New Trends in Mechanism Science

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New Trends in Mechanism Science

Analysis and Design

Springer

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Preface

The European Conference on Mechanism Science (EUCOMES 2010 Conference) is the third event of a series that has been started in 2006. EUCOMES has been started as a conference initiative to be a forum mainly for the European community working in Machine and Mechanism Science in order to facilitate aggregation and sharing interests and results for a better collaboration and activity visibility. The previous EUCOMES conferences were successfully held in Innsbruck, Austria (2006) and Cassino, Italy (2008). EUCOMES 2010 is taking place in Cluj-Napoca, Romania from 14 to 18 September 2010. EUCOMES 2010 is organized by the Center for Industrial Robots Simulation and Testing (CESTER) at the Faculty of Machine Building, Technical University of Cluj-Napoca under the patronage of IFToMM, the International Federation for the Promotion of Mechanism and Machine Science.

The aim of the conference is to bring together researchers, industry professionals and students from the broad ranges of disciplines referring to Mechanism Science, in an intimate, collegial and stimulating environment. The EUCOMES 2010 Conference aims to provide a special opportunity for the scientists to exchange their scientific achievements and build up national and international collaboration in the mechanism science field and its applications. This book presents the most recent research results in the mechanism science, intended to improve a variety of applications in daily life and industry. The book is published in the Springer series "Machine and Mechanism Science". The issues addressed are: Computational Kinematics, Micro-mechanisms, Mechanism Design, Mechanical Transmissions, Linkages and Manipulators, Mechanisms for Biomechanics, Experimental Mechanics, Mechanics of Robots, Dynamics, Control Issues of Mechanical Systems, Novel Designs, Applications and Teaching Methods.

EUCOMES 2010 received 100 papers and after careful review with at least two reviews for each paper, 79 papers have been accepted for publication and presentation at the conference.

We would like to express grateful thanks to IFToMM, the Romania IFToMM National Committee, the members of the International Steering Committee for the EUCOMES Conference for their co-operation: Marco Ceccarelli (University of Cassino, Italy), Burkhard Corves (University of Aachen, Germany), Manfred Husty

(University of Innsbruck, Austria), Jean-Pierre Merlet (INRIA, France), Doina Pisla (Technical University of Cluj-Napoca, Romania), Fernando Viadero (University of Cantabria, Spain) and Teresa Zielinska (Warsaw Technical University, Poland), the members of the International Award Committee and the members of the Honorary Committee.

We thank the authors who have contributed excellent papers on different subjects, covering many fields of Mechanism Science, and we are grateful to the reviewers for the time and effort they spent evaluating the papers.

We thank the Technical University of Cluj-Napoca and Faculty for Machine Building for hosting the EUCOMES 2010 Conference and we would like to thank our colleagues: Adrian Pisla, Tiberiu Itul, Tiberiu Antal, Calin Vaida, Bogdan Gherman, Marius Suciu, Dorin Lese, Ovidiu Detesan from the Local Organizing Committee, and the sponsors of this conference for their help.

Furthermore, we thank the Romanian National Authority for Scientific Research – ANCS for the financial support awarded for the EUCOMES 2010 Conference and the printing of this book.

We also thank the staff at Springer, including Nathalie Jacobs (editor), and Jolanda Karada (Karada Publishing Services) for their excellent technical and editorial support.

Cluj-Napoca, April 2010
The editors

Computational Kinematics

Design and Kinematic Analysis of a Multiple-Mode 5R2P Closed-Loop Linkage

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Abstract. This paper presents the design and kinematic analysis of a multiple-mode 5R2P linkage, which is a spatial closed-loop linkage built by combining two overconstrained mechanisms: the Bennett and RPRP linkages. In the paper, the construction of the multiple-mode 5R2P linkage and its variations is investigated. The analytical inverse kinematic solution of a 4R2P open chain is adopted to analyze the 5R2P linkage for a given driving joint angle. The analysis is then conducted for the whole revolution of the driving crank. The result suggests that the multiple-mode linkage consists of three operation modes: the 5R2P, Bennett, and RPRP modes. Transitional configurations between different modes are also identified in this paper. The multiple-mode linkage features the simpler motions of the Bennett and RPRP modes as well as the more complicated motion of the 5R2P mode without the need to disconnect the linkage.

Key words: kinematotropic mechanism, overconstrained linkage, Bennett, RPRP, operation mode

1 Introduction

Multiple-mode linkages are designed to carry out different tasks in different modes. This paper introduces a multiple-mode 5R2P linkage that possesses three operation modes by using only one actuator. The idea is to combine two overconstrained linkages, the Bennett and RPRP linkages, with a common *R* joint. The resulting linkage can move like a regular 5R2P linkage and features rather complicated motion. However, if one or more joints are locked, the 5R2P linkage can move like a Bennett or RPRP linkage. The simpler motions of the Bennett and RPRP modes can also be used to bridge different branches or circuits of the 5R2P mode.

