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Differential Equations, Chaos and Variational Problems

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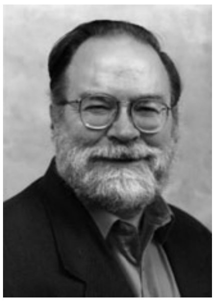
Editorial Introduction

This book is a collection of original papers and state-of-the-art contributions written by leading experts in the areas of differential equations, chaos and variational problems in honour of Arrigo Cellina and James A. Yorke, whose remarkable scientific carrier was a source of inspiration to many mathematicians, on the occasion of their 65th birthday.

Arrigo Cellina and James A. Yorke were born on the same day: August 3, 1941. Both received their Ph.D. degrees from the University of Maryland, where they met first in the late 1960s, at the Institute for Fluid Dynamics and Applied Mathematics. They had offices next to each other and though they were of the same age, Yorke was already Assistant Professor, while Cellina was a Graduate Student. Each one of them had a small daughter, and this contributed to their friendship.

Arrigo Cellina



James A. Yorke

Yorke arrived at the office every day with a provision of cans of Coca Cola, his daily ration, that he put in the air conditioning fan, to keep cool. Cellina says that he was very impressed by Yorke's way of doing mathematics; Yorke could prove very interesting new results using almost elementary mathematical tools, little more than second year Calculus.

From those years, he remembers for example the article *Noncontinuable solutions of differential-delay equations* where Yorke shows, in an elementary way but with a clever use of the extension theorem, that the basic theorem of continuation of solutions to ordinary differential equations cannot be valid for functional equations (at that time very fashionable). In the article *A continuous differential equation in Hilbert space without existence*, Yorke gave the first example of the nonexistence of solutions to Cauchy problems for an ordinary differential in a Hilbert space. Furthermore, in a joint paper with one of his students, Saperstone, he proved a controllability theorem without using the hypothesis that the origin belongs to the interior of the set of controls. This is just a sample of important problems to which Yorke made nontrivial contributions.

Yorke went around always carrying in his pocket a notebook where he annotated the mathematical problems that seemed important for future investigation. In those years Yorke's collaboration with Andrezj Lasota began, which produced outstanding results in the theory of "chaos". Yorke became famous even in non-mathematical circles for his mathematical model for the spread of gonorrhoea. While traditional models were not in accord with experimental data, he proposed a simple model based on the existence of two groups of people and proved that this model fits well the experimental data. Later, in a 1975 paper entitled *Period three implies chaos* with T.Y. Lee, Yorke introduced a rigorous mathematical definition of the term "chaos" for the study of dynamical systems. From then on, he played a leading role in the further research on chaos, including its control and applications.

Yorke's goals to explore interdisciplinary mathematics were fully realized after he earned his Ph.D. and joined the faculty of the Institute for Physical Science and Technology (IPST), an institute established in 1950 to foster excellence in interdisciplinary research and education at the University of Maryland. He said: *All along the goal of myself and my fellow researchers here at Maryland has been to find the concepts that the applied scientist needs.* His chaos research group introduced many basic concepts with exotic names like *crises*, the *control of chaos, fractal basin boundary, strange non-chaotic attractors*, and the *Kaplan–Yorke dimension.* One remarkable application of Yorke's theory of chaos has been the weather prediction.

In 2003 Yorke shared with Benoit Mandelbrot of Yale University the prize for *Science and Technology of Complexity* of the Science and Technology Foundation of Japan for the *Creation of Universal Concepts in Complex Systems-Chaos and Fractals.* With this prize, Jim Yorke was recognized for his outstanding achievements in nonlinear dynamics that have greatly advanced the frontiers of science and technology.

Yorke's research has been highly influential, with some of his papers receiving hundreds of citations. He is the author of three books on chaos, of a monograph on gonorrhoea epidemiology, and of more than 300 papers in the areas of ordinary differential equations, dynamical systems, delay differential equations, applied and random dynamical systems.

He believes that a Ph.D. in mathematics is a licence to investigate the universe, and he has supervised over 40 Ph.D. dissertations in the departments of mathematics, physics and computer science.

Currently, Jim Yorke is a Distinguished University Professor of Mathematics and Physics, and Chair of the Mathematics Department of the University of Maryland.

Arrigo Cellina received a Ph.D. degree in mathematics in 1968 and went back to Italy, where he was Assistant Professor and then Full Professor at the Universities of Perugia, Florence, and Padua, at the International School for Advanced Studies (SISSA) in Trieste, and at the University of Milan. He was a member of the scientific committee and then Director (1999–2001) of the International Mathematical Summer Centre (CIME) in Florence, Italy, and also a member of the scientific council of CIM (International Centre for Mathematics) seated in Coimbra, Portugal. Presently he is Professor at the University of Milan "Bicocca" and coordinator of the Doctoral Program of this university.

In Italy, the International School for Advanced Studies (SISSA) was established in 1978, in Trieste, as a dedicated and autonomous scientific institute to develop top-level research in mathematics, physics, astrophysics, biology and neuroscience, and to provide qualified graduate training to Italian and foreign laureates, to train them for research and academic teaching.

SISSA was the first Italian school to set up post-laurea courses aimed at a Ph. D. degree (Doctor Philosophiae). Cellina was one of the professors, founders and, for several years, the Coordinator of the Sector of Functional Analysis and Applications at SISSA, from 1978 until 1996.

I was lucky to have been initiated to mathematical research on Aubin–Cellina's book *Differential inclusions* in a research seminar at the University of Bucharest. Three years later I began my Ph.D. studies on differential inclusions at SISSA, under the supervision of Arrigo Cellina. I arrived at SISSA coming from Florence where I spent a very rewarding and training period of one year as a Research Fellow of GNAFA under the supervision of Roberto Conti, and I remember that Arrigo welcomed me with a kindness equal to his erudition.

Always available to discuss and to help his students to overcome difficulties, not only of mathematical orders, Arrigo taught me a lot more than differential inclusions. I remember with great pleasure his beautiful lessons, the long hours of reflection in front of the blackboard in his office, as well as the walks along the sea or in the park of Miramare.

I remember SISSA of those days as a very exciting environment. A community of researchers worked there, while several others were visiting SISSA and gave short courses or seminars concerning their new results. The Sector of Functional Analysis and Applications was located in a beautiful place, close to the Castle of Miramare, and near the International Centre for Theoretical Physics (ICTP), with an excellent library where we could spend much of our time. Without a doubt, this has been a very fruitful and rewarding period of my life, both as a scientific and as a life experience. Cellina's contribution has been significant.

Cellina's scientific work has always been highly original, introducing entirely new techniques to attack the difficult problems he considered. He introduced the notion of *graph approximate selection* for upper semicontinuous multifunctions, thus establishing a basic connection between ordinary differential inclusions and differential inclusions. He also introduced the *fixed-point approach* to prove the existence of differential inclusions based on *continuous selections from multifunctions with decomposable values*.

The Baire category method, for the analysis of differential inclusions without convexity assumptions, has been developed starting from Cellina's seminal paper *On the differential inclusion* $x' \in [-1, 1]$, published in 1980 by the *Rendiconti dell'Academia dei Lincei.* Eventually this method recently found applications to problems of the Calculus of Variations, without convexity or quasi-convexity assumptions, as well as to implicit differential equations. This year, it found even more striking new applications to the construction of deep counterexamples in the theory of multidimensional fluid flow.

