strongly nondegenerate.

**Note 2.2** (Terminological) As already remarked (see (2.8)), the positive definiteness of $\rho$ (see (2.5) or else (2.5′)) entails (2.8.1); yet, as explained above (see the comments following (2.8.1)), when referring to the classical theory, (2.8.1) is equivalent to (2.9). Thus, classically speaking, in the semi-Riemannian (or else pseudo-Riemannian) case (see B. O’Neill [1: pp. 54f], or even A.L. Besse [1: p. 29, Definition 1.33]), one generalizes, as it happens (e.g., in the General Theory of Relativity), by considering the nondegeneracy of $\rho$ [viz., (2.9), or, equivalently (for the finite-dimensional case), (2.8.1), in place of (2.5) (“positive definiteness” of $\rho$)].

Now, in this treatise, we first consider the Riemannian case (see Definition 2.1), while, later (see Chapt. IV, Section 2), we also employ the semi-Riemannian (in particular, Lorentz) case, where $\rho$ is not necessarily positive definite, as in (2.5) above (see Chapt. IV, Definition 2.1, in particular, (2.7): Lorentz condition), but is always (loc. cit. (2.6)) strongly nondegenerate. Thus, we define the notion of an $\mathcal{A}$-valued inner product $\rho$ on a given $\mathcal{A}$-module $\mathcal{E}$ by extending the classical situation to our abstract framework, according to the following.
Definition 2.1 Let \((X, \mathcal{A})\) be a (partially) ordered algebraized space (see (2.3)) and let \(E\) be an \(\mathcal{A}\)-module on \(X\). Now, an \(\mathcal{A}\)-metric, or even a Riemannian \(\mathcal{A}\)-metric, on \(E\) is an \(\mathcal{A}\)-bilinear symmetric positive definite and strongly nondegenerate map \(\rho\), as in (2.1). We also then speak of the pair

\[
(\mathcal{E}, \rho)
\]

as a metrized \(\mathcal{A}\)-module, or even as a Riemannian \(\mathcal{A}\)-module \(\mathcal{E}\) on \(X\).

Thus, given a (partially) ordered algebraized space \((X, \mathcal{A})\), as in (2.3), and an \(\mathcal{A}\)-module \(E\) on \(X\), an \(\mathcal{A}\)-valued inner product on \(E\) is, by definition, an \(\mathcal{A}\)-bilinear symmetric and positive definite map (sheaf morphism), as in (2.1). Consequently, in that sense, an \(\mathcal{A}\)-metric \(\rho\) on \(E\) (see Definition 2.1) is a strongly nondegenerate \(\mathcal{A}\)-valued inner product on \(E\).

Warning! In this connection, and, in conjunction with our previous scholium in Note 2.2, we still remark here that, in our case, an \(\mathcal{A}\)-valued inner product on \(E\), is not necessarily an \(\mathcal{A}\)-metric in the sense of Definition 2.1; only conversely, given that (2.8.1) does not always imply (2.9), viz., the “strong nondegeneracy” of \(\rho\), as in (2.1) (see the comments following (2.8.1)), by contrast with what happens in the classical theory (finite-dimensional case).

Now, further details pertaining to a local treatment in terms of local frames of Riemannian vector sheaves, which will also be considered in the sequel, are given in [VS: Chapt. IV, Section 8] (see, for instance, ibid., p. 322, Theorem 8.1).

Thus, after the preceding preliminary material, we come now to our main object.

2.1 Dual Differential Operators

We start by assuming that we have a differential triad

\[
(\mathcal{A}, \partial, \Omega^1)
\]

which, as usual, is associated with a given \(\mathbb{C}\)-algebraized space

\[
(X, \mathcal{A})
\]

(see [VS: Vol. II], or Volume I, Chapter I, Section 1 of this treatise). Furthermore, assume that we are given the sequence of differentials of the first kind (see Section 1),

\[
(d^n)_{n \in \mathbb{Z}_+}
\]

(to the extent, of course, that (1.7) and (1.7’) are valid, the rest of \(d^i\)‘s being entailed, according to (1.6), while (1.8) holds always true). Hence, we can further consider the concomitant abstract de Rham complex (see (1.10) and (1.12))

\[
\{(\Omega^n, d^n)_{n \in \mathbb{Z}_+}\}
\]

On the other hand, we still consider a pair

\[
(\mathcal{E}, D)
\]