Multiple-mode linkages stem from kinematotropic mechanisms [2, 5, 12] and belong to a class of reconfigurable mechanisms. The degree of freedom (DOF) of a kinematotropic linkage may change during operation; however, here we focus on the cases in which a multiple-mode mechanism has the same DOF in different modes. The use of overconstrained (paradoxic) linkages in the design of multiple-mode linkages was proposed only recently [6]. Building upon a previous development on

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multiple-mode 7R linkages [3], this research investigates spatial seven-link linkages with prismatic joints, with an emphasis on the 5R2P linkage. The Bennett and RPRP overconstrained linkages are used as building blocks. The proposed multiple-mode 5R2P linkage incorporates the kinematic properties of the Bennett, RPRP, and 5R2P linkages.

The kinematic analysis of a closed-loop 5R2P linkage can be modeled as the inverse kinematic analysis of a 4R2P open chain, in which we are concerned with finding all possible configurations of the linkage when given the position of its outermost body. The inverse kinematic solution of a 4R2P open chain has been obtained analytically [8], and it stems from the algorithm for solving the renowned 6R inverse kinematic problem [7, 9, 10]. In this paper, we adopt the solution method described in [8] to find all configurations of the 5R2P multiple-mode linkage for a full rotation of the input joint and identify different operation modes.

2 Construction of the Multiple-Mode Linkage

Figure 1a shows the schematic drawing of a Bennett linkage, whose joints are denoted by B (and 7), A (6), C (2), and D (1). The link lengths and joint twist angles designated in the figure must satisfy the following constraints [1]:

$$a_{76} = a_{21}; a_{62} = a_{17}$$
$$\alpha_{76} = \alpha_{21}; \alpha_{62} = \alpha_{17}$$
$$\frac{a_{76}}{\sin \alpha_{76}} = \frac{a_{62}}{\sin \alpha_{62}}$$

Figure 1b shows a RPRP linkage, whose joints are denoted by B (or 7), G (3), F (4), and E (5). The link lengths and joint twist angles designated in the figure must satisfy the following constraints [4, 11]:

$$\begin{aligned} a_{73} &= a_{57}; a_{34} = a_{45} \\ a_{73} &// a_{34}; a_{45} // a_{57} \\ \alpha_{73} &= \alpha_{45}; \alpha_{34} = \alpha_{57} \\ \alpha_{73} &+ \alpha_{34} = \alpha_{45} + \alpha_{57} = \pi \end{aligned}$$

To construct the multiple-mode linkage, we first combine the two closed-loop linkages, BACD, and BGFE, by aligning the common joint B, as shown in Figure 2a. Secondly, we remove all the links of the combined linkages and reconnect the joints in a different order to obtain a closed-loop linkage, BAGFECD. The newly constructed 5R2P linkage, as shown in Figure 2b, possesses one degree of freedom.

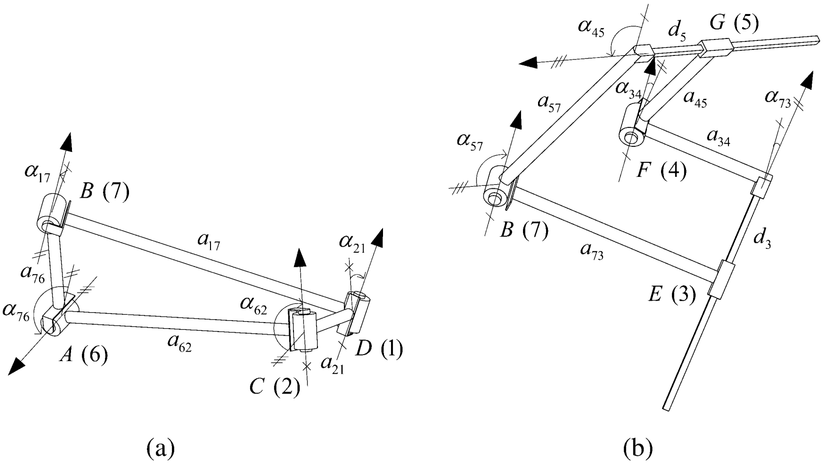

Fig. 1 (a) A Bennett linkage. (b) A RPRP overconstrained linkage.

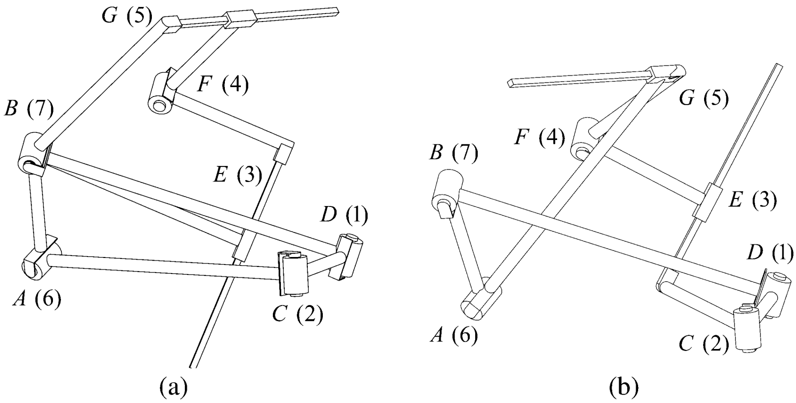

Fig. 2 (a) Combining the Bennett and RPRP linkages with a common R joint. (b) Connecting a multiple-mode 5R2P linkage.

