The Genius of Archimedes – 23 Centuries of Influence on Mathematics, Science and Engineering

HISTORY OF MECHANISM AND MACHINE SCIENCE
Volume 11

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The Genius of Archimedes – 23 Centuries of Influence on Mathematics, Science and Engineering

Proceedings of an International Conference held at Syracuse, Italy, June 8–10, 2010

Springer

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PREFACE

The idea of a Conference in Syracuse to honour Archimedes, one of the greatest figures in Science and Technology of all ages, was born during a Meeting in Patras, Greece, dealing with the cultural interaction between Western Greece and Southern Italy through History, organized by the Western Greece Region within the frame of a EU Interreg project in cooperation with several Greek and Italian institutions. Part of the Meeting was devoted to Archimedes as the representative figure of the common scientific tradition of Greece and Italy. Many reknown specialists attended the Meeting, but many more, who were unable to attend, expressed the wish that a respective Conference be organized in Syracuse. The present editors assumed the task of making this idea a reality by co-chairing a World Conference on 'The Genius of Archimedes (23 Centuries of Influence on the Fields of Mathematics, Science, and Engineering)', which was held in Syracuse, Italy, 8–10 June 2010, celebrate the 23th century anniversary of Archimedes' birth.

The Conference was aiming at bringing together researchers, scholars and students from the broad ranges of disciplines referring to the History of Science and Technology, Mathematics, Mechanics, and Engineering, in a unique multidisciplinary forum demonstrating the sequence, progression, or continuation of Archimedean influence from ancient times to modern era.

In fact, most the authors of the contributed papers are experts in different topics that usually are far from each other. This has been, indeed, a challenge: convincing technical experts and historian to go further in-depth into the background of their topics of expertise with both technical and historical views to Archimedes' legacy.

We have received a very positive response, as can be seen by the fact that these Proceedings contain contributions by authors from all around the world. Out of about 50 papers submitted, after thorough review, about 35 papers were accepted both for presentation and publication in the Proceedings. They include topics drawn from the works of Archimedes, such as Hydrostatics, Mechanics, Mathematical Physics, Integral Calculus, Ancient Machines & Mechanisms, History of Mathematics & Machines, Teaching of Archimedean Principles, Pycnometry, Archimedean Legends and others. Also, because of the location of the Conference, a special session was devotyed to Syracuse at the time of Archimedes. The figure on the cover is taken from the the book 'Mechanicorum Liber' by Guidobaldo Del Monte, published in Pisa on 1575 and represents the lever law of Archimedes as lifting the world through knowledge.

The world-wide participation to the Conference indicates also that Archimedes' works are still of interest everywhere and, indeed, an in-depth knowledge of this glorious past can be a great source of inspiration in developing the present and in shaping the future with new ideas in teaching, research, and technological applications.

We believe that a reader will take advantage of the papers in these Proceedings with further satisfaction and motivation for her or his work (historical or not). These papers cover a wide field of the History of Science and Mechanical Engineering.

We would like to express my grateful thanks to the members of the Local Organizing Committee of the Conference and to the members of the Steering Committee for co-operating enthusiastically for the success of this initiative. We are grateful to the authors of the articles for their valuable contributions and for preparing their manuscripts on time, and to the reviewers for the time and effort they spent evaluating the papers. A special thankful mention is due to the sponsors of the Conference: From the Greek part, the Western Greece Region, the University of Patras, the GEFYRA SA, the Company that built and runs the famous Rion-Antirrion Bridge in Patras, Institute of Culture and Quality of Life and last but not least the e-RDA Innovation Center, that offered all the necessary support in the informatics field. From the Italian part, the City of Syracuse, the University of Cassino, the School of Architecture of Catania University, Soprintendenza dei Beni Culturali e Archeologici di Siracusa, as well as IFToMM the International Federation for the Promotion of Mechanism and Machine Science, and the European Society for the History of Science.

The Editors are grateful to their families for their patience and understanding, without which the organization of such a task might be impossible. In particular, the first of us (M.C.), mainly responsible for the preparation of the present volume, wishes to thank his wife Brunella, daughters Elisa and Sofia, and young son Raffaele for their encouragement and support.

Cassino (Italy) and Patras (Greece): January 2010

Marco Ceccarelli, Stephanos A. Paipetis, Editors
Co-Chairmen for Archimedes 2010 Conference

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AN ARCHIMEDEAN RESEARCH THEME: THE CALCULATION OF THE VOLUME OF CYLINDRICAL GROINS

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ABSTRACT Starting from Archimedes' method for calculating the volume of cylindrical wedges, I want to get to describe a method of 18th century for cilindrical groins thought by Girolamo Settimo and Nicolò di Martino. Several mathematicians studied the measurement of wedges, by applying notions of infinitesimal and integral calculus; in particular I examinated Settimo's *Treatise on cylindrical groins*, where the author solved several problems by means of integrals.

KEYWORDS: Wedge, cylindrical groin, Archimedes' method, G. Settimo.

1. INTRODUCTION

"Cylindrical groins" are general cases of cylindrical wedge, where the base of the cylinder can be an ellipse, a parabola or a hyperbole. In the Eighteenth century, several mathematicians studied the measurement of vault and cylindrical groins by means of infinitesimal and integral calculus. Also in the Kingdom of Naples, the study of these surfaces was a topical subject until the Nineteenth century at least because a lot of public buildings were covered with vaults of various kinds: mathematicians tried to give answers to requirements of the civil society who vice versa submitted concrete questions that stimulated the creation of new procedures for extending the theoretical system.