More recently, Cellina's research activity was devoted to the area of the Calculus of Variations, where he obtained important results on the validity of the Euler-Lagrange equation, on the regularity of minimizers, on necessary and sufficient conditions for the existence of minima, and on uniqueness and comparison of minima without strict convexity.

The book *Differential Inclusions*, co-authored by Cellina with J. P. Aubin and published by Springer, as well as several of his eighty papers published in first-class journals, are now classic references to their subject. Cellina also edited several volumes with lectures and seminars of CIME sessions, published by Springer in the subseries *Fondazione C.I.M.E.* of the *Lecture Notes in Mathematics* series.

Cellina mentored ten Ph.D. students: seven of them while at SISSA and three others at the university of Milan. Among his former students, many are now Professors in prestigious universities in Italy, Portugal, Chile or other countries. Several more mathematicians continue to be inspired by his ground-breaking ideas.

Cellina and Yorke during the conference in Aveiro

In June 2006, I had the privilege to organize in Aveiro (Portugal) with my colleagues from the *Functional Analysis and Applications* research group, the conference *Views on ODEs*, in celebration of the 65th birthday of Arrigo Cellina and James A. Yorke. Several friends, former students and collaborators, presently leading experts in differential equations, chaos and variational problems, gathered in Aveiro on this occasion to discuss their new results. The present volume collects thirty-two original papers and state-of-the-art contributions of participants to this conference and brings the reader to the frontier of research in these modern fields of research.

I wish to thank Professor Haim Brezis for accepting to publish this book as a volume of the series *Progress in Nonlinear Differential Equations and Their Applications*. I also thank Thomas Hempfling for the professional and pleasant collaboration during the preparation of this volume. Finally, I gratefully acknowledge partial financial support from the Portuguese Foundation for Science and Technology (FCT) under the Project POCI/MAT/55524/2004 and from the *Mathematics and Applications* research unit of the University of Aveiro.

Aveiro, October 2007 Vasile Staicu

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Nodal and Multiple Constant Sign Solution for Equations with the p-Laplacian

Ravi P. Agarwal, Michael E. Filippakis,
Donal O'Regan, and Nikolaos S. Papageorgiou

Dedicated to Arrigo Cellina and James Yorke

Abstract. We consider nonlinear elliptic equations driven by the p-Laplacian with a nonsmooth potential (hemivariational inequalities). We obtain the existence of multiple nontrivial solutions and we determine their sign (one positive, one negative and the third nodal). Our approach uses nonsmooth critical point theory coupled with the method of upper-lower solutions.

Mathematics Subject Classification (2000). 35J60, 35J70.

Keywords. Scalar p-Laplacian, eigenvalues, $(S)_+$-operator, local minimizer, positive solution, nodal solution.

1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary ∂Z. We consider the following nonlinear elliptic problem with nonsmooth potential (hemivariational inequality):

$$\left\{ \begin{array}{l} -\mathrm{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) \in \partial j\big(z, x(z)\big) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0, \;\; 1 < p < \infty\,. \end{array} \right\} \tag{1.1}$$

Here $j(z,x)$ is a measurable function on $Z \times \mathbb{R}$ and $x \to j(z,x)$ is locally Lipschitz and in general nonsmooth. By $\partial j(z,\cdot)$ we denote the generalized subdifferential of $j(z,\cdot)$ in the sense of Clarke [3]. The aim of this lecture is to produce multiple nontrivial solutions for problem (1.1) and also determine their sign (positive, negative or nodal (sign-changing) solutions). Recently this problem was studied for equations driven by the p-Laplacian with a C^1-potential function (single-valued

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right hand side), by Ambrosetti-Garcia Azorero-Peral Alonso [1], Carl-Perera [2], Garcia Azorero-Peral Alonso [7], Garcia Azorero-Manfredi-Peral Alonso [8], Zhang-Chen-Li [15] and Zhang-Li [16]. In [1], [7], [8], the authors consider certain nonlinear eigenvalue problems and obtain the existence of two strictly positive solutions for all small values of the parameter $\lambda \in \mathbb{R}$ (i.e., for all $\lambda \in (0, \lambda^*)$). In [2], [15], [16] the emphasis is on the existence of nodal (sign changing) solutions. Carl-Perera [2] extend to the p-Laplacian the method of Dancer-Du [6], by assuming the existence of an ordered pair of upper-lower solutions. In contrast, Zhang-Chen-Li [15] and Zhang-Li [16], base their approach on the invariance properties of certain carefully constructed pseudogradient flow. Our approach here is closer to that of Dancer-Du [6] and Carl-Perera [2], but in contrast to them, we do not assume the existence of upper-lower solutions, but instead we construct them and we use a recent alternative variational characterization of the second eigenvalue λ_2 of $(-\triangle_p, W_0^{1,p}(Z))$ due to Cuesta-de Figueiredo-Gossez [5], together with a nonsmooth version of the second deformation theorem due to Corvellec [4].

2. Mathematical background

Let X be a Banach space and X^* its topological dual. By $\langle,\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X, X^*). Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz. The generalized directional derivative $\varphi^0(x; h)$ of φ at $x \in X$ in the direction $h \in X$, is given by

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

The function $h \to \varphi^0(x; h)$ is sublinear continuous and so it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq X^*$ defined by

$$\partial\varphi(x) = \left\{x^* \in X^* : \langle x^*, h\rangle \leq \varphi^0(x;h) \quad \text{for all } \ h \in X\right\}.$$

The multifunction $x \to \partial\varphi(x)$ is known as the generalized subdifferential or subdifferential in the sense of Clarke. If φ is continuous convex, then $\partial\varphi(x)$ coincides with the subdifferential in the sense of convex analysis. If $\varphi \in C^1(X)$, then $\partial\varphi(x) = \{\varphi'(x)\}$. We say that $x \in X$ is a critical point of φ, if $0 \in \partial\varphi(x)$. The main reference for this subdifferential, is the book of Clarke [3].

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we say that φ satisfies the nonsmooth Palais-Smale condition at level $c \in \mathbb{R}$ (the nonsmooth PS_c-condition for short), if every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\varphi(x_n) \to c$ and $m(x_n) = \inf\{\|x\| : x^* \in \partial\varphi(x_n)\} \to 0$ as $n \to \infty$, has a strongly convergent subsequence. If this is true at every level $c \in \mathbb{R}$, then we say that φ satisfies the PS-condition.

Definition 2.1. *Let Y be a Hausdorff topological space and E_0, E, D nonempty, closed subsets of Y with $E_0 \subseteq E$. We say that $\{E_0, E\}$ is linking with D in Y, if the following hold:*

(a) $E_0 \cap D = \emptyset$;

(b) *for any* $\gamma \in C(E,Y)$ *such that* $\gamma|_{E_0} = id|_{E_0}$, *we have* $\gamma(E) \cap D \neq \emptyset$.

Using this geometric notion, we can have the following minimax characterization of critical values for nonsmooth, locally Lipschitz functions (see Gasinski-Papageorgiou [9], p.139).

Theorem 2.2. *If* X *is a Banach space,* E_0, E, D *are nonempty, closed subsets of* X, $\{E_0, E\}$ *are linking with* D *in* X, $\varphi : X \to \mathbb{R}$ *is locally Lipschitz,* $\sup\limits_{E_0} < \inf\limits_{D} \varphi$, $\Gamma = \{\gamma \in C(E,X) : \gamma|_{E_0} = id|_{E_0}\}$, $c = \inf\limits_{\gamma\in\Gamma} \sup\limits_{v\in E} \varphi(\gamma(v))$ *and* φ *satisfies the nonsmooth* PS_c*-condition, then* $c \geq \inf\limits_{D} \varphi$ *and* c *is a critical value of* φ.

