Numerical Modeling in Open Channel Hydraulics

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Preface

Open channel hydraulics has always been a very interesting domain of scientific and engineering activity because of the great importance of water for human living. The free surface flow, which takes place in the oceans, seas and rivers, can be still regarded as one of the most complex physical processes in the environment. The first source of difficulties is the proper recognition of physical flow processes and their mathematical description. The second one is related to the solution of the derived equations. The equations arising in hydrodynamics are rather complicated and, except some much idealized cases, their solution requires application of the numerical methods. For this reason the great progress in open channel flow modeling that took place during last 40 years paralleled the progress in computer technique, informatics and numerical methods. It is well known that even typical hydraulic engineering problems need applications of computer codes. Thus, we witness a rapid development of ready-made packages, which are widely disseminated and offered for engineers. However, it seems necessary for their users to be familiar with some fundamentals of numerical methods and computational techniques applied for solving the problems of interest. This is helpful for many reasons. The ready-made packages can be effectively and safely applied on condition that the users know their possibilities and limitations. For instance, such knowledge is indispensable to distinguish in the obtained solutions the effects coming from the considered physical processes and those caused by numerical artifacts. This is particularly important in the case of hyperbolic equations, like the Saint-Venant equations or the advection equation.

In principle, numerical open channel hydraulics can be regarded as a sub-domain of Computational Fluid Dynamics (CFD) and the general methods and experiences of CFD are applicable in open channel flow modeling. Moreover, the open channel flow can be often treated as one-dimensional, which makes it relatively easy to solve compared to multidimensional flows considered in geophysics and industrial engineering. On the other hand, due to a range of specific issues, numerical open channel hydraulics developed into a branch of its own as early as in the years 1960s and 1970s. There exist a number of very good books on the subject, written at that time and later. A non-exhaustive list includes “Unsteady flow in open channel” edited by K. Mahmood and V. Yevjevich and containing the papers written by recognized experts, “Practical aspects of computational river hydraulics” by J.A. Cunge, F.M.

Holly and A. Verwey, “Computational hydraulics-Elements of the theory of free surface flow” by M.B. Abbott, “Dynamic hydrology” by P.S. Eagleson, “Open channel hydraulics” by V.T. Chow, “Open channel flow” by F.M. Henderson, “Kinematic wave modeling in water resources: surface water hydrology” by V.P. Singh, “The hydraulics of open channel flow: An introduction” by H. Chanson. These books cover most of the theoretical and practical issues related to open channel flow modeling and can be recommended for any engineer working in this field.

As far as the computational techniques are considered, one can recommend the following books: “Incompressible flow and the finite-element method” by P.M. Gresho and R.L. Sani, “Computational fluid dynamics” by M.B. Abbott and D.R. Basco, “Numerical head transfer and fluid flow” by S.V. Patankar, “Computational physics” by D. Potter, “Computational techniques for fluid dynamics” by C.A.J. Fletcher, “Finite volume methods for hyperbolic problems” by R.J. LeVeque, “The finite element method in engineering science” by O. C. Zienkiewicz. These books covering large area of the fluid dynamics and other engineering sciences are useful for open channel flow modeling as well.

In view of the continuous advance in numerical techniques, the present book is an attempt to complement the existing works with a more detailed and up-to-date discussion of selected numerical aspects of open channel hydraulics. It is largely based on author’s own research and focuses on one-dimensional models of steady and unsteady flow and transport in open channels and their networks.

The book is organized in nine chapters. Chapter 1 presents the background information on the open channel hydraulics and the derivation of the governing equations for both steady and unsteady flow, as well as for the transport of the constituents dissolved in the flowing water including the transport of thermal energy.

The next two chapters cover the basic numerical methods applicable for solving nonlinear equations and systems of linear and nonlinear equations (Chapter 2) and ordinary differential equations and their systems (Chapter 3). Implementation of the presented methods for solution the steady gradually varied flow in a single channel and in channel network is given in Chapter 4. These methods are also the basic building blocks for more complex numerical algorithms described in the following chapters.

Chapter 5 is an introduction to the partial differential equations of hyperbolic and parabolic types, frequently occurring in open channel hydraulic. It covers the classification of equations, formulation of solution problem, and introduction to the finite difference and element methods. This chapter ends by discussion of convergence, consistency and stability of the numerical methods.

In Chapters 6 and 7 the advection and advection-diffusion equations are considered. Basing on the solution using the finite difference box scheme, the main problems of numerical integration of the hyperbolic equations are discussed. The modified equation approach for the accuracy analysis of the numerical solution of hyperbolic equations is described. Besides the standard methods of solution, the splitting technique for the advection-diffusion transport equation is presented.

Chapter 8 is entirely devoted to the unsteady flow. Solution of the system of Saint Venant equations using both finite difference and element methods is described.

Solution of unsteady flow with moveable bed and the problem of propagation of steep waves are briefly described.

Chapter 9 covers the simplified models and theirs application for flood routing. Particular attention is focused on the close relation between the spatially lumped models and the discrete forms of distributed models. The conservative properties of the non-linear and linear simplified models are discussed.

The book includes numerous computational examples and step-by-step descriptions of numerical algorithms. It is the hope of the author that the reader will find it useful and easy to follow.