Archimedes studied the calculation of the volume of a cylindrical wedge, a result that reappears as theorem XVII of *The Method*:

If in a right prism with a parallelogram base a cylinder be inscribed which has its bases in the opposite parallelograms [in fact squares], and its sides [i.e., four generators] on the remaining planes (faces) of the

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prism, and if through the centre of the circle which is the base of the cylinder and (through) one side of the square in the plane opposite to it a plane be drawn, the plane so drawn will cut off from the cylinder a segment which is bounded by two planes, and the surface of the cylinder, one of the two planes being the plane which has been drawn and the other the plane in which the base of the cylinder is, and the surface being that which is between the said planes; and the segment cut off from the cylinder is one sixth part of the whole prism.

The method that Archimedes used for proving his theorem consist of comparing the area or volume of a figure for which he knew the total mass and the location of the centre of mass with the area or volume of another figure he did not know anything about. He divided both figures into infinitely many slices of infinitesimal width, and he balanced each slice of one figure against a corresponding slice of the second figure on a lever.

Using this method, Archimedes was able to solve several problems that would now be treated by integral and infinitesimal calculus.

The Palermitan mathematician Girolamo Settimo got together a part of his studies about the theory of vaults in his *Trattato delle unghiette cilindriche* (*Treatise on cylindrical groins*), that he wrote in 1750 about but he never published; here the author discussed and resolved four problems on cylindrical groins.

In his treatise, Settimo gave a significant generalization of the notion of groin and used the actual theory of infinitesimal calculus. Indeed, every one of these problems was concluded with integrals that were reduced to more simple integrals by means of decompositions in partial sums.

2. HOW ARCHIMEDES CALCULATED THE VOLUMES OF CYLINDRICAL WEDGES

The calculation of the volume of cylindrical wedge appears as theorem XVII of Archimedes' *The Method.* It works as follows: starting from a cylinder inscribed within a prism, let us construct a wedge following the statement of Archimedes' theorem and then let us cut the prism with a plane that is perpendicular to the diameter *MN* (see fig. 1.a). The section obtained is the rectangle *BAEF* (see fig. 1.b), where *FH'* is the intersection of this new plane with the plane generating the wedge, *HH'=h* is the height of the cylinder and *DC* is the perpendicular to *HH'* passing through its midpoint.

Then let us cut the prism with another plane passing through *DC* (see fig. 2). The section with the prism is the square *MNYZ*, while the section with the cylinder is the circle *PRQR'*. Besides, *KL* is the intersection between the two new planes that we constructed.

Let us draw a segment *IJ* parallel to *LK* and construct a plane through *IJ* and perpendicular to *RR'*; this plane meets the cylinder in the rectangle *S'T'I'T'* and the wedge in the rectangle *S'T'ST*, as it is possible to see in the fig. 3:

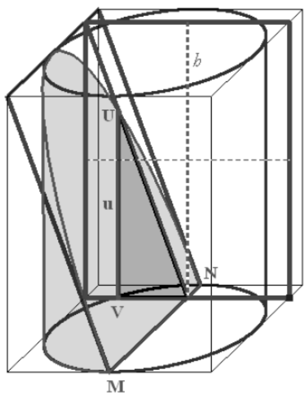

Fig. 1.a. Construction of the wedge.

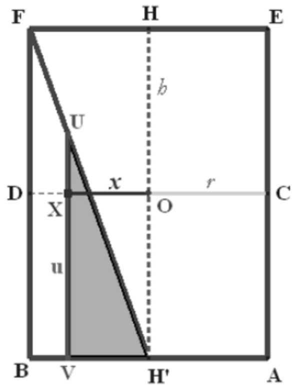

Fig. 1.b. Section of the cylinder with a plane perpendicular to the diameter MN.

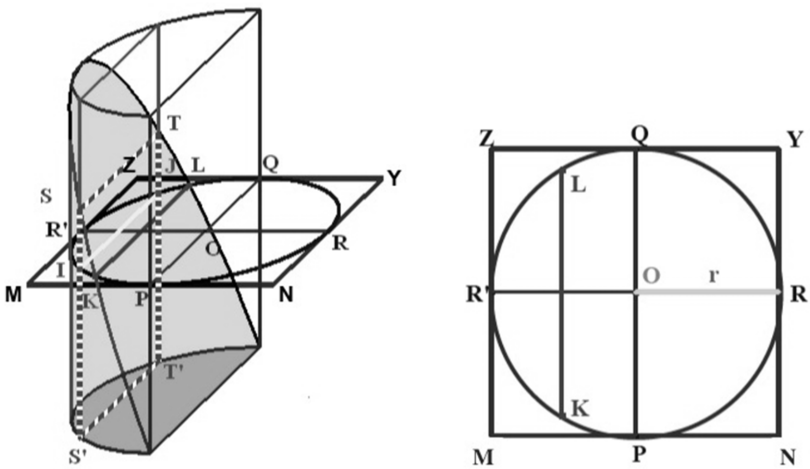

Fig. 2. Section of the cylinder with a plane passing through DC.

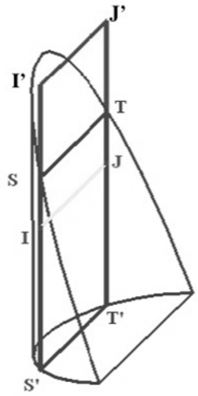

Fig. 3. Construction of the wedge.