Remark 2.3. By appropriate choices of the linking sets $\{E_0, E, D\}$, from Theorem 2.2, we obtain nonsmooth versions of the mountain pass theorem, saddle point theorem, and generalized mountain pass theorem. For details, see Gasinksi-Papageorgiou [9].

Given a locally Lipschitz function $\varphi : X \to \mathbb{R}$, we set

$\varphi^{\cdot c} = \big\{x \in X : \varphi(x) < c\big\}$ (the strict sublevel set of φ at $c \in \mathbb{R}$)

and $K_c = \big\{x \in X : 0 \in \partial\varphi(x), \varphi(x) = c\big\}$ (the critical points of φ at the level c).

The next theorem is a nonsmooth version, of the so-called "second deformation theorem" (see Gasinski-Papageorgiou [10], p.628) and it is due to Corvellec [4].

Theorem 2.4. *If* X *is a Banach space,* $\varphi : X \to \mathbb{R}$ *is locally Lipschitz, it satisfies the nonsmooth* PS*-condition,* $a \in \mathbb{R}$, $b \in \mathbb{R} \cup \{+\infty\}$, φ *has no critical points in* $\varphi^{-1}(a,b)$ *and* K_a *is discrete nonempty and contains only local minimizers of* φ, *then there exists a deformation* $h : [0,1] \times \varphi^{\cdot b} \to \varphi^{\cdot b}$ *such that*

(a) $h(t,\cdot)|_{K_a} = Id|_{K_a}$ *for all* $t \in [0,1]$;
(b) $h(t, \varphi^{\cdot b}) \subseteq \varphi^{\cdot a} \cup K_a$;
(c) $\varphi(h(t,x)) \leq \varphi(x)$ *for all* $(t,x) \in [0,1] \times \varphi^{\cdot b}$.

Remark 2.5. In particular then $\varphi^{\cdot b} \cup K_a$ is a weak deformation retract of $\varphi^{\cdot b}$.

Let us mention a few basic things about the spectrum of $(-\triangle_p, W_0^{1,p}(Z))$, which we will need in the sequel. So let $m \in L^\infty(Z)_+$, $m \neq 0$ and consider the following weighted eigenvalue problem:

$$\left\{\begin{array}{l} -\mathrm{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \lambda m(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0, \quad 1 < p < \infty\,. \end{array}\right\} \tag{2.1}$$

Problem (2.1) has at least eigenvalue $\widehat{\lambda}_1(m) > 0$, which is simple, isolated and admits the following variational characterization in terms of the Rayleigh quotient:

$$\widehat{\lambda}_1(m) = \min\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x \in W_0^{1,p}(Z)\,, \quad x \neq 0\right] \tag{2.2}$$

The minimum is attained on the corresponding one dimensional eigenspace $E(\lambda_1)$. By u_1 we denote the normalized eigenfunction, i.e., $\int_Z m|u_1|^p dz = 1$

(if $m \equiv 1$, then $\|u_1\|_p = 1$). We have $E(\lambda_1) = \mathbb{R}u_1$ and $u_1 \in C_0^1(\overline{Z})$ (nonlinear regularity theory, see Lieberman [13] and Gasinski-Papageorgiou [10], p.738). We set

$$C_+ = \left\{x \in C_0^1(\overline{Z}) : x(z) \geq 0 \text{ for all } z \in \overline{Z}\right\}$$
$$\text{and } \operatorname{int}C_+ = \left\{x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z\right\}.$$

The nonlinear strong maximum principle of Vazquez [14], implies that $u_1 \in \operatorname{int}C_+$.

Since $\widehat{\lambda}_1(m)$ is isolated, we can define the second eigenvalue of $(-\triangle_p, W_0^{1,p}(Z), m)$ by

$$\widehat{\lambda}_2^*(m) = \inf\left[\widehat{\lambda} : \widehat{\lambda} \text{ is an eigenvalue of (2.1), } \widehat{\lambda} \neq \widehat{\lambda}_1(m)\right] > \widehat{\lambda}_1(m)\,.$$

Also by virtue of the Liusternik-Schnirelmann theory, we can find an increasing sequence of eigenvalues $\{\widehat{\lambda}_k(m)\}_{k\geq 1}$ such that $\widehat{\lambda}_k(m) \to \infty$. These are the so-called LS-eigenvalues. We have

$$\widehat{\lambda_2}^*(m) = \widehat{\lambda}_2(m)\,,$$

i.e., the second eigenvalue and the second LS-eigenvalue coincide. The eigenvalues $\widehat{\lambda}_1(m)$ and $\widehat{\lambda}_2(m)$ exhibit the following monotonicity properties with respect to the weight function $m \in L^\infty(Z)_+$:

- If $m_1(z) \leq m_2(z)$ a.e. on Z, and $m_1 \neq m_2$, then $\lambda_1(m_2) < \lambda_1(m_1)$ (see (2.2)).
- If $m_1(z) < m_2(z)$ a.e. on Z, then $\lambda_2(m_2) < \lambda_2(m_1)$.

If $m \equiv 1$, then we write $\widehat{\lambda}_1(1) = \lambda_1$ and $\widehat{\lambda}_2(1) = \lambda_2$. Recently Cuesta-de Figueiredo-Gossez [5], produced the following alternative variational characterization of λ_2 :

$$\lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \sup\nolimits_{x \in \gamma_0([-1,1])} \|Dx\|_p^p \tag{2.3}$$

with $\Gamma_0 = \{\gamma_0 \in C([-1,1], S) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1\}$, $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ and $\partial B_1^{L^p(Z)} = \{x \in L^p(Z) : \|x\|_p = 1\}$.

Finally we recall the notions of upper and lower solution for problem (1.1).

Definition 2.6.

(a) *A function $\overline{x} \in W^{1,p}(Z)$ is an upper solution of (1.1), if $\overline{x}|_{\partial Z} \geq 0$ and*

$$\int_Z \|D\overline{x}\|^{p-2}(D\overline{x}, Dv)_{\mathbb{R}^N} dz \geq \int_Z uvdz$$

for all $v \in W_0^{1,p}(Z)$, $v \geq 0$ and all $u \in L^\eta(Z)$, $u(t) \in \partial j(t, \overline{x}(z))$ a.e. on Z for some $1 < \eta < p^*$.

(b) *A function $\underline{x} \in W^{1,p}(Z)$ is a lower solution of (1.1), if $\overline{x}|_{\partial Z} \leq 0$ and*

$$\int_Z \|D\underline{x}\|^{p-2}(D\underline{x}, Dv)_{\mathbb{R}^N} dz \leq \int_Z uvdz$$

for all $v \in W_0^{1,p}(Z)$, $v \geq 0$ *and all* $u \in L^\eta(Z)$, $u(z) \in \partial j(z, \underline{x}(z))$ *a.e. on* Z *for some* $1 < \eta < p^*$.

3. Multiple constant sign solutions

In this section, we produce multiple solutions of constant sign. Our approach is based on variational techniques, coupled with the method of upper lower solutions. We need the following hypotheses on the nonsmooth potential $j(z,x)$.