If we take link AB of the linkage shown in Figure 2b to be the ground link and joint B to be the driving joint, the linkage moves like a regular, one-degree-of-freedom, closed-loop linkage. What makes this linkage special is that it has two more operation modes, inherited from the two original four-link linkages. In Sections 4 and 5, we will conduct the position analysis of the linkage to confirm these operation modes.

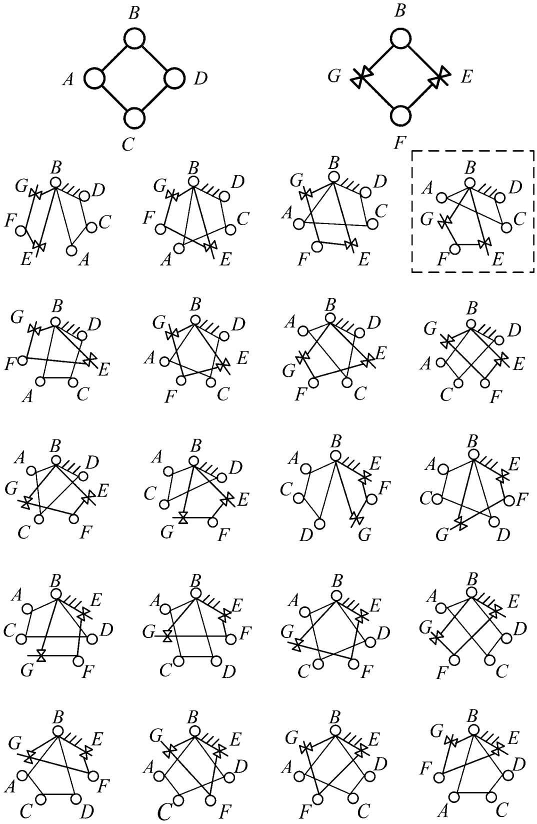

Fig. 3 Different combinations of the Bennett and RPRP linkages.

3 Variations in the Construction of the 5R2P Linkage

Before combining the two overconstrained linkages, each of the linkages, shown in Figure 1, can be in any of its feasible configurations. When combining the linkages, as shown in Figure 2a, we are free to rotate the two linkages relative to each other about joint B. Furthermore, an offset between the two linkages along joint B is allowed, though not shown in Figure 2a. Considering these variations, we will have an infinite number of different 5R2P linkages.

When reconnecting joints (the process shown in Figure 2), we must retain the orders of the original linkages. For example, the orders of the original linkages, B-A-C-D and B-G-F-E, are retained in the new linkage B-A-G-F-E-C-D. In fact, we can have 20 different combinations in constructing the 5R2P linkage. The 20 combinations are shown in Figure 3, of which the one constructed in Section 2 is identified at the upper right corner.

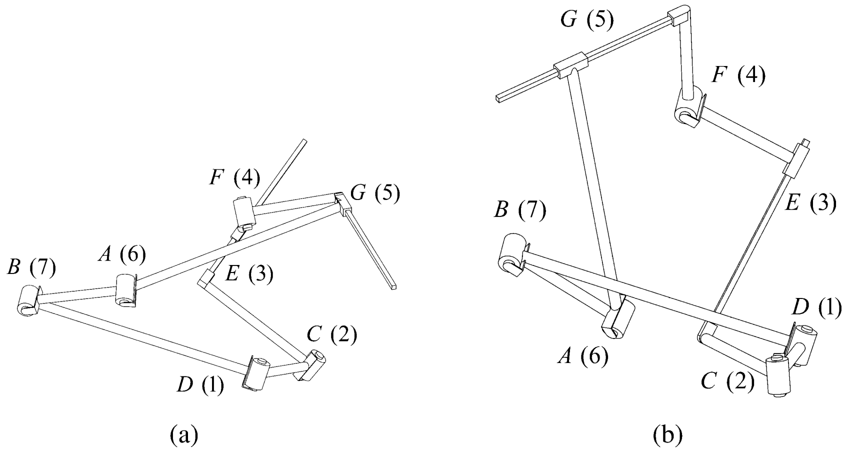

Fig. 4 (a) The 5R2P linkage in the Bennett mode. (b) The 5R2P linkage in the RPRP mode.

We can also reverse the orders of either one or both of the two original linkages to obtain different linkages. For example, we can reverse the order of the Bennett linkage when combining it with the RPRP linkage. In other words, we will combine linkages in the orders of B-D-C-A and B-G-F-E. As a result, another 20 combinations can be obtained.

4 Operation Modes and Transitional Configurations

As mentioned in Section 2, the linkage shown in Figure 2b possesses three operation modes: the Bennett, RPRP, and 5R2P modes. In addition to the regular 5R2P motion, the linkage shown in Figure 2b can be operated like a Bennett linkage or a RPRP linkage. There are several transitional configurations that allow the linkage to switch from one operation mode to another.