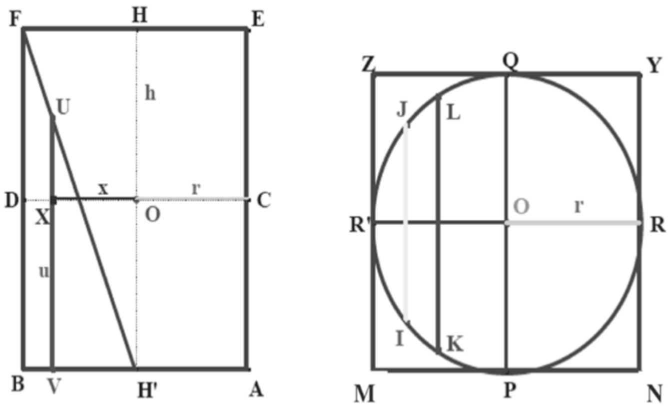

Fig. 4. Sections of the wedge.

Because *OH'* and *VU* are parallel lines cut by the two transversals *DO* and *H'F*, we have

$$DO : DX = H'B : H'V = BF : UV \text{ (see fig. 4)}$$

where *BF=h* and *UV* is the height, *u*, of the rectangle *S'T'ST*. Therefore

$$DO : DX = H'B : H'V = BF : UV = h : u = (h \bullet IJ) : (u \bullet IJ).$$

Besides *H'B=OD* (that is *r*) and *H'V=OX* (that is *x*). Therefore

$$(FB \bullet IJ) : (UV \bullet IJ) = r : x, \text{ and } (FB \bullet IJ) \bullet x = (UV \bullet IJ) \bullet r.$$

Then Archimedes thinks the segment *CD* as lever with fulcrum in *O*; he transposes the rectangle *UV•IJ* at the right of the lever with arm *r* and the rectangle *FB•IJ* at the left with the arm *x*. He says that it is possible to consider another segment parallel to *LK*, instead of *IJ* and the same argument is valid; therefore, the union of any rectangle like *S'T'ST* with arm *r* builds the wedge and the union of any rectangle like *S'T'I'T'* with arm *x* builds the half-cylinder.

Then Archimedes proceeds with similar arguments in order to proof completely his theorem.

Perhaps it is important to clarify that Archimedes works with right cylinders that have defined height and a circle as the base.

3. GIROLAMO SETTIMO AND HIS HISTORICAL CONTEST

Girolamo Settimo was born in Sicily in 1706 and studied in Palermo and in Bologna with Gabriele Manfredi (1681–1761). Niccolò De Martino (1701–1769) was born near Naples and was mathematician, and a diplomat. He was also one of the main exponents of the skilful group of Italian Newtonians, whereas the Newtonianism was diffused in the Kingdom of Naples. Settimo and De Martino met each other in Spain in 1740 and as a consequence of this occasion, when Settimo came back to Palermo, he began an epistolar relationship with Niccolò. Their correspondence collects 62 letters of De Martino and two draft letters of Settimo; its peculiar mathematical subjects concern with methods to integrate fractional functions, resolutions of equations of any degree, method to deduce an equation of one variable from a system of two equations of two unknown quantities, methods to measure surface and volume of vaults[1].

One of the most important arguments in the correspondence is also the publication of a book of Settimo who asked De Martino to publish in Naples his mathematical work: *Treatise on cylindrical groins* that would have to contain the treatise *Sulla misura delle Volte* ("On the measure of vaults"). In order to publish his book, Settimo decided to improve his knowledge of infinitesimal calculus and he needed to consult De Martino about this argument.

In his treatise, Settimo discussed and resolved four problems: calculus of areas, volumes, centre of gravity relative to area, centre of gravity relative to volume of cylindrical groins. The examined manuscript of

1 N. Palladino - A.M. Mercurio - F. Palladino, *La corrispondenza epistolare Niccolò de Martino-Girolamo Settimo. Con un saggio sull'inedito Trattato delle Unghiette Cilindriche di Settimo*, Firenze, Olschki, 2008.

Settimo, *Treatise on cylindrical groins,* is now stored at Library of *Società Siciliana di Storia Patria* in Palermo (Italy), *M.ss. Fitalia*, and it is included in the volume *Miscellanee Matematiche di Geronimo Settimo (M.SS. del sec. XVIII).*

4. GROINS IN SETTIMO'S TREATRISE

Settimo's *Treatise on cylindrical groins* relates four *Problems*. The author introduces every problem by *Definizioni*, *Corollari*, *Scolii* and *Avvertimenti*; adding also *Scolii*, *Corollari* and *Examples* after the discussion of it. On the whole, Settimo subdivides his manuscript into 353 *articles, Fig. 5.* The problems to solve are:

Problem 1: to determine the volume of a cylindrical groin;
Problem 2: to determine the area of the lateral surface of a cylindrical groin;
Problem 3: to determine the center of gravity relative to the solidity of a cylindrical groin;
Problem 4: to determine the center of gravity relative to the lateral surface of a cylindrical groin.

Settimo defines cylindrical groins as follows:

"*If any cylinder is cut by a plane which intersects both its axis and its base, the part of the cylinder remaining on the base is called a cylindrical groin*".

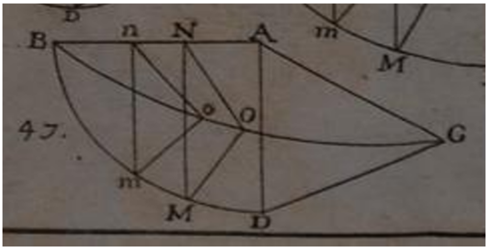

Fig. 5. Original picture by De Martino of cylindrical groin (in Elementi della Geometria così piana come solida coll'aggiunta di un breve trattato delle Sezioni Coniche, 1768).

Settimo concludes each one of these problems with integrals that are reduced to more simple integrals by means of decompositions in partial sums, solvable by means of elliptical functions, or elementary functions (polynomials, logarithms, circular arcs).