I am grateful to Springer and to the members of the Editorial Advisory Board of series Water Science and Technology Library for the possibility of publishing my work. My thanks are to Prof. Witold Strupczewski, for initiating the idea of this book, to the Editor-in-Chief Prof. Vijay. P. Singh for his valuable suggestions on its contents, and to Ms. Petra van Steenbergen and Ms. Cynthia de Jonge for their kind assistance in submitting the manuscript. I would also like to acknowledge the support received from Prof. Ireneusz Kreja, Dean of the Faculty of Civil and Environmental Engineering of the Gdańsk University of Technology. I owe a lot to the persons who assisted me in the work on the manuscript: Dr. Dariusz Gąsiorowski, who prepared many numerical examples, Ms. Katarzyna Olszonowicz, who prepared all the figures and my son Dr. Adam Szymkiewicz, whose remarks and suggestions helped to improve the text. Finally, I highly appreciate the effort of all members of staff involved at all stages of the editorial process of my book.

Gdansk, Poland Romuald Szymkiewicz

Contents

Chapter 1
Open Channel Flow Equations

1.1 Basic Definitions

Open channel is a conduit in which a part of the cross-section of the flowing stream, taken perpendicularly to the flow velocity vector, is exposed atmosphere. Open channels can be divided into natural and artificial ones. Natural channels, like rivers or creeks, result from the geophysical processes acting at the Earth surface, without essential participation of human activity. Conversely, artificial channels are built by men and comprise, among others, navigable, energetic and irrigation channels.

Another possible division of open channels is between prismatic and non-prismatic ones. A prismatic channel has constant cross-sectional shape and constant longitudinal bottom slope. Of course, such properties can be possessed only by artificial channels. All natural channels are non-prismatic. A schematic representation of a non-prismatic channel is shown in Fig. 1.1.

In natural channels the sections can be either compacted (Fig. 1.1a) or composite (Fig. 1.1b). In composite cross-section distinct parts exist (1, 2 and 3 in Fig. 1.1b), characterized by different bed elevations and possibly different bed roughness. This case is representative, for instance, of a river flowing over flooded terrace and calculation of the flow parameters requires appropriate treatment. Figure 1.1a also shows some basic variables used in open channel hydraulics. They are listed below.

Water stage, denoted by h, is the elevation of the free surface of water at a specific cross-section of the channel, measured with respect to the assumed datum. The elevation of the bottom measured with respect to the same datum is denoted by Z.

The depth of flow H is the vertical distance between the water surface and the channel bottom. In natural channels the depth and bed elevation have local meaning only, since they vary along the axis x, which is parallel to the flow direction and y axis, which is horizontal and normal to x. Conversely, in prismatic channels both Z and H are uniquely defined for each cross-section and depend on x only. Thus, the following relation holds:

$$h(x) = Z(x) + H(x). \tag{1.1}$$

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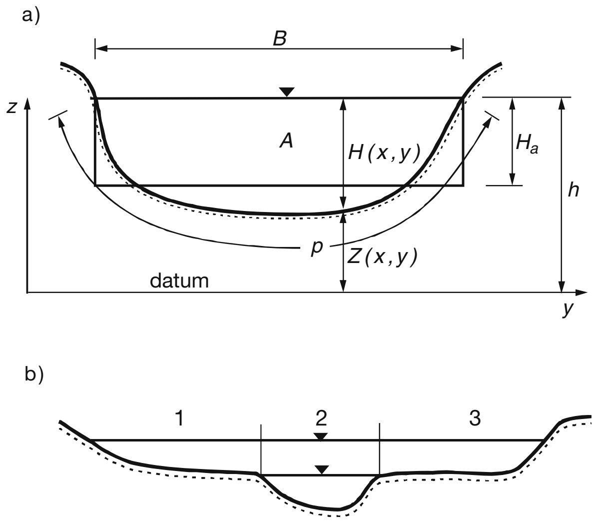

Fig. 1.1 Possible shapes of cross-sections of the non-prismatic channel (**a**) compacted, (**b**) composite

The *longitudinal bed slope* is defined as:

$$s = -\frac{\partial Z}{\partial x}, \tag{1.2}$$

Its value is usually small, which allows for considerable simplifications of the flow equations. In Fig. 1.2 a longitudinal profile of channel is shown.

The slope is uniquely defined for prismatic channels only. For natural channels the bed slope has rather local meaning, however its average value along channel axis taken over longer distances of practical interest is rough but important information.

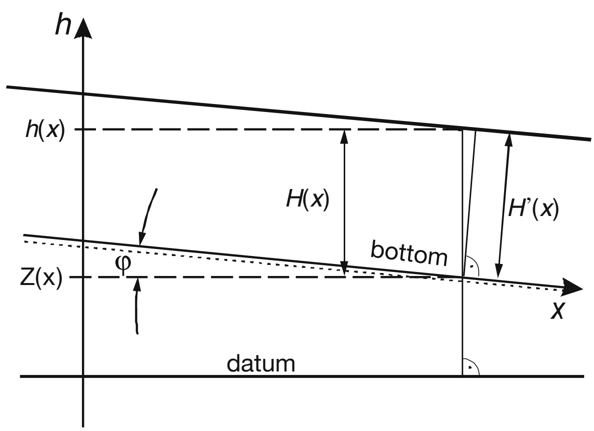

Fig. 1.2 Definition of the cross-section parameters for prismatic channel

The value of $H(x)$ should be distinguished from so-called depth of flow of section $H'(x)$, which is measured perpendicularly to the bottom (Fig. 1.2). Since both depths are related to each other by means of the bed slope $H'(x) = H(x)\cos(\varphi)$, then for small values of the longitudinal bed slope, when $\cos(\varphi) \approx 1$, one can assume that $\mathrm{H}(\mathrm{x}) \approx H'(x)$. Only in particular situations, for steep channels, this difference is appreciable (Akan 2006, French 1985). Consequently, usually it is reasonable to assume that the flow depth is measured vertically from bottom to the water surface, whereas the distance along channel axis is considered as horizontal one. The problem of co-ordinates system is summarized by Liggett (1975) as follows: "For open channel flow the system of co-ordinates is not entirely orthogonal in that x lies in the bed of the channel and z is vertical. This arrangement assumes that the cosine of the channel slope is approximately unity."