The configuration shown in Figure 2b is referred to as the assembly configuration of the multiple-mode linkage. It is the unique transitional configuration between the Bennett and RPRP modes. In the assembly configuration, if we lock any of joints A, C, and D, the other two joints will be automatically locked as well. As a result, the linkage can move as a RPRP linkage. On the other hand, if any of joints G, F, and E is locked in the assembly configuration, the other two joints will be locked too. Then the linkage will move like a Bennett linkage.

Note that the linkage cannot be operated in the 5R2P mode starting from the assembly configuration. We have to operate the linkage in either the Bennett or RPRP mode until it reaches a transitional configuration that allows the linkage to switch to the 5R2P mode. There is usually more than one transitional configuration for switching to the 5R2P mode. Figures 4a and 4b show the 5R2P linkage in the Bennett and RPRP modes, respectively.

Table 1 D–H parameters of the 5R2P linkage.

Joint i	α_i (deg)	a_i (unit)	d_i (unit)	θ_i (deg)
1	22.1838	11.6396	0	θ_1
2	34.0095	15.6255	2.1546	θ_2
3	47.0331	11.1573	d_3	78.2166
4	132.9669	11.1573	0	θ_4
5	142.24741	19.1424	d_5	149.7679
6	22.1838	11.6396	30.4649	θ_6
7	62.2676	27.2859	0	θ_7

The merit of the proposed multiple-mode linkage, as opposed a regular 5R2P linkage, is that it allows three different motion modes without increasing the number of actuators. Furthermore, some configurations of a regular 5R2P linkage may not be reachable without disconnecting and reassembling the linkage. With the help of the two additional operation modes, all configurations of the multiple-mode 5R2P linkage can be reached without having to disassemble or reconnect the linkage.

5 Kinematic Analysis and Numerical Example

In this section, we will briefly summarize the kinematic analysis of the multiple-mode linkage and give a numerical example to demonstrate the concepts discussed heretofore. We are concerned with finding all possible configurations of the linkage when the driving joint angle is known. Specifying the value of the input joint angle in a 5R2P linkage is equivalent to specifying the outermost body's position in a 4R2P open chain. We adopt the inverse kinematic solution procedures described in [8] without repeating them in this paper. We also utilize the Denavit and Hartenberg (D–H) convention for describing the geometry of links and joints of the linkage. The definitions of link parameters and coordinate systems are the same as those used in [10].

To present a numerical example, we follow the procedures described in Section 2 to construct a multiple-mode 5R2P linkage. The D–H parameter of the linkage is listed in Table 1. We conduct the position analysis of the linkage for the full rotation of the driving crank, using increments of one degree.

The results of the position analysis are illustrated by joint angle plots, as shown in Figure 5. Due to limited space, Figure 5 shows only the plots of angles of joints 1 and 4 against the driving joint angle. In the plots, we can see that the multiple-mode 5R2P linkage bears the three operation modes, designated by different curves. The thick solid curves correspond to the joint angles of the RPRP mode, while the thick dashed ones correspond to the joint angles of the Bennett mode. The curve of joint

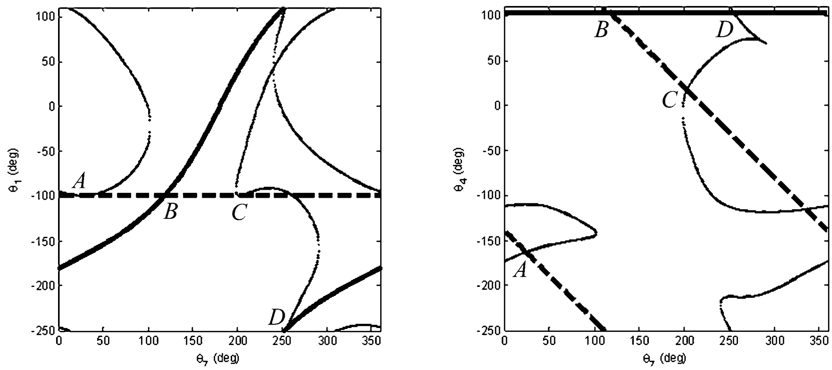

Fig. 5 Plots of two joint angles (1 and 4) against the driving joint angle (7).

Table 2 Transitional configurations of the 5R2P linkage.

Configurations	θ_7	Connected modes
A	23.75	RPRP & 5R2P
B	118.59	Bennett & RPRP
C	203.41	RPRP & 5R2P
D	256.15	Bennett & 5R2P

angle 1 is a horizontal line in the RPRP mode because joint 1 (and joints 2 and 6, not shown in the figure) is locked. Similarly, the curve corresponding to joint 4 (and joints 3 and 5, not shown in the figure) is a horizontal line in the Bennett mode. The thin dotted curves correspond to joint angles of the 5R2P mode.

Next, we need to identify the transitional configurations between different modes. By comparing the joint angle plots, we can see that there are two transitional configurations, indicated by points A and C, between the RPRP and 5R2P modes. There is one transitional configuration between the Bennett and 5R2P modes, indicated by point D in the plots. Finally, point B indicates the transitional configuration between the RPRP and Bennett modes. The corresponding driving joint angles of the transitional configurations are summarized in Table 2.