Settimo and de Martino had consulted also Euler to solve many integrals by means of logarithms and circular arcs[2].

Let us examine now how Settimo solved his first problem, *"How to determine volume of cylindrical groin"*.

He starts to build a groin as follows: let *AM* be a generic curve, that has the line *AB* as its axis of symmetry; on this plane figure he raises a cylinder; then on *AB* he drew a plane parallel to the axis of the cylinder; this plane is perpendicular to the plane of the basis (see fig. 6).

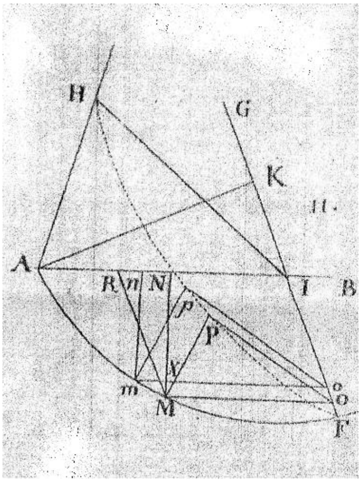

Fig. 6. Original picture of groin by Settimo.

Let *AH* be the intersection between this plane and the cylinder; *BAH* is the angle that indicates obliqueness of the cylinder; the perpendicular line from *H* to the cylinder's basis falls on the line *AB*.

Let's cut the cylinder through the plane *FHG*, that intersects the plane of basis in the line *FG*. Since we formed the groin *FAGH*, the line *FG* is the directrix line of our groin. If *FG* is oblique, or perpendicular, or parallel to *AB*, then the groin *FAGH* is "obliqua" (oblique), or "diretta" (direct), or "laterale" (lateral). To solve the problem:

1. firstly, Settimo supposes that the directrix *FG* intersects *AB* obliquely;
2. then, he supposes that *FG* intersects *AB* forming right angles;
3. finally, he supposes that *FG* is parallel to *AB*.

---

[2] In particular see L. Euler, *Introductio in analysin infinitorum*, Lausannae, Apud Marcum-Michaelem Bousquet & Socios, 1748 and G. Ferraro - F. Palladino, *Il calcolo sublime di Eulero e Lagrange esposto col metodo sintetico nel progetto di Nicolò Fergola*, Istituto Italiano per gli Studi Filosofici, Napoli, La Città del Sole, 1995.

The directrix *FG* and the axis *AI* intersect each other in *I*. On the line *FG* let's raise the perpendicular line *AK*. Let's put *AI=f, AK=g, KI=h*. From the generic point *M*, let's draw the distance *MN* on *AB* and then let's draw the parallel line *MR* to *FG*. Let us put *AN=x e MN=y*. Then, *NI* is equal to *f-x*. We have *AK:KI=MN:NR* and so $NR = \frac{hy}{g}$. Then, let's draw the parallel *MO* to *AB* and $MO = RI = f - x + \frac{hy}{g}$.

Let *Mm* be an *infinitely small arc*; let *mo* be parallel to *AB* and infinitely near *MO*; *mo* intersects *MN* in *X*. On *MO* let's raise the plane *MPO* and on *mo* let's raise the plane *mpo*, both parallel to *AHI*. *MPO* intersects the groin in the line *PO* and *mpo* intersects the groin in the line *po*.

The prism that these planes form is the "elemento di solidità" (*element of solidity*) of the groin. Its volume is the area of *MPO* multiplied by *MX* (where *MX=dy*). So, we are now looking for the area of *MPO*.

Let's put *AH=c*. Since *AHI* and *MPO* are similar, we have a proportion: *AI* is to *AH* as *MO* is to *MP*, and $MP = \frac{c}{f}\left(f - x + \frac{hy}{g}\right)$. The planes are parallel, *MP* is to the perpendicular line on *MO* from *P*, as radius is to sine of *BAH*. Let *r* be the radius and let *s* be the sine.

The dimension of the perpendicular is $MP = \frac{cs}{fr}\left(f - x + \frac{hy}{g}\right)$. Let us multiply it by $MO = f - x + \frac{hy}{g}$ and divide by 2. Therefore the area of the triangle is $\frac{cs}{2fr}\left(f - x + \frac{hy}{g}\right)^2$. Finally, we found the element of solidity of the groin multiplying by *dy*: $\frac{csdy}{2fr}\left(f - x + \frac{hy}{g}\right)^2$.

Since we know the curve of the groin, we can eliminate a variable in our equation $\frac{csdy}{2fr}\left(f - x + \frac{hy}{g}\right)^2$ and the element becomes "integrable".

Then, Settimo applies the first problem on oblique groins and on the elliptical cylinder

$$\frac{hy^2}{a} = bx - x^2 \Rightarrow x = \frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{hy^2}{a}}.$$

He writes the differential term like

$$\frac{csdy}{2rf}\left[p^2-2p\sqrt{\frac{b^2}{4}-\frac{hy^2}{a}}+\frac{b^2}{4}+\frac{hy^2}{a}+\frac{2phy}{g}-\frac{2hy}{g}\sqrt{\frac{b^2}{4}-\frac{hy^2}{a}}+\frac{h^2y^2}{g^2}\right]$$

and says that the problem of searching the volume of the groin is connected with the problem of squaring the ellipse.

At last, he talks about lateral groins, by analogous procedures.

In the second example, Settimo considers a hyperbolic cylinder and an oblique, direct or lateral groin. He says here that calculating volumes is connected with squaring hyperbolas. In the third example, he considers a parabolic cylinder and an oblique, direct or lateral groin, solving the problems of solidity for curves of equation $y^m=x$ that he calls "infinite parabolas".