The cross-section can be characterized by the following geometric parameters:

- *Top width B* is the width of the channel at the level of water surface (Fig. 1.1).
- *Flow area A* is the wetted cross-sectional area measured perpendicularly to the vector of channel flow velocity.
- *Wetted perimeter P* is the length of the interface between the water and the channel bed.
- *Hydraulic radius R* is the ratio of the wetted flow area and the wetted perimeter:

$$R = \frac{A}{P}. \tag{1.3}$$

  This parameter is meaningful only for compact cross-sections as shown in Fig. 1.1a.
- *Hydraulic depth* H_a is the ratio of the wetted flow area A and the top width B:

$$H_a = \frac{A}{B}. \tag{1.4}$$

  H_a and B represent the dimensions of a rectangle of area A equivalent to the natural cross-section (Fig. 1.1a).

The cross-sections of prismatic channels are typically triangular, rectangular or trapezoidal so the parameters listed above can be expressed as analytical functions of the depth H. Consider for example a trapezoidal channel (Fig. 1.3) characterized by the channel width at the level of bottom b, and the side slope m, which is the cotangent of the angle ψ between the side and horizontal plane.

For such type of channel the cross-section parameters A, P and B are given by the following formulas:

$$A = b \cdot H + m \cdot H^2 \tag{1.5}$$

$$P = b + 2H\sqrt{1 + m^2} \tag{1.6}$$

$$B = b + 2m \cdot H \tag{1.7}$$

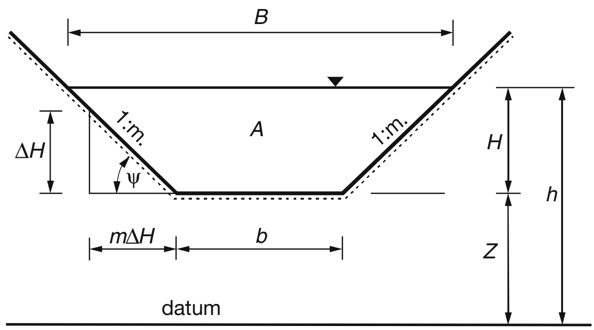

Fig. 1.3 Trapezoidal channel cross-section

For $m = 0$ the trapezoidal cross-section becomes rectangular, whereas for $b = 0$ it becomes triangular. For natural channels, the geometric parameters are usually expressed as function of the water stage, presented in tabularized form.

The basic quantities characterizing liquid flow are: pressure, density, velocity and acceleration. The flow is described by the scalar or vector fields of these variable parameters. In general all those variables depend on three spatial coordinates and time. However, in open channel flow the component of the velocity vector parallel to the channel axis is typically dominating and in addition it has relatively uniform distribution over the cross-section area. Thus, it is assumed that the flow parameters are functions of two variables only: x and t, which represent the spatial coordinate related to the channel axis and time.

The flowing stream is conveniently characterized by the following averaged quantities:

- *Discharge*, which represents the mass or volume of water flowing through considered cross-section per unit time. The volume discharge is defined as:

$$Q = \iint_A u \cdot dA \tag{1.8}$$

where:

u – normal velocity to the cross-section,
A – wetted cross-sectional area,
Q – flow discharge.

- *Average flow velocity* in the cross-section, which ensures the same discharge as actual velocity distribution over cross-section. It is defined as follows:

$$U = \frac{Q}{A} = \frac{1}{A} \iint_A u \cdot dA \tag{1.9}$$

where: U – average velocity in a cross-section.

- *Volume flux per unit time and unit width* of the vertical cross-section measured between the water surface and the bottom, given by the formula:

$$q = \int_{Z}^{h} u \cdot dz = U \cdot H \tag{1.10}$$

where q is flow discharge related to the width unit of a channel. For channel having a rectangular cross-section q is equal:

$$q = \frac{Q}{B} \tag{1.11}$$

The total energy of water particle traveling along a streamline (a line constructed in the velocity field, so that at its every point the velocity vector is tangential to this line) is given by the Bernoulli equation:

$$E_{sl} = z_{sl} + \frac{p_{sl}}{\gamma} + \frac{u_{sl}^2}{2g} \tag{1.12}$$

where:

E_{sl} – total energy of traveling particle along a streamline,
z_{sl} – elevation of the streamline above the assumed datum,
p_{sl} – local pressure,
γ – specific weight,
u_{sl} – local velocity,
g – local acceleration due to gravity.

In Eq. (1.12) the term (p_{sl}/γ) is the pressure head, whereas the term $(u_{sl}^2/2\,g)$ represents velocity head. The term $(p_{sl}/\gamma + z_{sl})$ is called the piezometric head (Chanson 2004).