$\underline{H(j)_1}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t,0) = 0$ and $\partial j(z,0) = \{0\}$ a.e. on Z, and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;

(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;

(iii) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have

$$|u| \leq a(z) + c|x|^{p-1} \text{ with } a \in L^\infty(Z)_+,\ c > 0\,;$$

(iv) there exists $\theta \in L^\infty(Z)_+$, $\theta(z) \leq \lambda_1$ a.e. on Z, $\theta \neq \lambda_1$ such that

$$\limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \leq \theta(z)$$

uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(v) there exists $\eta, \widehat{\eta} \in L^\infty(Z)_+$, $\lambda_1 \leq \eta(z) \leq \widehat{\eta}(z)$ a.e. on Z, $\lambda_1 \neq \eta$ such that

$$\eta(z) \leq \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \widehat{\eta}(z)$$

uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$;

(vi) for a.a. $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z,x)$, we have $ux \geq 0$ (sign condition).

Let $\varepsilon > 0$ and $\gamma_\varepsilon \in L^\infty(Z)_+$, $\gamma_\varepsilon \neq 0$ and consider the following auxiliary problem:

$$\left\{ \begin{array}{l} -\operatorname{div}\big(\|Dx(z)\|^{p-2}Dx(z)\big) = \big(\theta(z)+\varepsilon\big)|x(z)|^{p-2}x(z) + \gamma_\varepsilon(z) \text{ a.e. on } Z\,, \\ x|_{\partial Z} = 0\,. \end{array} \right\} \quad (3.1)$$

In what follows by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z))$ $(\frac{1}{p} + \frac{1}{p'} = 1)$. Let $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$\big\langle A(x), y \big\rangle = \int_Z \|Dx\|^{p-2}(Dx, Dy)_{\mathbb{R}^N} dz \text{ for all } x, y \in W_0^{1,p}(Z)\,.$$

We can check that A is monotone, continuous, hence maximal monotone. In particular then we can deduce that A is pseudomonotone and of type $(S)_+$.

Also let $N_\varepsilon : L^p(Z) \to L^{p'}(Z)$ be the bounded, continuous map defined by

$$N_\varepsilon(x)(\cdot) = \big(\theta(\cdot) + \varepsilon\big)|x(\cdot)|^{p-2}x(\cdot)\,.$$

Evidently due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, we have that $N_\varepsilon|_{W_0^{1,p}(Z)}$ is completely continuous. Hence $x \to A(x) - N_\varepsilon(x)$ is pseudomonotone. Moreover, from the hypothesis on θ (see $H(j)_1(iv)$), we can show that there exists $\xi_0 > 0$ such that

$$\|Dx\|_p^p - \int_Z \theta |x|^p dz \geq \xi_0 \|Dx\|_p^p \quad \text{for all } \ x \in W_0^{1,p}(Z)\,. \tag{3.2}$$

Therefore for $\varepsilon > 0$ small the pseudomonotone operator $x \to A(x) - N_\varepsilon(x)$ is coercive. But a pseudomonotone coercive operator is surjective (see Gasinski-Papageorgiou [10], p.336). Combining this fact with the nonlinear strong maximum principle, we are led to the following existence result concerning problem (3.1).

Proposition 3.1. *If $\theta \in L^\infty(Z)_+$ is as in hypothesis $H(j)_1(iv)$, then for $\varepsilon > 0$ small problem* (3.1) *has a solution $\overline{x} \in intC_+$.*

Because of hypothesis $H(j)_1(iv)$, we deduce easily the following fact:

Proposition 3.2. *If hypotheses $H(j)_1 \to (iv)$ hold and $\varepsilon > 0$ is small, then the solution $\overline{x} \in intC_+$ obtained in Proposition* 3.1 *is a strict upper solution for* (1.1) *(strict means that $\overline{x}$ is an upper solution which is not a solution).*

Clearly $\underline{x} \equiv 0$ is a lower solution for (1.1).

Let $C = [0, \overline{x}] = \{x \in W_0^{1,p}(Z) : 0 \leq x(z) \leq \overline{x}(z) \ \text{ a.e. on } \ Z\}$. We introduce the truncation function $\tau_+ : \mathbb{R} \to \mathbb{R}$ defined by

$$\tau_+(x) = \begin{cases} 0 & \text{if } \ x \leq 0 \\ x & \text{if } \ x > 0 \end{cases}\,.$$

We set $j_1(z,x) = j(z, \tau_+(x))$. This is still a locally Lipschitz integrand. We introduce $\varphi_+ : W_0^{1,p}(Z) \to \mathbb{R}$ defined by

$$\varphi_+(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_+\big(z, x(z)\big)dz \quad \text{for all } x \in W_0^{1,p}(Z)\,.$$

The function φ_+ is Lipschitz continuous on bounded sets, hence locally Lipschitz. Using hypothesis $H(j)_1(iv)$ and (3.2), we can show that φ_+ is coercive. Moreover, due to the compact embedding of $W_0^{1,p}(Z)$ into $L^p(Z)$, φ_+ is weakly lower semicontinuous. Therefore by virtue of Weierstrass theorem, we can find $x_0 \in C$ such that

$$\varphi_+(x_0) = \inf_C \varphi_+\,. \tag{3.3}$$

Hypothesis $H(j)_1(v)$ implies that for $\mu > 0$ small we have $\varphi_+(\mu u_1) < 0 = \varphi_+(0)$. Since $\mu u_1 \in C$, it follows that $x_0 \neq 0$. Moreover, from (3.3) we have

$$0 \leq \big\langle A(x_0), y - x_0 \big\rangle - \int_Z u_0(z)(y - x_0)(z)dz \quad \text{for all } \ y \in C\,, \tag{3.4}$$

with $u_0 \in L^{p'}(Z)$, $u_0(z) \in \partial j_+(z, x_0(z)) = \partial j(z, x_0(z))$ a.e. on Z. For $h \in W_0^{1,p}(Z)$ and $\varepsilon > 0$, we define

$$y(z) = \begin{cases} 0 & \text{if } z \in \{x_0 + \varepsilon h \le 0\} \\ x_0(z) + \varepsilon h(z) & \text{if } z \in \{0 < x_0 + \varepsilon h \le \overline{x}\} \\ \overline{x}(z) & \text{if } z \in \{\overline{x} \le x_0 + \varepsilon h\} \end{cases} .$$

Evidently $y \in C$ and so we can use it as a test function in (3.4). Then we obtain

$$0 \le \langle A(x_0) - u_0, h \rangle . \tag{3.5}$$

Because $h \in W_0^{1,p}(Z)$ was arbitrary, from (3.5) we conclude that

$$A(x_0) = u_0 \quad \Rightarrow x_0 \in W_0^{1,p}(Z) \ \text{ is a solution of (1.1)}. \tag{3.6}$$

Nonlinear regularity theory implies that $x_0 \in C_0^1(\overline{Z})$, while the nonlinear strong maximum principle of Vazquez [14], tell us that $x_0 \in \text{int}C_+$.