We note that in the first problem, Settimo is able to solve and calculate each integral, but in the second problem, Settimo shows that its solution is connected with rectification of conic sections. He gives complicated differential forms like sums of more simple differentials that are integrable by elementary functions or connected with rectification of conic sections.

In the "first example" of the "second Problem", the oblique groin is part of an elliptical cylinder, where the equation of the ellipse is known; "the element of solidity" is the differential form:

$$\frac{c}{f}\left(f-x+\frac{hy}{g}\right)\sqrt{dy^2+\frac{s^2dx^2}{r^2}} \Rightarrow \frac{c}{f}\left(p+\sqrt{\frac{b^2}{4}-\frac{by^2}{a}}+\frac{hy}{g}\right)\frac{dy\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}+\frac{s^2}{r^2}y^2}}{\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}}}$$

that is decomposition of three differentials:

$$\frac{cp}{f}\frac{dy\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}+\frac{s^2}{r^2}y^2}}{\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}}}+\frac{bc}{af}dy\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}+\frac{s^2}{r^2}y^2}+\frac{chydy}{fg}\frac{\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}+\frac{s^2}{r^2}y^2}}{\sqrt{\frac{a^2}{4}-\frac{ay^2}{b}}}$$

Settimo starts studying the second differential: when he supposes the inequality $\frac{s^2}{r^2}<\frac{a}{b}$, he makes some positions and then makes a transformation on the differential that he rewrites like

$$\frac{bcm}{afr}\frac{\frac{1}{2}q^5du-\frac{1}{2}q^3u^2du}{\left(q^2+u^2\right)^2}+\frac{bcm}{afr}\frac{\frac{1}{2}q^5du+\frac{1}{2}q^3u^2du}{\left(q^2+u^2\right)^2}.$$

Settimo "constructs the solution", according to the classical method; i.e. he graphically resolves the arc that denotes the logarithm of imaginary numbers and shows that this solution solves the problem to search the original integral.

He calculates the integral of the first addend and transforms the second addend, but here he makes an important observation:

"*[this formula] includes logarithms of imaginary numbers [...]; now, since logarithms of imaginary numbers are circular arcs, in this case, from a circular arc the integral of the second part repeats itself. This arc, by 'il metodo datoci dal Cotes' [i.e. Cotes' method] has q as radius and u as tangent*".

Roger Cotes' method is in *Harmonia Mensurarum*[3]; there are also 18 tables of integrals; these tables let to get the "fluens" of a "fluxion" (i.e., the integral of a differential form) in terms of quantities, which are sides of a right triangle. Roger Cotes spent a good part of his youth (from 1709 to 1713) drafting the second edition of Newton's Principia. He died before his time, leaving incomplete and important researches that Robert Smith (1689–1768), cousin of Cotes, published in Harmonia Mensurarum, in 1722, at Cambridge.

In the first part of Harmonia Mensurarum, the Logometria, Cotes shows that problems that became problems on squaring hyperbolas and ellipses, can be solved by measures of ratios and angles; these problems can be solved more rapidly by using logarithms, sines and tangents. The "Scolio Generale", that closes the Logometria, contains a lot of elegant solutions for problems by logarithms and trigonometric functions, such as calculus of measure of lengths of geometrical or mechanical curves, volumes of surfaces, or centers of gravity.

We report here Cotes' method that Settimo uses in his treatise (see fig. 7).

Starting from the circle, let $CA=q$ and $TA=u$ the tangent*;* therefore $CT=\sqrt{q^2+u^2}$. Let's put $Tt=du$. Settimo investigates the arc that is the

[3] R. Cotes, *Harmonia Mensurarum, sive Analysis & Synthesis per Rationum & Angulorum Mensuras Promotae: Accedunt alia Opuscula Mathematica: per Rogerum Cotesium. Edidit & Auxit Robertus Smith, Collegii S. Trinitatis apud Cantabrigienses Socius; Astronomiae & Experimentalis Philosophiae Post Cotesium Professor, Cantabrigiae,* 1722. See also R. Cotes, *Logometria*, «Philosophical Transactions of the Royal Society of London», vol. 29, n° 338, 1714.

logarithm of imaginary numbers and showed that this solution solves the problem of searching the original integral $\frac{bcm}{afr}\frac{\frac{1}{2}q^3du}{q^2+u^2}$.

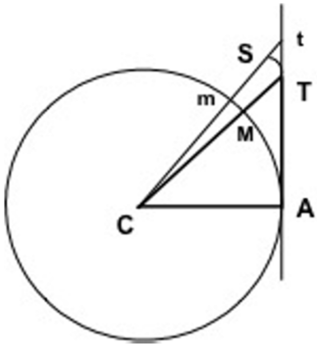

Fig. 7. Figure to illustrate Cotes' method.

The triangles *StT* and *ATC* are similar, therefore

$$Tt : TS = CT : CA \text{ and } TS = \frac{CA \cdot Tt}{CT} = \frac{qdu}{\sqrt{q^2+u^2}}.$$

CTS and *CMm* are also similar, therefore

$$TS : Mm = CT : CM \text{ and } Mm = \frac{TS \cdot CM}{CT} = \frac{q^2du}{q^2+u^2}.$$

Since the arc AM represents the integral of *Mm,* Cotes finds the original integral $\frac{bcm}{afr}\frac{\frac{1}{2}q^3du}{q^2+u^2}$. From $AM = \alpha q$ with $\alpha = \arctan\frac{u}{q}$, then

$$\frac{bcm}{afr}\frac{1}{2}q \times AM = \frac{bcm}{afr}\frac{1}{2}q^2 \arctan\frac{u}{q}$$

and its derivative is $\frac{bcm}{afr}\frac{\frac{1}{2}q^3du}{q^2+u^2}$.