When the flow in open channel with cross-sectional average velocity is considered, the total energy of flow referred to the assumed datum is:

$$E = Z + H + \frac{\alpha \cdot U^2}{2\,g} = h + \frac{\alpha \cdot U^2}{2\,g}, \tag{1.13}$$

whereas the specific energy of the open channel flow relative to the bottom of a channel is defined by formula (Fig. 1.4):

$$E_S = H + \frac{\alpha \cdot U^2}{2g} \tag{1.14}$$

where:

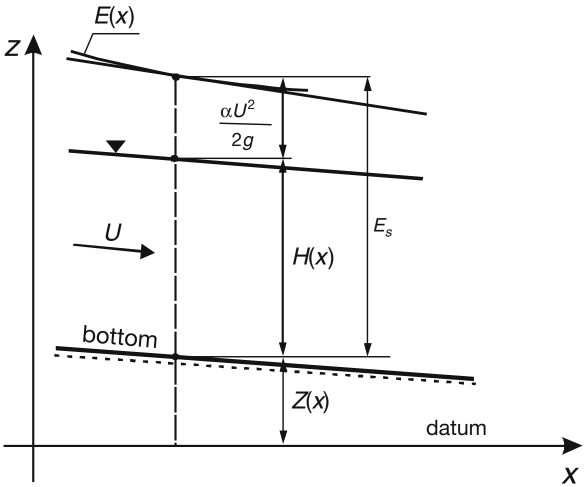

Fig. 1.4 Representation of the energy equation's terms

E – total energy related to the assumed datum,
E_s – specific energy related to the bottom,
Z – elevation of the bed above the assumed datum,
H – depth,
h – elevation of the water surface above the assumed datum,
U – average flow velocity,
α – kinetic energy correction factor (Coriolis coefficient).

The Coriolis coefficient α is introduced to correct the kinetic energy calculated using the average flow velocity U instead of the actual velocities, variable over wetted cross-sectional area A. Comparison of both energies yields:

$$\alpha = \frac{1}{U^3 \cdot A} \iint_A u^3 \cdot dA \tag{1.15}$$

Coefficient α is never less than 1. Its value increases with the increasing differentiation of flow velocity distribution over cross-section.

Analysis of Eq. (1.13) for horizontal channel with $\alpha = 1$ shows that the specific energy is a function of the flow depth H only, i.e. $E_s = E_s\ (H)$. From its examination results that E_s tends to infinity for H tending either to 0 or to infinity. This means that in the interval $\langle 0, \infty)$ the function $E_s(H)$ has an extreme minimum point. At this point the following condition resulting from Eq. (1.14) is valid:

$$\frac{\alpha \cdot Q^2}{g} = \frac{A^3}{B} \tag{1.16}$$

Then except this point, there are two alternate depths H for which $E_s(H)$ takes the same values. The depth corresponding to the extreme minimum is *critical depth*. This depth, designed as H_c, ensures that the flowing stream with flow rate Q has least specific energy or for given specific energy the flow rate is greatest. The critical depth divides the interval $\langle 0, \infty)$ into two parts. The channel flow with $H < H_c$ is so called supercritical or torrential flow, whereas the flow with $H > H_c$ is called subcritical or fluvial one. To distinguish these cases the following dimensionless Froude number is introduced:

$$F_r = \frac{U}{\sqrt{g \cdot H}} \tag{1.17}$$

This number, representing the ratio of inertial and gravity forces, is expressed by the average flow velocity U and the celerity of gravity wave in shallow water $(g\ H)^{1/2}$. Using the Froude number one can distinguish:

- critical flow when F_r=1,
- supercritical flow when $F_r > 1$,
- subcritical flow when $F_r < 1$.

The subcritical flow is typical form of channel flow. It is characterized by relatively high depths and small velocities. This form of flow is required as the most suitable one in natural channel. The supercritical flow is characterized by relatively small depths and great velocities. In natural conditions it can occur in upper parts of rivers. This kind of flow is typically observed on the weirs of dams.

Since Eq. (1.13) represents the total mechanical energy related to the assumed datum, then its derivative with regard to x represents local energy grade line slope S. Therefore one can write:

$$\frac{dE}{dx} = -S \tag{1.18}$$

where S is energy grade line slope or friction slope. The negative sign at right side of Eq. (1.18) takes into account the fact that energy decreases with increasing of x (Fig. 1.4).

The water flow in open channel can be classified with regard to various criteria. Taking into account the time variability one can distinguish:

- steady flow, when the flow parameters do not vary in time,
- unsteady flow when the flow parameters vary in time.

With the time variation of flow is connected the notion of time-invariant process or system. A channel reach can be considered as a physical system, which is capable to transform the flood wave occurring at the upstream end to the one observed at the downstream end. If the same waves entering the channel at given time intervals produce the same response, the considered channel reach is a time-invariant system.

It means that its main properties affecting the flow process, such as the shape of cross-sections and hydraulic roughness do not vary in time. In the opposite case, when the channel properties vary in time, the system is called time-variable.

Steady flows can be further divided with respect to the spatial variability of the parameters. From this point of view one can distinguish uniform and non-uniform flows. A uniform flow is characterized by spatially constant flow parameters such as velocities and depths, so that the water surface and the energy line are parallel to the bottom. Thus uniform flow can occur in prismatic channels only, in which the shape of cross-sections does not vary and the bed slope is constant. In natural channels such kind of flow does not exist. For steady uniform flow the average velocity is given by the well known empirical formulas:

- The Chézy equation:

$$U = C_C (R \cdot s)^{1/2} \tag{1.19}$$

where:

C_C – the Chézy coefficient,
R – hydraulic radius,
s – longitudinal channel bed slope, equal to the energy line slope S.

- The Manning equation

$$U = \frac{1}{n_M} R^{2/3} \cdot s^{1/2} \tag{1.20}$$

This equation can be considered as the Chézy equation with C_C defined as follows:

$$C_C = \frac{1}{n_M} R^{1/6} \tag{1.21}$$

where n_M is the Manning roughness coefficient. In SI units this coefficient is expressed in $s/m^{1/3}$.