Using the comparison principles of Guedda-Veron [11], we can show that

$$\overline{x} - x_0 \in \text{int}C_+ .$$

Therefore x_0 is a local $C_0^1(\overline{Z})$-minimizer of φ, hence x_0 is a local $W_0^{1,p}(Z)$-minimizer of φ (see Gasinski-Papageorgiou [9], pp.655–656 and Kyritsi-Papageorgiou [12]). Therefore we can state the following result:

Proposition 3.3. *If hypotheses $H(j)_1$ hold, then there exists $x_0 \in C$ which is a local minimizer of φ_+ and of φ.*

If instead of (3.1), we consider the following auxiliary problem

$$\left\{ \begin{array}{l} -\text{div}\big(\|Dv(z)\|^{p-2}Dv(z)\big) = \big(\theta(z) + \varepsilon\big)|v(z)|^{p-2}v(z) - \gamma_\varepsilon(z) \text{ a.e. on } Z\,, \\ v|_{\partial Z} = 0\,. \end{array} \right\} \tag{3.7}$$

then we obtain as before a solution $\underline{v} \in -\text{int}C_+$ of (3.7). We can check that this $\underline{v} \in -\text{int}C_+$ is a strict lower solution for problem (1.1). Now we consider the set

$$D = \big\{x \in v \in W_0^{1,p}(Z) : \underline{v}(z) \le v(z) \le 0 \ \text{ a.e. on } \ Z\big\} .$$

We introduce the truncation function $\tau_- : \mathbb{R} \to \mathbb{R}_-$. defined by

$$\tau_-(x) = \begin{cases} x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0 \end{cases} .$$

Then $j_-(z,x) = j(z, \tau(x))$ and $\varphi_-(x) = \frac{1}{p}\|Dx\|_p^p - \int_Z j_-(z, x(z))dz$ for all $x \in W_0^{1,p}(Z)$. We consider the minimization problem $\inf_D \varphi_-$. Reasoning as with φ_+ on C, we obtain:

Proposition 3.4. *If hypotheses $H(j)_1$ hold, then there exists $v_0 \in D$ which is a local minimizer of φ_- and of φ.*

Propositions 3.3 and 3.4, lead to the following multiplicity theorem for solutions of constant sign for problem (1.1).

Theorem 3.5. *If hypotheses $H(j)_1$ hold, then problem* (1.1) *has at least two constant sign smooth solutions $x_0 \in intC_+$ and $v_0 \in -intC_+$.*

Remark 3.6. Since x_0, v_0 are both local minimizers of φ, from the mountain pass theorem, we obtain a third critical point y_0 of φ, distinct from x_0, v_0. However, at this point we can not guarantee that $y_0 \neq 0$, let alone that it is nodal. This will be done in the next section under additional hypotheses.

4. Nodal solutions

In this section we produce a third nontrivial solution for problem (1.1) which is nodal (i.e., sign-changing). Our approach was inspired by the work of Dancer-Du [6]. Roughly speaking the strategy is the following: Continuing the argument employed in Section 3, we produce a smallest positive solution y_+ and a biggest negative solution y_-. In particular $\{y_\pm\}$ is an ordered pair of upper-lower solutions. So, if we form the order interval $[y_-, y_+]$ and we argue as in Section 3, we can show that problem (1.1) has a solution $y_0 \in [y_-, y_+]$ distinct from y_-, y_+. If we can show that $y_0 \neq 0$, then clearly y_0 is a nodal solution of (1.1). To show the nontriviality of y_0, we use Theorem 2.4 and (2.3).

We start implementing the strategy, by proving that the set of upper (resp. lower) solutions for problem (1.1), is downward (resp. upward) directed. The proof relies on the use of the truncation function

$$\xi_\varepsilon(s) = \begin{cases} -\varepsilon & \text{if } s < \varepsilon \\ s & \text{if } s \in [-\varepsilon, \varepsilon] \\ \varepsilon & \text{if } s > \varepsilon \end{cases} .$$

Note that

$$\frac{1}{\varepsilon}\xi_\varepsilon\big((y_1 - y_1)^-(z)\big) \to \chi_{\{y_1<y_2\}}(z) \quad \text{a.e. on } Z \text{ as } \quad \varepsilon \downarrow 0 .$$

So we have the following lemmata

Lemma 4.1. *If $y_1, y_2 \in W^{1,p}(Z)$ are two upper solutions for problem* (1.1) *and $y = \min\{y_1, y_2\} \in W^{1,p}(Z)$, then y is also an upper solution for problem* (1.1).

Lemma 4.2. *If $v_1, v_2 \in W^{1,p}(Z)$ are two lower solutions for problem* (1.1) *and $v = \max\{v_1, v_2\} \in W^{1,p}(Z)$, then v is also a lower solution for problem* (1.1).

In Section 3 we used zero as a lower solution for the "positive" problem and as an upper solution for the "negative" problem. However, this is not good enough for the purpose of generating a smallest positive and a biggest negative solution, as described earlier. For this reason, we strengthen the hypotheses on $j(z, x)$ as follows:

$\underline{H(j)_2}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(t, 0) = 0$ a.e. on Z, $\partial j(z, 0) = \{0\}$ a.e. on Z, hypotheses $H(j)_2(i) \to (iv)$ and (vi) are the same as hypotheses $H(j)_1(i) \to (iv)$ and (vi) and

(iv) there exists $\widehat{\eta} \in L^\infty(Z)_+$, such that

$$\lambda_1 < \liminf_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \limsup_{x\to 0} \frac{u}{|x|^{p-2}x} \leq \widehat{\eta}(z)$$

uniformly for a.a. $z \in Z$ and all $u \in \partial j(z,x)$.

Using this stronger hypothesis near origin, we can find $\mu_0 \in (0,1)$ small such that $\underline{x} = \mu_0 u_1 \in \mathrm{int}C_+$ is a strict lower solution and $\overline{v} = \mu_0(-u_1) \in -\mathrm{int}C_+$ is a strict upper solution for problem (1.1). So we can state the following lemma:

Lemma 4.3. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has a strict lower solution $\underline{x} \in intC_+$ and a strict upper solution $\overline{v} \in -intC_+$.*

We consider the order intervals

$$[\underline{x}, \overline{x}] = \left\{ x \in W_0^{1,p}(Z) : \underline{x}(z) \leq x(z) \leq \overline{x}(z) \ \text{ a.e. on } \ Z \right\}$$

$$\text{and } \ [\underline{v}, \overline{v}] = \left\{ v \in W_0^{1,p}(Z) : \underline{v}(z) \leq v(z) \leq \overline{v}(z) \ \text{ a.e. on } \ Z \right\}.$$

Using Lemmata 4.1 and 4.2 and Zorn's lemma, we prove the following result:

Proposition 4.4. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *admits a smallest solution in the order interval $[\underline{x}, \overline{x}]$ and a biggest solution in the order interval $[\underline{v}, \overline{v}]$.*

Now let $\underline{x}_n = \varepsilon_n u_1$ with $\varepsilon_n \downarrow 0$ and let $E_+^n = [\underline{x}_n, \overline{x}]$. Proposition 4.4 implies that problem (1.1) has a smallest solution x_*^n in E_+^n. Clearly $\{x_*^n\}_{n\geq 1} \subseteq W_0^{1,p}(Z)$ is bounded and so by passing to a suitable subsequence if necessary, we may assume that

$$x_*^n \xrightarrow{w} y_+ \ \text{ in } \ W_0^{1,p}(Z) \quad \text{and} \quad x_*^n \to y_+ \ \text{ in } \ L^p(Z) \ \text{ as } \ n \to \infty.$$

Arguing by contradiction and using hypothesis $H(j)_2(v)$, we can show that $y_+ \neq 0$ and of course $y_+ \geq 0$. Here we use the strict monotonicity of the principal eigenvalue on the weight function (see Section 2). Moreover, by Vazquez [14], we have $y_+ \in \mathrm{int}C_+$ and using this fact it is not difficult to check that y_+ is in fact the smallest positive solution of problem (1.1).