Becoming again to Settimo's treatise, when Settimo supposes the inequality $\frac{s^2}{r^2} > \frac{a}{b}$, he solves the integral by means of logarithms of imaginary numbers, then (by using Cotes' method) with circular arcs.

Finally, Settimo shows problems on calculus of centre of gravity relative to area and volume of groins.

5. CONCLUSION

Various authors have eredited Archimedes, but we know that Prof. Heiberg found the Palimpsest containing *Archimedes' method* only in 1907, and therefore it is practically certain that Settimo did not know Archimedes' work.

Archimedes' solutions for calculating the volume of cylindrical wedges can be interpreted as computation of integrals, as Settimo really did, but both methods of Archimedes and Settimo are missing of generality: there is no a general computational algorithm for the calculations of volumes. They base the solution of each problem on a costruction determined by the special geometric features of that particular problem; Settimo however is able to take advantage of prevoious solutions of similar problems.

It is important finally to note that Settimo, who however has studied and knew the modern infinitesimal calculus (he indeed had to consult Roger Cotes and Leonhard Euler with De Martino in order to calculate integrals by using logarithms and circular arcs), considers the construction of the infinitesimal element similarly Archimedes.

Wanting to compare the two methods, we can say that both are based on geometrical constructions, from where they start to calculate infinitesimal element (that Settimo calls "elemento di solidità"): Archimedes' mechanical method was a precursor of that techniques which led to the rapid development of the calculus.

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ON ARCHIMEDEAN ROOTS IN TORRICELLI'S MECHANICS

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ABSTRACT In recent papers we analyzed the historical development of the foundations of the centres of gravity theory during the Renaissance. Using these works as a starting point, we shall briefly present a progression of knowledge with cultural and mathematical Archimedean roots in Torricelli's mechanics.

1. INTRODUCTION

Archimedes (287–212 B.C.) was a deeply influential author for Renaissance mathematicians according to the two main traditions. The humanistic tradition, adhering strictly to philological aspects, followed by Willem van Moerbeke (1215–1286), Regiomontanus (1436–1476) and Federigo Commandinus (1509–1575). The pure mathematical tradition followed by Francesco Maurolico (1694–1575), Luca Valerio (1552–1618), Galileo Galilei (1564–1642), Evangelista Torricelli (1608–1647).

The investigation into Archimedes's influence on Torricelli has a particular relevance because of its depth. Also it allows us to understand in which sense Archimedes' influence was still relevant for most scholars of the seventeenth century (Napolitani 1988). Besides there being a general influence on the geometrization of physics, Torricelli was particularly influenced by Archimedes with regard to mathematics of indivisibles. Indeed, it is Torricelli's attitude to confront geometric matter both with the methods of the ancients, in particular the exhaustion method, and with the indivisibles, so attempting to compare the two, as is clearly seen in his letters with Cavalieri (Torricelli 1919–1944; see mainly vol. 3). Torricelli, in particular, solved twenty one different ways the squaring a parabola (Heath 2002; *Quadrature of the parabola*, Propositio 17 and 24, p. 246;

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p. 251), a problem already studied by Archimedes: eleven times with exhaustion, ten with indivisibles. The *reductio ad absurdum* proof is always present.

Based on previous works (Pisano 2008) we can claim that the Archimedean approach to geometry is different from the Euclidean one. The object is different, because Archimedes mainly deals with metric aspects, which was quite new, also the aim is different, being more oriented towards solving practical problems. In addition, mainly the theory organization is different, because Archimedes does not develop the whole theory axiomatically, but sometimes uses an approach for problems, characterized by *reductio ad absurdum*. Moreover, the epistemological status of the principles is different. Archimedean principles are not always as self evident as those of the Euclidean tradition and may have an empirical nature. Some of the Archimedean *principles* have a clear methodological aim, and though they may express the daily feeling of the common man, they have a less cogent evidence then the principles of Euclidean geometry.

Knowledge of Archimedes' contribution is also fundamental to an historical study of Torricelli's mechanics. Archimedes was the first scientist to set *rational criteria* for determining centres of gravity of bodies and his work contains physical concepts formalised on mathematical basis. In *Book I* of the *On Plane Equilibrium* (Heath 2002) Archimedes, besides studying the rule governing the law of the lever also finds the centres of gravity of various geometrical plane figures (Heath 2002, Clagett 1964–1984; Heiberg 1881). By means of his *Suppositio* (principles) Archimedes (Heath 2002, pp. 189–202) is able to prove *Propositio* (theorems) (Heath 2002, pp. 189–202) useful in finding the centres of gravity of composed bodies. In particular, the sum of all the components may require the adoption of the method of exhaustion.

Archimedes's typical method of arguing in mechanics was by the use of the reduction *ad absurdum*, and Torricelli in his study on the centres of gravity resumes the same approach.

With regard to Torricelli's works, we studied mainly his mechanical theory (Capecchi and Pisano 2004; Idem 2007; Pisano 2009) in the *Opera geometrica* (Torricelli 1644), Table 1 and Fig. 1. We focused in detail on his discourses upon centres of gravity (Pisano 2007) where he enunciated his famous principle:

"It is impossible for the centre of gravity of two joined bodies in a state of equilibrium to sink due to any possible movement of the bodies".

The *Opera geometrica* is organized into four parts. Particularly, parts 1, 2, 3, are composed of *books* and part 4 is composed of an *Appendix*. Table 1 shows the index of the text:

Table 1. An index of *Opera geometrica* (Torricelli's manuscripts are now preserved at the Biblioteca Nazionale of Florence. Galilean Collection, n° 131–154).