6 Conclusion

This paper presents the construction of a multiple-mode 5R2P linkage by using two overconstrained linkages: the Bennett and RPRP linkages. In addition to the 5R2P operation mode, this linkage also inherits the kinematic characteristics from the two overconstrained linkages. The proposed multiple-mode linkage uses only one actuator, and it eliminates the need to disconnect and reassemble the linkage in order to operate in different modes. When operating in either the Bennett or RPRP mode, three of the joints will be automatically locked. To switch to the 5R2P mode, we need to trigger one of the locked joints at a transitional configuration.

This paper then conducts the kinematic analysis of the 5R2P multiple-mode linkage by using the inverse kinematic solution of a 4R2P linkage. The result of the analysis confirms the three operation modes and identifies the transitional configurations between different modes.

Acknowledgments

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Synthesis of Spatial RPRP Loops for a Given Screw System

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Abstract. The dimensional synthesis of spatial chains for a prescribed set of positions can be used for the design of parallel robots by joining the solutions of each serial chain at the end effector. In some cases, this may yield a system with negative mobility. The synthesis of some overconstrained but movable linkages can be done if the finite screw system associated to the motion of the linkage is known. For these cases, the screw system could be related to the finite tasks positions traditionally defined in the synthesis theory. This paper presents the simplest case, that of the spatial RPRP closed chain, for which one solution exists.

Key words: dimensional synthesis, overconstrained linkages, finite screw systems

1 Introduction

Synthesis of parallel robots has focused mainly on type or structural synthesis, using group theory, screw theory, or geometric methods, see for instance [3]. Dimensional synthesis examples exist, which focus on optimizing performance indices [7, 10] or on reachable workspace sizing [2, 12]; see also [14].

The dimensional synthesis of spatial serial chains for a prescribed set of positions can be used for the design of parallel robots by synthesizing all supporting legs for the same set of positions. There are a few examples of finite-position dimensional synthesis of parallel robots in the literature, most of them doing partial synthesis. Wolbrecht et al. [21] perform synthesis of 3-RRS, 4-RRS and 5-RRS symmetric parallel manipulators; Kim and Tsai [11] and Rao [20] solve the partial kinematic synthesis of a 3-RPS parallel manipulator. This method yields, in some cases, a system with negative mobility.

One interesting question is whether the finite-screw surfaces generated by the task positions can give any information for the synthesis of the overconstrained closed linkages. Using Parkin's definition for pitch [15], the screws corresponding to finite displacements can form screw systems. Huang [4] showed that the single RR chain forms a finite screw system of third order; however, the set of finite displacements of the coupler of the Bennett linkage form a cylindroid, which is a gen-

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eral 2-system of screws [5]. Baker [1] has also studied the motion of the Bennett linkage. Perez and McCarthy [16] used two arbitrary displacements to generate the cylindroid associated to the Bennett linkage in order to perform dimensional synthesis. Husty et al. [9] use the geometry of the Study quadric to obtain simpler equations for the synthesis and analysis. Following this approach, Pfurner and Husty [19] present the constraint manifold of overconstrained 2-3R parallel robots as 6R closed chains.

In this paper, the focus is on the simplest of the overconstrained linkages, the closed spatial RPRP linkage. Recently, Huang [6] has shown that the set of screws corresponding to displacements of this linkage forms a 2-screw system. We use this result in order to synthesize RPRP linkages with positive mobility and for a given shape of the screw system of the relative displacements. In order to do so, we state the design equations using the Clifford algebra of dual quaternions [18], which has a direct relation to the screw system. The design yields a single RPRP linkage.

2 Clifford Algebra Equations for the Synthesis

2.1 Forward Kinematics

The approach used in this paper for stating design equations is based on the method of Lee and Mavroidis [13]. They equate the forward kinematics of a serial chain to a set of goal displacements and consider the Denavit-Hartenberg parameters as variables. A more efficient formulation for our purpose consists of stating the forward kinematics of relative displacements using the even Clifford subalgebra $C^+(P^3)$, also known as dual quaternions. In this section, we follow the approach presented in [18].

The Plücker coordinates $\mathsf{S} = (\mathbf{s}, \mathbf{c}\times\mathbf{s})$ of a line can be identified with the Clifford algebra element $\mathsf{S} = \mathbf{s} + \varepsilon\mathbf{c}\times\mathbf{s}$. Similarly, the screw $\mathsf{J} = (\mathbf{s}, \mathbf{v})$ becomes the element $\mathsf{J} = \mathbf{s} + \varepsilon\mathbf{v}$. Using the Clifford product we can compute the exponential of the screw $\frac{\theta}{2}\mathsf{J}$,

$$e^{\frac{\theta}{2}\mathsf{J}} = \left(\cos\frac{\theta}{2} - \frac{d}{2}\sin\frac{\theta}{2}\varepsilon\right) + \left(\sin\frac{\theta}{2} + \frac{d}{2}\cos\frac{\theta}{2}\varepsilon\right)\mathsf{S} = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\mathsf{S}. \quad (1)$$

The exponential of a screw defines a unit dual quaternion, which can be identified with a relative displacement from an initial position to a final position in terms of a rotation around and slide along an axis.