The depth H_n which satisfies Eqs. (1.19) or (1.20) is called *normal depth*. In the case of uniform flow, when the flow velocities and wetted cross-sectional areas do not vary, the energy grade line, the hydraulic grade line and bottom line are parallel one another.

In particular cases it can occur that the normal depth is equal to the critical depth. The channel slope, for which the uniform flow is critical, is called the *critical slope* (Chanson 2004, French 1985). Usually it is denoted s_c. Relation between the normal and critical depths can be used to characterize the channel slope. One can distinguish (Chanson 2004):

- mild slope when $H_n > H_c$,
- critical slope when $H_n = H_c$,
- steep slope when $H_n < H_c$.

The non-uniform flow does not satisfy the conditions of uniform flow, so the depths and velocities vary along channel axis. Consequently the bottom, water surface and energy line are not parallel to each other. Depending on the degree of the spatial variability one can distinguish gradually and rapidly varied flows. To explain the essential difference between both types of flow, let us recall the Bernoulli equation (Eq. 1.12). In open channel gradually varied flow the water particles move along nearly straight streamlines. Since the streamlines have insignificant curvature, the cross-sectional surface, orthogonal to the streamline, are nearly flat. It means that they are almost normal to the velocity vectors and the vertical components of velocity do not exist. These assumptions imply that the sum $(z_{sl} + p/\gamma)$ representing the elevation of the hydraulic grade line above the assumed datum, can be considered as constant. Consequently the pressure distribution in the stream flowing in a channel with small bed slope, in which the streamlines are nearly parallel to the bottom, is governed by the hydrostatic law. Conversely, when rapid flow is considered, one needs to take into account the curvature of the streamlines and cross-sections. The law of hydrostatic distribution of pressure along the depth is no longer valid.

In non-uniform flow the friction slope S is usually also expressed with the Manning formula. Although this formula has been derived for steady uniform flow, it is applicable for non-uniform flow as well. It is assumed that the error generated due to this approximation is small comparing with the one resulting from the estimation of the coefficient n_M. The reformed formula (1.20) is used:

$$S = \frac{n_M^2 \cdot U^2}{R^{4/3}} = \frac{n_M^2 \cdot Q^2}{R^{4/3} \cdot A^2} \tag{1.22}$$

For steady uniform flow the friction slope is constant, whereas for non-uniform flow it varies along channel axis.

To solve practical open channel flow problems the energy equations written for two neighboring stations is usually applied. For a channel reach presented in Fig. 1.5 it takes the following form:

$$h_i + \frac{\alpha \cdot U_i^2}{2g} = h_{i+1} + \frac{\alpha \cdot U_{i+1}^2}{2g} + \Delta x_i \cdot \overline{S} \tag{1.23}$$

where:

h_i, h_{i+1} – elevations of the water surface above the assumed datum in cross-section i and $i+1$ respectively,
U_1, U_{i+1} – average flow velocities in cross-section i and $i+1$ respectively,
$\overline{S}$ – average friction slope between both cross-sections,
Δx_i – distance between considered stations.

The average value of the friction slope is determined using its local values in both cross-sections calculated using Eq. (1.22). Equation (1.23) allows us to compute the water surface profile in open channel for steady gradually varied flow.

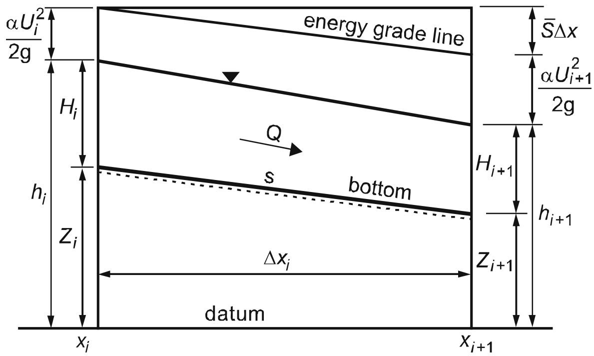

Fig. 1.5 Sketch of a channel reach

This section covers only basic information on the open channel flow, which is necessary for further presentation of numerical applications. For more comprehensive discussion of the theoretical background for the open channel hydraulics the reader is directed to the works of Akan (2006), Chanson (2004), Chow (1959), French (1985), Henderson (1966), Jain (2000), Singh (1996) and others.

1.2 General Equations for Incompressible Liquid Flow

Domination of one component of the velocity vector is a typical attribute of the open channel flow. This feature allows us to consider the flow process as spatially one-dimensional phenomenon. Similarly to the other types of surface water flows, the governing equations are derived from two principles of conservation:

- momentum conservation law,
- mass conservation law.

The derivation can be performed in various ways. The approaches found in the literature differ with respect to the formulation of the principle of conservation, and to the stage at which the one-dimensionality assumption is introduced. For example, the open channel flow equations are very often derived by balancing the fluxes and forces acting on the considered control volume, with an a priori assumption of uniform velocity flow distribution over a channel cross-section. Next, when the governing equations are derived, some additional factors are introduced to correct the balanced quantity. In such a way the correction parameters α or β appear in the 1D dynamic equation.

A more consistent approach is to derive the unsteady open channel flow equations from general equations of hydrodynamics. It allows us to show clearly that this kind of flow is governed by equations being a particular case of general equations for 3D unsteady flow. In addition, the effects of introduced simplifications and assumptions can be easily followed. This point of view coincides with the suggestion given by Abbott and Basco (1989). These authors cite Bird, Steward and Lightfoot (1960), who stated: "It is not... necessary to formulate a momentum balance (or mass balance for continuity) whenever one begins to work on a new flow problem. In fact, it is seldom desirable to do so. It is quicker, easier and safer to start with the equations of conservation of mass and momentum in general form and to simplify these equations to fit the problem at hand". Moreover Abbott and Basco (1989) state: "The advantage of this procedure is that when we are finished discarding terms and simplifying the equations (based on intuition, experiments, field data, experience, etc.) we will have available a complete list of assumptions." This approach will be followed here.