Similarly, working on $E_-^n = [\underline{v}, \overline{v}_n]$ with $\overline{v}_n = \varepsilon_n(-u_1)$, $\varepsilon_n \downarrow 0$, we obtain $y_- \in -\mathrm{int}C_+$ the biggest negative solution of (1.1). So we can state the following proposition:

Proposition 4.5. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has a smallest positive solution $y_+ \in intC_+$ and a biggest negative solution $y_- \in -intC_+$.*

According to the scheme outlined in the beginning of the section, using this proposition, we can establish the existence of a nodal solution. As we already mentioned, a basic tool to this end, is equation (2.3). But in order to be able to use (2.3), we need to strengthen further our hypothesis near the origin. Also we need to restrict the kind of locally Lispchitz functions $j(z,x)$, we have. Namely, let $f : Z \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that for every $r > 0$ there exists $a_r \in L^\infty(Z)_+$ such that

$$|f(z,x)| \leq a_r(z) \ \text{ for a.a. } \ z \in Z \quad \text{and all } \ |x| \leq r.$$

We introduce the following two limit functions:

$$f_1(z,x) = \liminf_{x'\to x} f(z,x') \quad \text{and} \quad f_2(z,x) = \limsup_{x'\to x} f(z,x')\,.$$

Both functions are $\mathbb{R}$-valued for a.a. $z \in Z$. In addition we assume that they are sup-measurable, meaning that for every $x : Z \to \mathbb{R}$ measurable function, the functions $z \to f_1(z, x(z))$ and $z \to f_2(z, x(z))$ are both measurable. We set

$$j(z,x) = \int_0^x f(z,s)ds\,. \tag{4.1}$$

Evidently $(z,x) \to j(z,x)$ is jointly measurable and for a.a. $z \in Z$, $x \to j(z,x)$ is locally Lipschitz. We have

$$\partial j(z,x) = \big[f_1(z,x), f_2(z,x)\big] \quad \text{for a.a. } z \in Z\,, \quad \text{for all } x \in \mathbb{R}\,.$$

Clearly $j(z,0) = 0$ a.e. on Z and if for a.a. $z \in Z$, $f(z,\cdot)$ is continuous at 0, then $\partial j(z,0) = \{0\}$ for a.a. $z \in Z$. The hypotheses on this particular nonsmooth potential function $j(z,x)$ are the following:

$\underline{H(j)_3}$: $j : Z \times \mathbb{R} \to \mathbb{R}$ is defined by (4.1) and

(i) $(z,x) \to f(z,x)$ is measurable with f_1, f_2 sup-measurable;

(ii) for a.a. $z \in Z$, $x \to f(z,x)$ is continuous at $x = 0$;

(iii) $|f(z,x)| \le a(z) + c|x|^{p-1}$ a.e. on Z, for all $x \in \mathbb{R}$,with $a \in L^\infty(Z)_+$, $c > 0$;

(iv) there exists $\theta \in L^\infty(Z)_+$ satisfying $\theta(z) \le \lambda_1$ a.e. on Z, $\theta \ne \lambda_1$ and

$$\limsup_{|x|\to\infty} \frac{f_2(z,x)}{|x|^{p-2}x} \le \theta(z)$$

uniformly for a.a. $z \in Z$;

(v) there exists $\widehat{\eta} \in L^\infty(Z)_+$ such that

$$\lambda_2 < \liminf_{x\to 0} \frac{f_1(z,x)}{|x|^{p-2}x} \limsup_{x\to 0} \frac{f_2(z,x)}{|x|^{p-2}x} \le \widehat{\eta}(z)$$

uniformly for a.a. $z \in Z$;

(vi) for a.a. $z \in Z$ and all $x \in \mathbb{R}$, we have $f_1(z,x)x \ge 0$ (sign condition).

From Proposition 4.5, we have a smallest positive solution $y_+ \in \mathrm{int}C_+$ and a biggest negative solution $y_- \in -\mathrm{int}C_+$ for problem (1.1). We have

$$A(y_\pm) = u_\pm \;\text{ with }\; u_\pm \in L^{p'}(Z),\; u_\pm(z) \in \partial j\big(z, x_\pm(z)\big) \quad \text{a.e. on } Z\,.$$

We introduce the following truncations of the functions $f(z,x)$:

$$\widehat{f}_+(z,x)=\begin{cases}0 & \text{if } x<0\\ f(z,x) & \text{if } 0\le x\le y_+(z)\\ u_+(z) & \text{if } y_+(z)<x\end{cases},$$

$$\widehat{f}_-(z,x)=\begin{cases}u_-(z) & \text{if } x<y_-(z)\\ f(z,x) & \text{if } y_-(z)\le x\le 0\\ 0 & \text{if } 0<x\end{cases},$$

$$\widehat{f}(z,x)=\begin{cases}u_-(z) & \text{if } x<y_-(z)\\ f(z,x) & \text{if } y_-(z)\le x\le y_+(z)\\ u_+(z) & \text{if } y_+(z)<x\end{cases},$$

Using them, we define the corresponding locally Lipschitz potential functions, namely $\widehat{j}_+(z,x)=\int_0^x \widehat{f}_+(z,s)ds$, $\widehat{j}_-(z,x)=\int_0^x \widehat{f}_-(z,s)ds$ and $\widehat{j}(z,x)=\int_0^x \widehat{f}(z,s)ds$ for all $(z,x)\in Z\times\mathbb{R}$.

Also, we introduce the corresponding locally Lipschitz Euler functionals defined on $W_0^{1,p}(Z)$. So we have

$$\widehat{\varphi}_+(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z \widehat{j}_+\big(z,x(z)\big)dz,\ \widehat{\varphi}_-(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z \widehat{j}_-\big(z,x(z)\big)dz$$

$$\text{and }\ \widehat{\varphi}(x)=\frac{1}{p}\|Dx\|_p^p-\int_Z \widehat{j}\big(z,x(z)\big)dz\ \text{ for all } x\in W_0^{1,p}(Z)\,.$$

Finally, we set

$$T_+=[0,y_+],\ T_-=[y_-,0] \text{ and } T=[y_-,y_+]\,.$$

We can show that the critical points of φ_+ (resp. of φ_-,φ) are in T_+ (resp. in T_-,T). So the critical points of $\widehat{\varphi}_+$ (resp. $\widehat{\varphi}_-$) are $\{0,y_+\}$ (resp. $\{0,y_-\}$). Moreover,

$$\widehat{\varphi}_+(y_+)=\inf\widehat{\varphi}_+<0=\widehat{\varphi}_+(0)\quad\text{and}\quad\widehat{\varphi}_-(y_-)=\inf\widehat{\varphi}_-<0=\widehat{\varphi}_-(0)\,.$$

Clearly y_+,y_- are local $C_0^1(\overline{Z})$-minimizers of $\widehat{\varphi}$ and so they are also local $W_0^{1,p}(Z)$-minimizers. Without any loss of generality, we may assume that they are isolated critical points of $\widehat{\varphi}$. So we can find $\delta>0$ small such that

$$\widehat{\varphi}(y_-)<\inf\big[\widehat{\varphi}(x):x\in\partial B_\delta(y_-)\big]\le 0\,,$$

$$\widehat{\varphi}(y_+)<\inf\big[\widehat{\varphi}(x):x\in\partial B_\delta(y_+)\big]\le 0\,,$$

where $\partial B_\delta(y_\pm)=\{x\in W_0^{1,p}(Z):\|x-y_\pm\|=\delta\}$. Assume without loss of generality that $\widehat{\varphi}(y_-)\le\widehat{\varphi}(y_+)$.