For a serial chain with n joints, in which each joint can rotate an angle θ_i around, and slide the distance d_i along, the axis S_i, for $i = 1, \ldots, n$, the forward kinematics of relative displacements (with respect to a reference position) can be expressed as the composition of Clifford algebra elements. Let θ_0 and $\mathbf{d}_0$ be the joint parameters of this chain when in the reference configuration, so we have $\Delta\hat{\theta} = (\theta - \theta_\mathbf{0} + (\mathbf{d} - \mathbf{d_0})\varepsilon)$. Then, the movement from this reference configuration is defined by

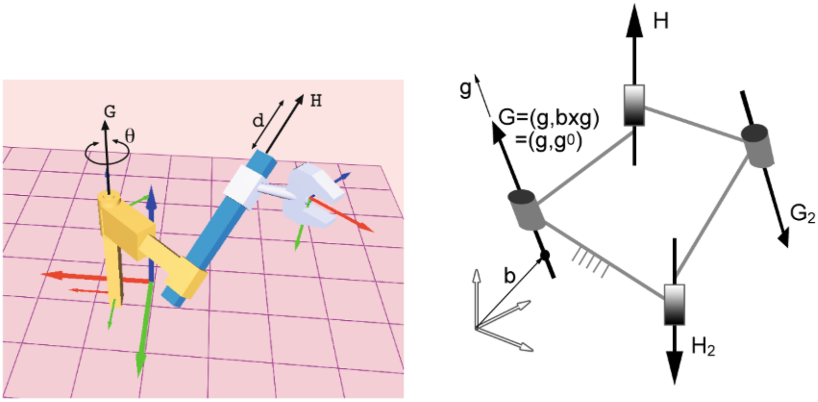

Fig. 1 The RP serial chain, left; the RPRP closed linkage, right.

$$\begin{aligned}\hat{Q}(\Delta\hat{\theta}) &= e^{\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n}, \\ &= \left(c\frac{\Delta\hat{\theta}_1}{2} + s\frac{\Delta\hat{\theta}_1}{2}\mathsf{S}_1\right)\left(c\frac{\Delta\hat{\theta}_2}{2} + s\frac{\Delta\hat{\theta}_2}{2}\mathsf{S}_2\right)\cdots\left(c\frac{\Delta\hat{\theta}_n}{2} + s\frac{\Delta\hat{\theta}_n}{2}\mathsf{S}_n\right). \end{aligned} \quad (2)$$

Note that s and c denote the sine and cosine functions, respectively.

The RPRP linkage has a mobility $M = -2$ using the Kutzbach–Groebler formula; however, for certain dimensions of the links, it moves with one degree of freedom. The RPRP linkage can be seen as a serial RP chain and a serial PR chain joined at their end-effectors.

The RP serial chain consists of a revolute joint followed by a prismatic joint. Figure 1 shows the RP serial chain and a sketch of the RPRP linkage with its axes. In the PR serial chain, the order of the joints in the chain is switched. For both the RP and PR serial chains, let $\mathsf{G} = \mathbf{g} + \varepsilon\mathbf{g}^0$ be the revolute joint axis, with rotation θ, and $\mathsf{H} = \mathbf{h} + \varepsilon\mathbf{h}^0$ the prismatic joint axis, with slide d. Notice that, for synthesis purposes, the location of the slider, given by $\mathbf{h}^0$, is irrelevant. The Clifford algebra forward kinematics equations of the RP chain are

$$\begin{aligned}\hat{Q}_{RP}(\Delta\theta, \Delta d) &= \left(\cos\frac{\Delta\theta}{2} + \sin\frac{\Delta\theta}{2}\mathsf{G}\right)\left(1 + \varepsilon\frac{\Delta d}{2}\mathsf{H}\right) \\ &= \left(c\frac{\Delta\theta}{2} + s\frac{\Delta\theta}{2}\mathbf{g}\right) + \varepsilon\left(-\frac{\Delta d}{2}s\frac{\Delta\theta}{2}\mathbf{g}\cdot\mathbf{h} + \frac{\Delta d}{2}c\frac{\Delta\theta}{2}\mathbf{h} + s\frac{\Delta\theta}{2}\mathbf{g}^0 + \frac{\Delta d}{2}s\frac{\Delta\theta}{2}\mathbf{g}\times\mathbf{h}\right). \end{aligned} \quad (3)$$

For the *PR* chain, the only difference is a negative sign in the cross product.

2.2 Design Equations and Counting

The design equations are created when a set of task positions are defined. The design variables that determine the dimensions of the chain are the position of the joint axes in the reference configuration.

Given a set of task positions expressed as relative displacements, $\hat{P}_{1j} = \cos\frac{\Delta\hat{\phi}_{1j}}{2} + \sin\frac{\Delta\hat{\phi}_{1j}}{2}\mathsf{P}_{1j}$, $j = 2,\ldots,m$, we equate them to the forward kinematics in Eq. (2),

$$\hat{P}_{1j} = e^{\frac{\Delta\hat{\theta}_{1j}}{2}\mathsf{S}_1} e^{\frac{\Delta\hat{\theta}_{2j}}{2}\mathsf{S}_2} \cdots e^{\frac{\Delta\hat{\theta}_{nj}}{2}\mathsf{S}_n}, \quad j = 2,\ldots,m. \tag{4}$$

The result is $8(m-1)$ design equations. The unknowns are the n joint axes S_i, $i = 1,\ldots,n$, and the $n(m-1)$ pairs of joint parameters $\Delta\hat{\theta}_{ij} = \Delta\theta_{ij} + \Delta d_{ij}\varepsilon$.