De sphaera et solidis sphaeralibus, Liber primus, 3–46; Liber secondus, 47–94.
De motu gravium naturaliter descendentium et proiectorum, Liber Primus, 97–153; Liber secundus, 154–243.
De dimensione parabolae Solidique Hyperbolici, 1–84.
Appendix: *De Dimensione Cycloidis*, 85–92.
De Solido acuto Hyperbolico, 93–135.
De Dimensione Cochlea, 136–150.


OPERA
GEOMETRICA
EVANGELISTÆ
TORRICELLII

De Solidis Sphæralibus
De Motu.
De Dimensione Parabolæ

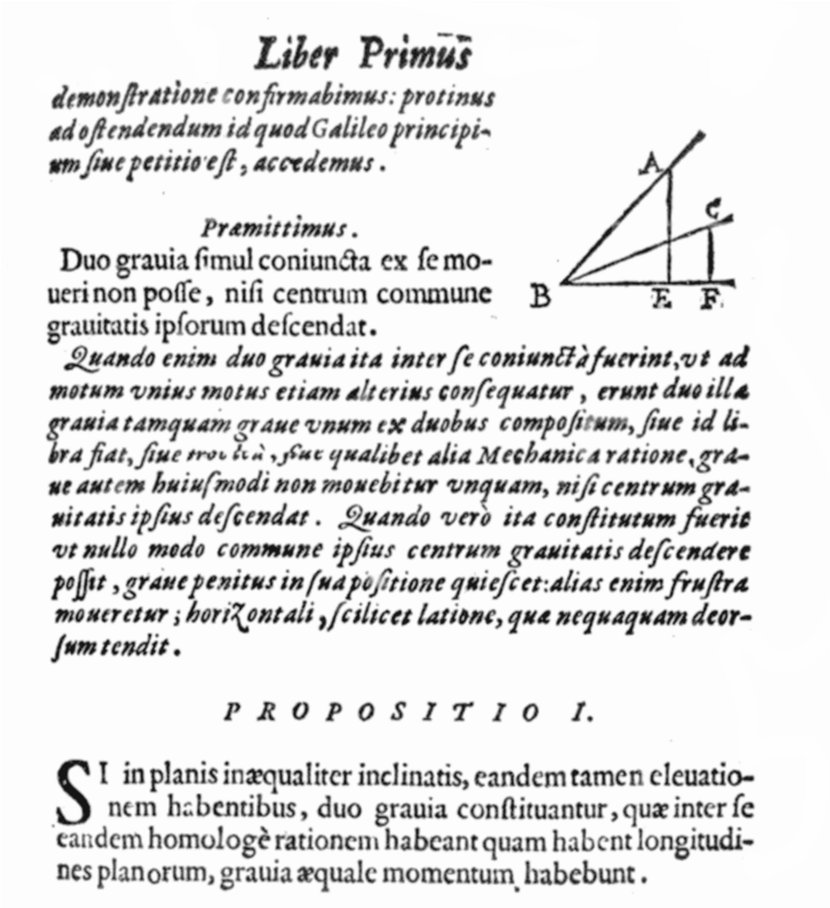
De solido Hyperbolico
Cum Appendicibus de Cycloide, & Cochlea.

Fig. 1. The front page of Torricelli's Opera geometrica with the index of content.

Torricelli in his theory on the centre of gravity, following Archimedes' approach, uses

a) *Reductio ad absurdum* as a particular instrument for mathematical proof.
b) Geometrical representation of physical bodies: weightless beams and reference in geometrical form to the law of the lever.
c) Empirical evidence to establish principles.

We focused mostly upon the exposition of studies contained in *Liber primis. De motu gravium naturaliter descendentium,* where Torricelli's principle is exposed, Fig. 2 and 3. In Galileo's theory on dynamics, Torricelli present problems which, according to him, remain unsolved. His main concern is to prove a Galileo's supposition, which states: velocity degrees for a body are directly proportional to the inclination of the plane over which it moves:


Liber Primus

demonſtratione confirmabimus: protinus ad oſtendendum id quod Galileo principium ſiue petitio eſt, accedemus.

Præmittimus.

Duo grauia ſimul coniuncta ex ſe moueri non poſſe, niſi centrum commune grauitatis ipſorum deſcendat.

Quando enim duo grauia ita inter ſe coniunctà fuerint, vt ad motum vnius motus etiam alterius conſequatur, erunt duo illa grauia tamquam graue vnum ex duobus compoſitum, ſiue id libra fiat, ſiue troc lea, ſiue qualibet alia Mechanica ratione, graue autem huiuſmodi non mouebitur vnquam, niſi centrum grauitatis ipſius deſcendat. Quando verò ita conſtitutum fuerit vt nullo modo commune ipſius centrum grauitatis deſcendere poſſit, graue penitus in ſua poſitione quieſcet: alias enim fruſtra moueretur; horizontali, ſcilicet latione, qua nequaquam deorſum tendit.

PROPOSITIO I.

SI in planis inæqualiter inclinatis, eandem tamen eleuationem habentibus, duo grauia conſtituantur, quæ inter ſe eandem homologè rationem habeant quam habent longitudines planorum, grauia æquale momentum habebunt.

Fig. 2. Torricelli's principle. Opera Geometrica. De motu gravium naturaliter descendentium et proiectorum, p. 99.