If we set $S=\partial B_\delta(y_+)$, $T_0=\{y_-,y_+\}$ and $T=[y_-,y_+]$, then we can check that the pair $\{T_0,T\}$ is linking with S in $W_0^{1,p}(Z)$. So by virtue of Theorem 2.2, we can find $y_0\in W_0^{1,p}(Z)$ a critical point of $\widehat{\varphi}$ such that

$$\widehat{\varphi}(y_\pm)<\widehat{\varphi}(y_0)=\inf_{\overline{\gamma}\in\Gamma}\max_{t\in[-1,1]}\widehat{\varphi}\big(\gamma(t)\big) \tag{4.2}$$

where $\overline{\Gamma}=\{\overline{\gamma}\in C([-1,1],W_0^{1,p}(Z)):\overline{\gamma}(-1)=y_-,\overline{\gamma}(1)=y_+\}$. Note that from (4.2) we infer that $y_0\ne y_\pm$.

We will show that $\widehat{\varphi}(y_0) < \widehat{\varphi}(0) = 0$ and so $y_0 \neq 0$. Hence y_0 is the desired nodal solution. To establish the nontriviality of y_0, it suffices to construct a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\widehat{\varphi}(\gamma_0(t)) < 0 \quad \text{for all } \ t \in [0,1] \ \ (\text{see (4.2)}) .$$

Using (2.3), we can produce a continuous path γ_0 joining$-\varepsilon u_1$ and εu_1 for $\varepsilon > 0$ small. Note that if $S_c = C_0^1(\overline{Z}) \cap \partial B_1^{L^p(Z)}$ and $S = W_0^{1,p}(Z) \cap \partial B_1^{L^p(Z)}$ are equipped with the relative $C_0^1(\overline{Z})$ and $W_0^{1,p}(Z)$ topologies respectively, then

$$C\big([-1,1], S_c\big) \quad \text{is dense in } \ C\big([-1,1], S\big) .$$

Also we have

$$\widehat{\varphi}|_{\gamma_0} < 0 . \tag{4.3}$$

Using Theorem 2.4, we can generate the continuous path

$$\gamma_+(t) = h(t, \varepsilon u_1) , \quad t \in [0,1] ,$$

with $h(t,x)$ the deformation of Theorem 2.4. This path joins εu_1 and y_+. Moreover, we have

$$\widehat{\varphi}|_{\gamma_+} < 0 . \tag{4.4}$$

In a similar fashion we produce a continuous path γ_- joining y_- with $-\varepsilon u_1$ such that

$$\widehat{\varphi}|_{\gamma_-} < 0 . \tag{4.5}$$

Concatinating γ_-, γ_0 and γ_+, we produce a path $\overline{\gamma}_0 \in \overline{\Gamma}$ such that

$$\widehat{\varphi}|_{\overline{\gamma}_0} < 0 \ \ (\text{see (4.3),(4.4) and (4.5)}) .$$

This proves that $y_0 \neq 0$ and so y_0 is a nodal solution. Nonlinear regularity theory implies that $y_0 \in C_0^1(\overline{Z})$.

Therefore we can state the following theorem on the existence of nodal solutions

Theorem 4.6. *If hypotheses $H(j)_3$ hold, then problem* (1.1) *has a nodal solution $y_0 \in C_0^1(\overline{Z})$.*

Combining Theorems 3.5 and 4.6, we can state the following multiplicity result for problem (1.1).

Theorem 4.7. *If hypotheses $H(j)_3$ hold, then problem* (1.1) *has at least three nontrivial solutions, one positive $x_0 \in intC_+$, one negative $v_0 \in -intC_+$ and the third $y_0 \in C_0^1(\overline{Z})$ nodal.*

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A Young Measures Approach to Averaging

Zvi Artstein

Dedicated to Arrigo Cellina and James Yorke

Abstract. Employing a fast time scale in the Averaging Method results in a limit dynamics driven by a Young measure. The rate of convergence to the limit induces quantitative estimates for the averaging. Advantages that can be drawn from the Young measures approach, in particular, allowing time-varying averages, are displayed along with a connection to singularly perturbed systems.

1. Introduction

The Averaging Method suggests that a time-varying yet small perturbation on a long time interval, can be approximated by a time-invariant perturbation obtained by "averaging" the original one. The method has been introduced in the 19th Century as a practical device helping computations of stellar motions. Its rigorous grounds have been affirmed in the middle of the 20th Century. Many applications, including to fields beyond computations, make the field very attractive today. For a historical account and many applications consult Lochak and Meunier [10], Sanders and Verhulst [14], Verhulst [17], and references therein.

In this paper we make a connection between the averaging method and another useful tool, namely, probability measure-valued maps, called Young measures. These were introduced by L.C. Young as generalized curves in the calculus of variations; other usages are as relaxed controls, worked out by J. Warga, limits of solutions of partial differential equations, and many more. For an account of some of the possible applications consult the monographs and surveys Young [19], Warga [18], Valadier, [16], Pedregal [12, 13], Balder [6], and references therein. For a connection to singular perturbations extending, in particular, the Levinson-Tikhonov scope, see Artstein [3].

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The qualitative consequences of the averaging method played a role in all the aforementioned applications of Young measures. The purpose of this note is to show that the Young measures approach can contribute to the considerations of averaging, including to the quantitative estimates the theory offers.

In the next section we explain how Young measures arise in the averaging considerations. A general estimate based on the distance in the sense of Young measures is displayed in Section 3. Applications to the classical averaging, along with some examples, are given in Section 4. Averaging considerations relative to subsequences, resulting, in particular, in time-varying averages, is a feature Young measures help to clarify; it is displayed in Section 5 along with a comment on the connection to singularly perturbed systems.

2. The connection

In this section we provide the basic definitions of Young measures and explain how they arise in considerations of averaging. We start actually with the latter, namely, provide the motivation first.

Averaging of ordinary differential equations is concerned with an equation which depends on a small positive parameter ε and given by

$$\frac{dx}{dt} = \varepsilon f(t, x, \varepsilon)\,, \quad x(0) = x_0\,. \tag{2.1}$$

We assume, throughout, continuity of $f(t, x, \varepsilon)$ in x and measurability in t (continuity in ε is not needed in general; it is explicitly assumed below when used). In many applications one has to carry out a change of variables in order to arrive to the form (2.1); in fact, the form (2.1) already depicts the small perturbation; see Verhulst [17] for an elaborate discussion. Of interest is the limit behaviour of solutions of (2.1) as $\varepsilon \to 0$. A typical result assures, under appropriate conditions, that the solution, say $x(\cdot)$, of (2.1) (it depends on ε) is close to the solution, say $x_0(\cdot)$, of the averaged equation, namely, the equation

$$\frac{dx}{dt} = \varepsilon f^0(x)\,, \quad x(0) = x_0\,; \tag{2.2}$$

here the time-invariant right hand side of (2.2) is the limit average of the original equation, namely,

$$f^0(x) = \lim_{T\to\infty,\ \varepsilon\to 0} \frac{1}{T}\int_0^T f(t, x, \varepsilon)dt\,, \tag{2.3}$$

assuming, of course, that the limit exists. (The order of convergence between T and ε in (2.3) may play a role; we do not address this issue in this general discussion.) Furthermore, the theory assures that the two solutions, $x(\cdot)$ and $x_0(\cdot)$, are uniformly close on an interval of length of order ε^{-1}, say uniformly on $[0, \varepsilon^{-1}]$. Estimating the order of approximation is a prime goal of the theory. Discussions and examples can be found in Arnold [1], Bogoliubov and Mitropolsky [8], Guckenheimer and Holmes [9], Lochak and Meunier [10], Sanders and Verhulst [14], Verhulst [17]. We provide some concrete examples later on.