For the RP (and similarly the PR) serial chains, the design equations are

$$\hat{Q}_{RP}(\Delta\theta^j, \Delta d^j) = \hat{P}_{1j}, \quad j = 1,\ldots,m. \tag{5}$$

The counting of independent equations and unknowns allows to define the maximum number of arbitrary positions m that can be reached, based only on the type and number of joints of the serial chain, see [17] for details. Consider a serial chain with r revolute and p prismatic joints. The maximum number of task positions is given by m in Eq.(6).

For serial chains with less than three revolute joints, the structure of semi-direct product of the composition of displacements needs to be considered, and the maximum number of rotations m_R needs to be calculated too. Assuming that the orientations are given and that both the directions of the revolute joints and the angles to reach the task orientations are known, we can count, in a similar fashion, the number of translations m_T that the chain can be defined for,

$$m = \frac{3r+p+6}{6-(r+p)}, \qquad m_R = \frac{3+r}{3-r}, \qquad m_T = \frac{2r+p+3-c}{3-p}. \tag{6}$$

In order to determine the maximum number of task positions for the RP and PR chains, we apply Eq. (6), to obtain $m = 2.5$ task positions. Additional information is obtaining when computing $m_R = 2$ task rotations, and $m_T = 3$ task translations. Hence, we can define one arbitrary relative displacement and a second relative displacement whose orientation is not general.

3 Screw System for the RPRP Linkage

In the context of this paper, a screw surface is a ruled surface in which the lines correspond to relative displacements. A screw surface will be a screw system if it is closed under addition and scalar multiplication, that is, if it can be written as a linear combination of screws.

The linear combination of two arbitrary screws representing relative displacements form a 2-system known as the cylindroid, which is the manifold for the relative displacements of the closed 4R linkage. Huang [6], by intersecting the 3-systems associated with the RP and PR dyads, shows that the screw surface of the closed RPRP linkage forms a 2-system of a special type, the fourth special type according to Hunt [8], also known as 2-IB [22]. The screws of this system are parallel, coplanar screws whose pitches vary linearly with their distance.

This screw system can be generated by two screws with same direction and finite pitches. Notice that this coincides with the results of the counting for the synthesis of the RP (or PR) serial chain. This allows us to define and use the screw system as input for the dimensional synthesis of the closed RPRP chain.

We have several strategies for doing so. For instance, we can select a first relative displacement, $\hat{S}_{12} = \cos\frac{\Delta\hat{\psi}}{2} + \sin\frac{\Delta\hat{\psi}}{2}(\mathbf{s}_{12} + \varepsilon\mathbf{s}^0_{12})$. The rotation axis of the displacement, $\mathbf{s}_{12}$ is common to both $\hat{S}_{12}$ and the second relative displacement. We set $\mathbf{s}_{12} = \mathbf{s}_{13}$ and select a rotation angle to define the relative rotation $\hat{s}_{13}$.

We can then set the slope of the pitch distribution in order to shape the screw system. The pitch for the finite displacement screws is [15]

$$p_{1i} = \frac{\frac{\Delta t_{1i}}{2}}{\tan\frac{\Delta\psi_{1i}}{2}}, \tag{7}$$

directly calculated from the dual quaternion using $p_{1i} = \frac{\mathbf{s}_{1i}\cdot\mathbf{s}^0_{1i}}{\mathbf{s}_{1i}\cdot\mathbf{s}_{1i}}$. Similarly, a point on the screw axis is calculated as

$$\mathbf{c}_{1i} = \mathbf{s}_{1i} \times \mathbf{s}^0_{1i}. \tag{8}$$

Define the slope of the distribution as

$$K = \frac{p_{13} - p_{12}}{||\mathbf{c}_{13} - \mathbf{c}_{12}||} \tag{9}$$

If we set the value of K, we can solve for Δt_{13} in order to define the pitch of the second relative displacement, the location of its screw axis being defined. This defines the screw system; by converting to absolute displacements, we can easily check whether the trajectory of the end-effector is acceptable.