It is well known that in general case the flow of viscous and incompressible fluid is described by the Navier–Stokes equations and the continuity equation. These equations are derived from the momentum and mass conservation principles, respectively. Although the Navier–Stokes equations describe all forms of flow, in practice they are useful for laminar flow only, while every large scale geophysical surface flow is turbulent (Eagleson 1970). Therefore the open channel flow must be considered as turbulent. In such kind of flow random fluctuations of velocities and pressure are present. Since it is impossible to describe these fluctuations, the instantaneous velocity is expressed in terms of a time-averaged velocity and its random part, according to the Reynolds hypothesis. The same is applied for pressure. Inserting these relations into the Navier–Stokes equations one obtains the Reynolds equations. As a result of this operation an additional tensor of turbulent stresses appears. Applying the Boussinesq concept of the eddy viscosity one obtains the following system of equations (French 1985):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}(\mu + \mu^T)\Delta u, \tag{1.24}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}(\mu + \mu^T)\Delta v, \tag{1.25}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}(\mu + \mu^T)\Delta w, \tag{1.26}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1.27}$$

where:

t – time,
x, y, z – spatial coordinates,
u, v, w – components of the velocity vector in x, y and z direction respectively,
ρ – water density,
μ – coefficient of dynamic viscosity,
μ^T – coefficient of turbulent viscosity (eddy viscosity),
p – pressure,

F_x, F_y, F_z – components of gravitational forces in x, y and z direction respectively,
Δ – Laplace operator.

The system of Eqs. (1.24), (1.25), (1.26) and (1.27) is formally very similar the system of Navier–Stokes equations. However, it is interpreted in a different way. Eqs. (1.24), (1.25), (1.26) and (1.27) describe the relations between time averaged values of the dependent variable, while the Navier–Stokes equations relate their instantaneous values. Conversely to the coefficient μ depending on the physical properties of the water, the coefficient μ^T is a function of the flow turbulence only. As the turbulent diffusion is much more important than the molecular one because $\mu^T \gg \mu$ (Martin and McCutcheon 1999), is reasonable to neglect the dynamic viscosity.

The system of Eqs. (1.24), (1.25), (1.26) and (1.27) representing the momentum and mass conservation equations written in general form will be used to derive the equations for open channel flow.

1.3 Derivation of 1D Dynamic Equation

Equations (1.23), (1.24), (1.25) and (1.26) can be simplified when the specific features of the open channel flow problem are taken into account. Usually open channel flow can be considered as a propagation of long waves in shallow water. These waves have relatively small amplitudes compared to their lengths so the accelerations and velocities in vertical direction are negligibly small in relation to the accelerations and velocities in horizontal directions. Let us assume the co-ordinate system as shown in Fig. 1.6, in which after Liggett (1975), x corresponds to the direction of the primary flow, y is the horizontal direction normal to primary flow, whereas z is the vertical direction.

The gravity force is the only force acting along z axis:

$$F_z = -g, \tag{1.28}$$

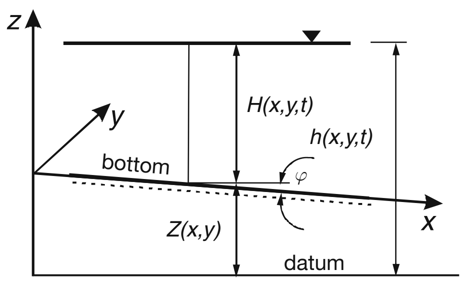

Fig. 1.6 Assumed system of co-ordinates

Consequently Eq. (1.26) takes the following form:

$$-g - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0. \tag{1.29}$$

Integration of this equation with regard to z from the bottom to the water surface (Fig. 1.6)

$$\int_Z^h \frac{\partial p}{\partial z} dz = -\int_Z^h \rho \cdot g \cdot dz \tag{1.30}$$

yields:

$$p(h) - p(Z) = -\rho \cdot g(h - Z). \tag{1.31}$$

Since at the water surface the pressure is equal to the atmospheric one $p(h) = P_a$ then the following formula representing pressure variation along z axis accordingly to the hydrostatic pressure law, is obtained:

$$p(z) = P_a + \rho \cdot g(h - z) \text{ for } Z \le z \le h \tag{1.32}$$

where:

P_a – atmospheric pressure,
ρ – water density,
g – acceleration due to gravity,
h – elevation of the water surface above the datum,
Z – elevation of the bottom above the datum,
z – vertical co-ordinate.