The speeds acquired by one and the same body moving down planes of different inclinations are equal when the heights of these planes are equal (Galilei 1890–1909, Vol., VIII, p. 205)

Torricelli seems to suggest that this supposition may be *proved* beginning with a "theorem" according to which "the momentum of equal bodies on planes unequally inclined are to each other as the perpendicular lines of equal parts of the same planes" (Torricelli 1644, *De motu gravium naturaliter descendentium et proiectorum*, p. 99). Moreover, Torricelli also assumes that this theorem has not yet been demonstrated (Note, in the first edition of the Galileo's *Discorsi* in 1638, there is no proof of the "theorem". It was added only in 1656 to the *Opere di Galileo Galilei linceo*, (Galilei 1656). However Torricelli knew it, as is clear in some letters from Torricelli to Galileo regarding the "theorem"; Torricelli 1919–1944, Vol. III, p. 48, pp. 51, 55, 58, 61).

2. ARCHIMEDEAN THINKING

Torricelli frequently declares and explains his Archimedean background.

> Inter omnia opera Mathematics disciplinas pertinentia, iure optimo Principem sibi locum vindicare videntur Archimedis; quae quidem ipso subtilitatis miraculo terrent animos (Torricelli 1644, *Proemium*, p. 7).

Archimedes, in the *Quadratura parabolae*, first obtains results using the mechanical approach and then reconsiders the discourse with the classical methods of geometry to confirm in a rigorous way the correctness of his results (Heath 2002). Similarly, Torricelli, with the compelling idea of duplicating the procedure, devotes many pages to proving certain theorems on the "parabolic segment", by following, the geometry used in pre-history ancients (Torricelli (1644), *Quadratura parabolae pluris modis per duplicem positionem more antiquorum absoluta*, pp. 17–54)[1] and then proving the validity of the thesis also with the "indivisibilium" (Heath 2002, *Quadratura parabolae,* pp. 253–252; pp. 55–84; Torricelli 1644, *De solido acuto hyperbolico problema alterum*, pp. 93–135). In this respect, it is interesting to note that he underlines the "concordantia" (Torricelli 1644, *De solido acuto hyperbolico problema alterum*, p. 103) of methods of varying rigour.

> Hactenus de dimensione parabolae more antiquorum dictum sit; Reliquum est eandem parabolae mensuram nova quedam, sed mirabili ratione aggrediamur; ope scilicet Geometriae Indivisibilium, et hoc diversis modis: Suppositis enim praecipui Theorematib. antiquorum tam Euclidis, quam Archimedis, licet de rebus inter se diversissimis sint, mirum est ex unoquoque eorum quadraturam parabolae facili negotio elici posse; et vive versa. Quasi ea sit commune quoddam vinculum veritatis. […] Contra vero: supposita parabolae quadratura, praedicta omnia Theoremata facile demonstrari possunt. Quod autem haec indivisibilium Geometria novum penitus inventum sit equidem non ausim affirmare. Crediderim potius veteres Geometras hoc metodo usos in inventione Theorematum difficillimorum quamquam in demonstrationibus aliam viam magis probaverint, sive ad occultandum artis arcanum, sive ne ulla invidis detractoribus proferretur occasio contradicendi (Torricelli 1644, *Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, p. 55, *op. cit*.).

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[1] In the original manuscripts of *Opera geometrica* there are some glosses to Eulid's *Elements*, to Apollonius' *Conic sections*, to Archimedes, Galileo, Cavalieri's works, et al., autograph by Torricelli.

From the previous passage there appears not only the desire to give the reader results and methods, but also to say that the indivisibles technique was not completely unknown to the ancient Greek scholars. Besides, Torricelli seems to hold onto the idea that the method of demonstration of the ancients, such as the Archimedes' method, was intentionally kept secret. He states that the ancient geometers worked according to a method "in invenzione" suitable "ad occultandum artis arcanum" (Torricelli 1644, *Quadratura parabolae per novam indivisibilium Geometriam pluribus modis absoluta*, p. 55).

However the Archimedean influence in Torricelli goes further. The well known books *De sphaera et solidis sphaeralibus* (Torricelli 1644, *Liber primus*, 3–46) present an enlargement of the Archimedean proofs of books I–II of *On the sphere and cylinder* (Heath 2002, pp. 1–90).

> […] In quibus Archimedis Doctrina de sphaera & cylindro denuo componitur, latius promovetur, et omni specie Solidorum, quae vel circa, vel intra, Sphaeram, ex conversione polygonorum regularium gigni possint, universalius Propagatur (Torricelli 1644, *De sphaera et solidis sphaeralibus*, p. 2).

In other parts Torricelli faces problems not yet solved by Archimedes, or by the other mathematicians of antiquity. With the same style as Archimedes, he does not try to arrive at the first principles of the theory and does not limit himself to a single way of demonstrating a theory.

> Veritatem praecedentis Theorematis satis per se claram, et per exempla ad initium libelli proposita confirmatam satis superque puto. Tamen ut in hac parte satisfaciam lectori etiam Indivisibilium parum amico, iterabo hanc ipsam demonstrationis in calce operis, per solitam veterum Geometrarum viam demonstrandi, longiorem quidem, sed non ideo mihi certiorem (Torricelli 1644, *De solido hyperbolico acuto problema secundum,* p. 116).

We note that the exposition of the mechanical argumentation present in Archimedes's *Method* was not known at Torricelli's time because Johan Heiberg only discovered it in 1906 (Heath 1912). Therefore, in Archimedes's writing there were lines of reasoning which, because a lack of justification, were labelled as mysterious by most scholars. Thus in such instances it was necessary to assure the reader of the validity of the thesis and also to convince him about the strictness of Archimedes' approaches, particularly exhaustion reasoning and *reductio ad absurdum*, by proving his results with some other technique.

> The appearance of approximation [in Archimedes's proofs] is surely a substantial innovation in the mathematical demonstrations and the difference between