4 Dimensional Synthesis of the RPRP Linkage for a Prescribed Screw System

The solution of the RP, and similarly, PR chains is simple and yields one solution. Given an arbitrary relative displacement $\hat{Q}_{12} = (q^w_{12} + \mathbf{q}_{12}) + \varepsilon(q^{w0}_{12} + \mathbf{q}^0_{12})$ and a second displacement $\hat{Q}_{13}$ such that both have same direction and a given pitch distribution, as explained in previous section, we equate them to the forward kinematics in Eq.(3). We can solve for the direction of the revolute joint $\mathbf{g}$ and the rotation angles,

$$\mathbf{g} = \frac{\mathbf{q}_{12}}{||\mathbf{q}_{12}||}, \qquad \tan\frac{\Delta\theta_{1i}}{2} = \frac{||\mathbf{q}_{1i}||}{q^w_{1i}}, \quad i = 2,3. \tag{10}$$

The equations for the dual part are linear in the moment of the revolute joint, $\mathbf{g}^0$,

$$\mathbf{g}^0 = \frac{1}{\sin\frac{\Delta\theta_{1i}}{2}}\left(\mathbf{q}^0_{1i} - \frac{\Delta d_{1i}}{2}\left(\cos\frac{\Delta\theta_{1i}}{2}\mathbf{h} + \sin\frac{\Delta\theta_{1i}}{2}\mathbf{g}\times\mathbf{h}\right)\right), \quad i = 1,2. \tag{11}$$

Equating the solution of $\mathbf{g}^0$ for both relative displacements, we can solve linearly for $\mathbf{h}$ as a function of the slides Δd_{12}, Δd_{13}. The relation between the slides is given by the pitch condition,

$$\frac{q^{w0}_{12}}{\frac{\Delta d_{12}}{2}\sin\frac{\Delta\theta_{12}}{2}} = \frac{q^{w0}_{13}}{\frac{\Delta d_{13}}{2}\sin\frac{\Delta\theta_{13}}{2}} \tag{12}$$

Imposing $||h|| = 1$, we can solve for the slides to obtain one solution.

Using the same process, we can solve for the PR serial chain.

5 Example

The dual quaternions in Table 1 were generated as explained. $\hat{S}_{12}$ has been randomly generated, while the rotation in $\hat{S}_{13}$ is such that it belongs to the workspace of the chain. We select $\mathbf{c}_{13} = (-1.237, 2.541, -1.601)$, randomly generated. Set a value of the slope of the pitch distribution, $K = 0.48$, to create the second displacement.

These two screws generate the screw system in Figure 2, where the length of each screw is proportional to its pitch. Some of the corresponding absolute displacements of the trajectory are included in the same figure.

Table 1 Goal relative displacements for the RP and PR chains.

$(0.459, -0.133, -0.565, -0.672) + \varepsilon(1.660, 0.343, -0.019, 1.082)$
$(0.135, -0.039, -0.167, 0.976) + \varepsilon(0.023, -0.570, -0.923, -0.184)$

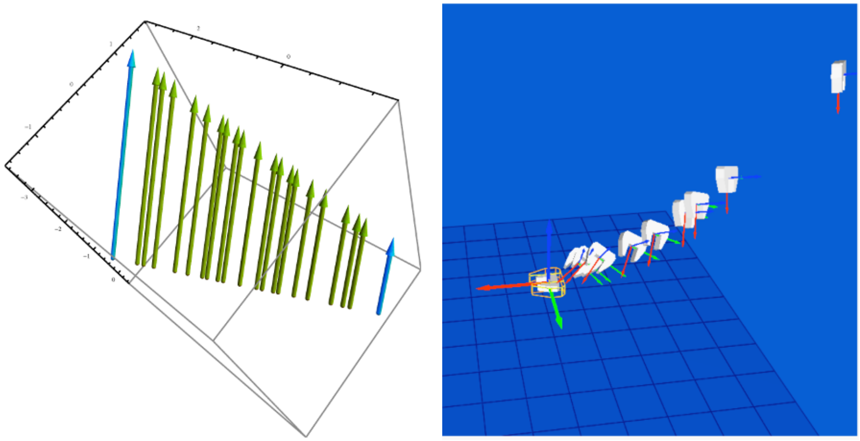

Fig. 2 Left: screw system generated by S_{12} and S_{13} (shown in corners). Right: corresponding absolute displacements.

Table 2 Joint axes for the RPRP linkage at the reference configuration.

Chain	Revolute joint G	Prismatic joint **h**	Rotations $(\theta_{12},\theta_{13})$	Slides (d_{12},d_{13})
RP	$(-0.620, 0.179, 0.763)+$ $\varepsilon(-0.436,-3.166,0.389)$	$(-0.087, 0.893, 0.442)$	(-264.5,-25.2)	(5.30,-3.05)
PR	$(0.620,-0.179,-0.763)+$ $\varepsilon(2.316,-1.826,2.310)$	$(-0.274, 0.065,-0.959)$	(264.5,25.2)	(-5.30,3.05)

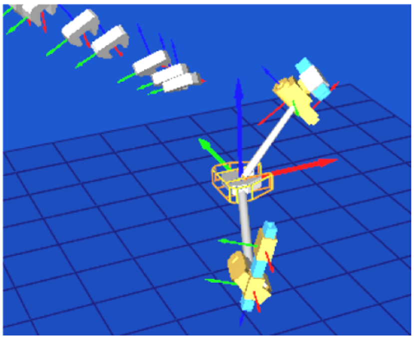

Fig. 3 The RPRP linkage reaching the first position.

We obtain one solution for the RPRP linkage, specified in Table 2 as the Plücker coordinates of the axes and the joint variables to reach the positions.

Comparing these results to the joint variable conditions in [6] we can see that our solution corresponds to the unfolded RPRP linkage. Figure 3 shows the chain reaching the three displacements, using the identity as reference displacement.