Let us assume that the velocity vector is parallel to x axis – it has one component only $u = u(z, x, t)$. Consequently Eq. (1.25) for the component v disappears, whereas Eq. (1.24) becomes:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu^T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right). \tag{1.33}$$

In addition, let us assume that the channel bed is inclined towards x axis and its slope is given by the angle φ (Fig. 1.6). The only force acting along x axis is the component of the gravitational force parallel to the bottom:

$$F_x = -g \cdot \sin(\varphi). \tag{1.34}$$

For a small bottom slope which is usually assumed for open channels (see Section 1.1), one has $\sin(\varphi) \approx \text{tg}(\varphi)$. Then the component F_x is expressed as:

$$F_x = -g\frac{\partial Z}{\partial x}, \tag{1.35}$$

where $\partial Z/\partial x$ accordingly to Eq. (1.2) is the longitudinal channel bed slope. With Eqs. (1.32) and (1.35), Eq. (1.33) is rewritten as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial Z}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(P_a + \rho \cdot g \cdot H) - \frac{\mu^T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0. \tag{1.36}$$

In turbulent open channel flow the vertical velocity distribution is rather uniform, so the actual velocity $u(x,z,t)$ can be replaced by the depth-averaged one $U(x,t)$. Let us introduce:

- the average horizontal velocity $U(x,t)$:

$$U(x,t) = \frac{1}{H}\int_Z^h u(x,z,t)\,dz, \tag{1.37}$$

  where $H = h-Z$ designates the depth;
- the deviation $u''(z)$ of the actual velocity from the average one (Fig. 1.7):

$$u''(z) = u(z) - U. \tag{1.38}$$

From Eq. (1.37) results the following condition:

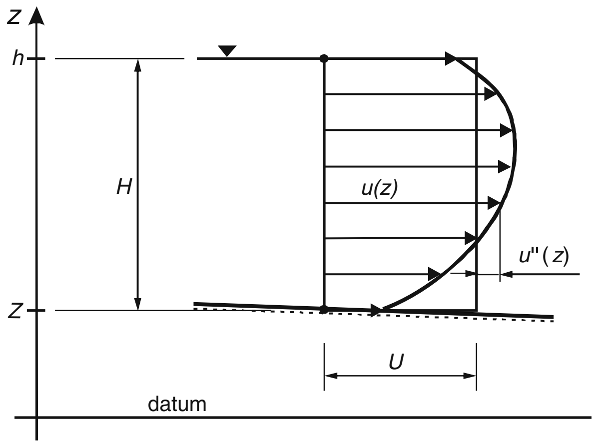

Fig. 1.7 Actual and averaged distribution of horizontal velocity u along the vertical axis

$$\int_{Z}^{h} u''(z)\,dz = 0. \tag{1.39}$$

Averaging the horizontal velocity over depth allows us to eliminate the variability in z direction and consequently to reduce the number of dimensions. To do it, Eq. (1.36) must be integrated over depth:

$$\int_{Z}^{h} \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial Z}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(P_a + \rho \cdot g \cdot H) - \frac{\mu^T}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)\right) dz = 0. \tag{1.40}$$

The integration limits, i.e. the water stage h and the bottom elevation Z, in a general case are functions of the horizontal spatial coordinates. Therefore while integrating Eq. (1.40) we will face a problem of differentiation inside the integral with variable limits. According to the Leibniz rule (Korn and Korn 1968, McQuarrie 2003) in such a case the differentiation of any function ϕ is carried out as follows:

$$\int_{Z(x)}^{h(x)} \frac{\partial \phi}{\partial x} dz = \frac{\partial}{\partial x}\int_{Z(x)}^{h(x)} \phi \cdot dz - \phi(h)\frac{\partial h}{\partial x} + \phi(\mathrm{Z})\,\frac{\partial Z}{\partial x}. \tag{1.41}$$

Let us integrate subsequent terms of Eq. (1.40). The first term is integrated as follows:

$$I_1 = \int_{Z}^{h} \frac{\partial u}{\partial t} dz = \frac{\partial}{\partial t}\int_{Z}^{h} u \cdot dz - u(h)\frac{\partial h}{\partial t} + u(\mathrm{Z})\frac{\partial Z}{\partial t}. \tag{1.42}$$

If we:

- make use of the definition (1.37),
- assume that the channel bed does not change its position with time,
- assume that the velocity at the water surface is equal to the average one then Eq. (1.42) will take the following final form:

$$I_1 = \frac{\partial}{\partial t}(U \cdot H) - U\frac{\partial H}{\partial t} = H\frac{\partial U}{\partial t}. \tag{1.43}$$

The integral of the second term with Eq. (1.38) is rearrangered to the form:

$$I_2 = \int_Z^h u \frac{\partial u}{\partial x} dz = \frac{1}{2} \int_Z^h \frac{\partial (U + u'')^2}{\partial x} d\,z$$
$$= \frac{1}{2} \left[\frac{\partial}{\partial x} \int_Z^h (U + u'')^2 dz - (U + u'')^2_{z=h} \frac{\partial h}{\partial x} + (U + u'')^2_{z=Z} \frac{\partial Z}{\partial x} \right]. \tag{1.44}$$

Developing the square of sum and assuming that the flow velocities at the water surface and at the channel bottom are equal to the average velocity U, one obtains:

$$I_2 = \frac{1}{2} \left[\frac{\partial}{\partial x} \left(\int_Z^h U^2 \cdot dz + \int_Z^h 2U \cdot u'' \cdot dz + \int_Z^h \left(u''\right)^2 dz \right) - U^2 \frac{\partial \, (h - Z)}{\partial x} \right]. \tag{1.45}$$

Integrals in the brackets are calculated term by term as follows:

$$\int_Z^h U^2 \cdot d\,z = U^2 \cdot H, \tag{1.46}$$

$$\int_Z^h 2U \cdot u'' \cdot dz = 2U \int_Z^h u'' \cdot dz = 0. \tag{1.47}$$

The last integral at the right hand side of Eq. (1.45) $\int_Z^h (u'')^2 dz$ cannot be calculated since the actual distribution of the velocity over the depth is unknown. However, a rough estimation of this integral in relation to the integral (1.46) can be attempted. To this end let us assume a hypothetical vertical distribution of $u(z)$ as presented in Fig. 1.8.