A standard approach to verifying the approximation and establishing the order of approximation is via differential or integral inequalities, e.g., Gronwall inequalities, carefully executed so to produce the appropriate estimates. We offer another approach which starts with a change of time scales, namely, $s = \varepsilon t$. In the "fast" time variable s equations (2.1) and (2.2) take the form

$$\frac{dx}{ds} = f(s/\varepsilon, x, \varepsilon),\ \ x(0) = x_0 \tag{2.4}$$

and, respectively,

$$\frac{dx}{ds} = f^0(x),\ \ x(0) = x_0\,. \tag{2.5}$$

Verifying an approximation estimate for solutions of (2.1) and (2.2) uniformly on $[0, \varepsilon^{-1}]$ amounts to verifying the same estimate for solutions of (2.4) and (2.5) uniformly on $[0, 1]$.

When attempting to apply limit considerations to the right hand side of (2.4) a difficulty arises, namely, to determine the limit, as $\varepsilon \to 0$, of the function $f(\frac{s}{\varepsilon}, x, \varepsilon)$, as a function of s for a fixed x. Indeed, the point-wise limit may not exist, while weak limits, although resulting in the desired average, are not easy to manipulate when quantitative estimates are sought. What we suggest is to employ the *Young measures limit*, as follows.

The best way to explain the idea is via a concrete example. Suppose that the right hand side in (2.4) is the function $\sin(\frac{s}{\varepsilon})$. As $\varepsilon \to 0$ the function oscillates more and more rapidly. What the Young measure limit captures is the distribution of the values of the function. Indeed, on any fixed small interval, say $[s_1, s_2]$ in $[0, 1]$, when ε is small the values of $\sin(\frac{s}{\varepsilon})$ are distributed very closely to the distribution of the values of the sin function over one period; namely, the distribution is $\mu_0(d\xi) = \pi^{-1}(1 - \xi^2)^{-\frac{1}{2}} d\xi$ which is a probability measure over the space of values of the mapping $\sin(\cdot)$. A way to depict the limit is to identify it with the probability measure-valued map, say $\mu(\cdot)(d\xi)$ which assigns to each $s \in [0, 1]$ the probability distribution $\mu_0(d\xi)$ just defined. In the example, the same probability distribution is assigned to all s in the interval. The general definition of a Young measure allows probability measure-valued maps which may not be constant over the interval. Later we take advantage of this possibility when allowing time-varying averages.

A probability measure on R^n is a σ-additive mapping, say μ, from the Borel subsets of R^n into $[0, 1]$ such that $\mu(R^n) = 1$. The space of probability measures is endowed with the weak convergence of measures, namely, μ_i converge to μ_0 if $\int h(\xi)\mu_i(d\xi)$ converge to $\int h(\xi)\mu_0(d\xi)$ for every bounded and continuous mapping $h(\cdot) : R^n \to R$. Here ξ is an element of R^n. The space of probability measures on R^n is denoted $\mathcal{P}(R^n)$. In the next section we recall the Prohorov metric; it makes the space $\mathcal{P}(R^n)$ with the weak convergence of measures a complete metric space. On this space see Billingsley [7].

A measurable mapping $\mu(\cdot) : [0,1] \to \mathcal{P}(R^n)$ is called a Young measure, the measurability being with respect to the weak convergence. A Young measure, say $\mu(\cdot)$, is associated with a measure, marked in this paper in bold face font, say $\boldsymbol{\mu}$, on $[0,1] \times R^n$ defined on rectangles $E \times B$ by $\boldsymbol{\mu}(E \times B) = \int_E \mu(s)(B)ds$. The resulting measure is also a probability measure (since the base space has Lebesgue measure one, otherwise we get a probability measure multiplied by the Lebesgue measure of that base). The convergence in the space of Young measures is now derived from the convergence on $\mathcal{P}([0,1] \times R^n)$, and likewise the Prohorov metric. A useful consequence is that the space of Young measures with values supported on a compact subset of R^n is a compact set in the space of Young measures.

An R^n-valued function, say $f(s)$, is identified with the Young measure whose values are the Dirac measures supported on the singletons $\{f(s)\}$. The convergence of functions in the sense of Young measures, say of $\{f_i(\cdot)\}$, is taken to be the convergence of the associated Young measures. More on the basic theory of Young measures and their convergence see Balder [6], Valadier [16].

The application of the Young measure convergence to the averaging problem is via the convergence, as $\varepsilon \to 0$, in the sense of Young measures of the functions $f(\frac{s}{\varepsilon}, x, \varepsilon)$. We shall also consider convergence of $f(\frac{s}{\varepsilon_j}, x, \varepsilon_j)$ for a subsequence $\varepsilon_j \to 0$. The limit in the general case is a Young measure, say $\mu_0(s,x)(d\xi)$ (here x is the parameter carried over from the function $f(\frac{s}{\varepsilon}, x, \varepsilon)$, and $d\xi$ is an infinitesimal element in R^n). The resulting limit differential equation is defined by

$$\frac{dx}{ds} = E\big(\mu_0(s,x)(d\xi)\big)\,, \quad x(0) = x_0\,, \tag{2.6}$$

where $E(\mu_0(s,x)(d\xi))$ is the expectation with respect to ξ of the measure, namely, it is equal to $\int_{R^n} \xi \mu_0(s,x)(d\xi)$. Thus, the differential equation (2.6) is an ordinary differential equation whose right hand side is determined via an average of values. When the measure $\mu_0(s,x)(d\xi)$ is a Dirac measure, namely a function, the equation reduces to the form in (2.4). We abuse rigorous terminology and refer to $\mu_0(s,x)$ as the right hand side of the differential equation (2.6). It is easy to see that when the convergence holds when $\varepsilon \to 0$ (rather than for a subsequence $\varepsilon_j \to 0$), the limit Young measure is constant-valued, see Remark 5.3.

It should be pointed out that, throughout the derivations, it is the expectation of the Young measure which plays a role, and not the Young measure itself. Considering the entire Young measure does not, however, restrict the scope of the applications and, in turn, helps in the analysis.

It has been known for a long time that, under appropriate conditions, if the right hand side, say $f_i(s,x)$, of a differential equation converges in the sense of Young measures to, say, $\mu_0(s,x)$, then the corresponding solutions converge uniformly on bounded intervals. This may be considered a qualitative aspect of averaging. It was exploited in many frameworks. One such application is to relaxed controls, see Warga [18], Young [19]. Applications more related to the averaging principle were to systems with oscillating parameter and to singularly perturbed systems, see Artstein [2, 3], Artstein and Vigodner [5]. In the present paper the