Over the entire depth the deviation of actual velocity is taken as $u''(z) = \pm 0.2U$. Then we have:

$$\int_Z^h (u'')^2 dz = \int_Z^h (0.2U)^2 d\,z = 0.04\, U^2 \int_Z^h d\,z = 0.04\, U \cdot H. \tag{1.48}$$

The result of integration shows that even for the assumed significant non-uniformity of the vertical distribution of velocity, the value of integral (1.48) constitutes only 4% of the value of integral (1.46). To take into account these integral we can consider it as a fraction of integral (1.46) and to add its value to (1.46). This allows us to write:

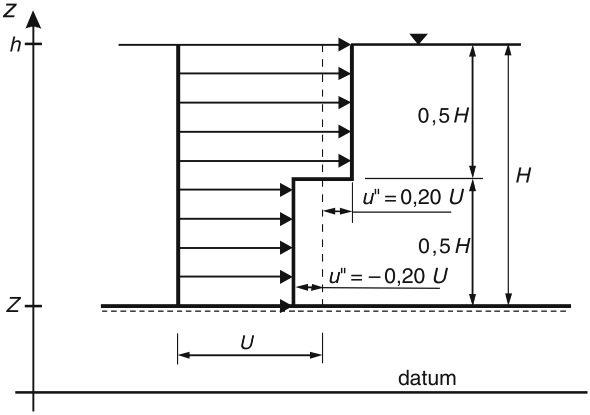

Fig. 1.8 Assumed hypothetical vertical distribution of flow velocity

$$\int_Z^h U^2 \cdot dz + \int_Z^h (u'')^2 dz = U^2 \cdot H + \int_Z^h (u'')^2 dz = \left(1 + \frac{\int_Z^h (u'')^2 dz}{U^2 \cdot H} \right) U^2 \cdot H = \beta \cdot U^2 \cdot H. \tag{1.49}$$

In this equation we recognize

$$\beta = 1 + \frac{\int_Z^h (u'')^2 dz}{U^2 \cdot H}. \tag{1.50}$$

as a correction parameter. Coming back to Eq. (1.45) we have:

$$I_2 = \frac{1}{2} \left[\frac{\partial}{\partial x} \left(\beta \cdot U^2 \cdot H \right) - U^2 \frac{\partial H}{\partial x} \right]. \tag{1.51}$$

Developing of the derivative in Eq. (1.51) yields:

$$I_2 = \frac{1}{2} \left[H \frac{\partial \left(\beta \cdot U^2 \right)}{\partial x} + \beta \cdot U^2 \frac{\partial H}{\partial x} - U^2 \frac{\partial H}{\partial x} \right]. \tag{1.52}$$

Neglecting the difference of the last two terms as small value, which additionally can be corrected by chosen value of β, the integral of the second term of Eq. (1.40) takes its following final form:

$$I_2 = \beta \cdot U \frac{\partial U}{\partial x} H. \tag{1.53}$$

Introduction of the correction factor β requires a comment. Assumption of the uniform flow velocity distribution instead of the actual one always requires a correction of the conservative quantity, which depends on the velocity. In the case of the discrete energy equation (1.23) applied for channel flow, comparison of the actual kinetic energy with that calculated with the average velocity shows that it is necessary to introduce the correction factor α, given by Eq. (1.15). This coefficient was introduced into the dynamic equation of the Saint Venant system by Chow (1959). However, Abbott (1979) in this same equation introduces the factor β, which corrects the momentum. It is worth emphasizing that usually at first the unsteady flow equations are derived assuming a priori one dimensionality and afterwards the effect of averaging is corrected. In the derivation from general equations as we have just applied, the correction factor appears naturally as a result of the averaging process. Such a way of derivation ensures a simple interpretation of the corrective parameter β and explains the real reasons of its appearance. It corrects the balance of momentum affected by the average flow velocity introduced instead of the actual velocity distribution over the cross-sectional area. This parameter allows us to include the estimated value of integral (1.48), impossible to be calculated directly. For $u(z)=$ const., i.e. when $u''(z) = 0$ for $Z \leq z \leq h$ one obtains $\beta = 1$.

The terms of Eq. (1.40) representing gravitation and pressure are integrated as follows:

$$I_3 = \int_Z^h g \frac{\partial Z}{\partial x} dz = g \frac{\partial Z}{\partial x} H, \tag{1.54}$$

$$I_4 = \frac{1}{\rho} \int_Z^h \frac{\partial}{\partial x} (P_a + \rho \cdot g \cdot H)\, dz = \frac{1}{\rho} \frac{\partial P_a}{\partial x} H + g \frac{\partial H}{\partial x} H. \tag{1.55}$$

The term describing the turbulent diffusion of momentum in x direction takes the form:

$$\begin{aligned} I_5 &= \int_Z^h \frac{\mu^T}{\rho} \frac{\partial^2 u}{\partial x^2} dz = \frac{\mu^T}{\rho} \int_Z^h \frac{\partial^2}{\partial x^2} (U + u'')\, dz = \\ &= \frac{\mu^T}{\rho} \frac{\partial^2 U}{\partial x^2} H + \frac{\mu^T}{\rho} \int_Z^h \frac{\partial^2 u''}{\partial x^2} dz = \frac{\mu^D}{\rho} \frac{\partial^2 U}{\partial x^2} H, \end{aligned} \tag{1.56}$$

where μ^D is the coefficient of dispersive transport of momentum in x direction. It represents the combined effect of the turbulent viscosity with coefficient μ^T and vertical velocity averaging. This is an analogous situation to the one discussed previously, where integrating the term of velocity advection led to the correction factor β (see Eq. 1.50).

The term describing the momentum transfer in vertical is integrated as follows: