Exotic Option Pricing and Advanced Lévy Models

Edited by

Andreas E. Kyprianou, Wim Schoutens and Paul Wilmott

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Contents

Contributors

Hansjörg Albrecher
Department of Mathematics, Graz University of Technology, Steyrergasse 30, A-8010 Graz, Austria

Ariel Almendral
Norwegian Computing Center, Gaustadalleen 23, Postbox 114, Blindern, N-0314 Oslo, Norway

Pauline Barrieu
Statistics Department, London School of Economics, Houghton Street, London, WC2A 2AE, UK

Nadine Bellamy
Equipe d'Analyse et Probabilités, Université d'Evry Val d'Essonne, Rue du Père Jarlan, 91025 Evry Cedex, France

Peter Carr
Bloomberg LP, 731 Lexington Avenue, New York, NY 10022, USA

Terence Chan
School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh, EH14 4AS, Scotland, UK

José Manuel Corcuera
Facultat de Matematiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain

Ernst Eberlein
Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, D-79104, Freiburg, Germany

Pavel V. Gapeev
Institute of Control Sciences, Russian Academy of Sciences, Profsoyuznaya Str. 65, 117997 Moscow, Russia

Hélyette Geman
University of Paris Dauphine, Paris, France and ESSEC-Finance Department, 95021 Cergy-Pontoise, France

Ali Hirsa
Caspian Capital Management, LLC, 745 Fifth Avenue, 28th Floor, New York, NY 10151, USA

Jan Kallsen
HVB-Institute for Mathematical Finance, Munich University of Technology, D-85747 Garching, Germany

Christoph Kuehn
Johann Wolfgang Goethe-Universität, Fachbereich Mathematik (Fach 187), D-60054 Frankfurt am Main, Germany

Andreas E. Kyprianou
School of Mathematical and Computer Sciences, Heriot Watt University, Edinburgh, EH14 4AS, Scotland, UK

R. Loeffen
Department of Mathematics, University of Utrecht, PO Box 80.010, 3508 TA Utrecht, The Netherlands

Dilip B. Madan
Department of Finance, Robert H. School of Business, Van Munching Hall, University of Maryland, College Park, MD 20742, USA

David Nualart
Facultat de Matematiques, Universitat de Barcelona, Gran Via de les Corts Catalanes 585, E-08007 Barcelona, Spain

Antonis Papapantoleon
Department of Mathematical Stochastics, University of Freiburg, Eckerstraße 1, D-79104, Freiburg, Germany

Goran Peskir
Department of Mathematical Sciences, University of Aarhus, Ny Munkegade, DK-8000, Aarhus, Denmark

Wim Schoutens
Katholieke Universiteit Leuven - U.C.S, W. de Croylaan 54, B-3001 Leuven, Belgium

Irwin Simons
ING SWE, Financial Modelling, Marnixlaan 24, B-1000 Brussels, Belgium

Jurgen Tistaert
ING SWE, Financial Modelling, Marnixlaan 24, B-1000 Brussels, Belgium

Nadia Uys
Programme for Advanced Mathematics of Finance, School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Witwatersrand 2050, South Africa

Nick Webber
Warwick Business School, University of Warwick, Coventry CV4 7AL, UK

Preface

Since around the turn of the millennium there has been a general acceptance that one of the more practical improvements one may make in the light of the shortfalls of the classical Black–Scholes model is to replace the underlying source of randomness, a Brownian motion, by a Lévy process. Working with Lévy processes allows one to capture distributional characteristics in the stock returns such as semi-heavy tails and asymmetry, as well as allowing for jumps in the price process with the interpretation as market shocks and effects due to trading taking place in business time rather than real time. In addition, Lévy processes in general, as well as having the same properties as Brownian motion in the form of stationary independent increments, have many well understood probabilistic and analytical properties which make them attractive as mathematical tools.

At the same time, exotic derivatives are gaining increasing importance as financial instruments and are traded nowadays in large quantities in over the counter markets. The consequence of working with markets driven by Lévy processes forces a number of new mathematical challenges with respect to exotic derivatives. Many exotic options are based on the evolving historical path of the underlying. In terms of pricing and hedging, this requires an understanding of fluctuation theory, stochastic calculus and distributional decompositions associated with Lévy processes. This current volume is a compendium of articles, each of which consists of a discursive review and recent research on the topic of *Exotic Option Pricing and Advanced Lévy Models* written by leading scientists in this field.

This text is organized as follows. The first two chapters can be seen as an introduction to Lévy processes and their applications. The first chapter, by A. E. Kyprianou and R. Loeffen, gives a brief introduction to Lévy processes, providing several examples which are commonly used in finance, as well as examining in more detail some of their fine and coarse path properties. To apply Lévy processes in practice one needs good numerics. In Chapter 2, N. Webber discusses recent progress in the development of simulation methods suitable for most of the widely used Lévy processes. Speed-up methods, bridge algorithms and stratified sampling are some of the many ingredients. These techniques are applied in the context of the valuation of different kinds of exotic options.

In the second part, one can see Lévy-driven equity models at work. In Chapter 3, H. Geman and D. Madan use pure jump models, in particular from the CGMY class, for the evolution of stock prices and investigate in this setting the relationship between the statistical and risk-neutral densities. Statistical estimation is conducted on different world indexes. Their conclusions depart from the standard applications of utility theory to asset pricing which assume a representative agent who is long the market. They argue that one must have at a minimum a two-agent model in which some weight is given to an agent who is short the market. In Chapter 4, W. Schoutens, E. Simons and J. Tistaert calibrate different Lévy-based stochastic volatility models to a real market option surface and price by Monte Carlo techniques a range of exotics options. Although the different models discussed can all be nicely calibrated to the option surface – leading to almost identical vanilla prices – exotic option prices under the different models discussed can differ considerably. This investigation is pushed further by looking at the prices of moment derivatives, a new kind of derivative paying out realized higher moments. Even more pronounced differences are reported in this case. The study reveals that there is a clear issue of model risk and warns of blind use of fancy models in the realm of exotic options.

The third part is devoted to pricing, hedging and general theory of different exotics options of a European nature. In Chapter 5, E. Eberlein and A. Papapantoleon consider time-inhomogeneous Lévy processes (or additive processes) to give a better explanation of the so-called 'volatility smile', as well as the 'term structure of smiles'. They derive different kinds of symmetry relations for various exotic options. Their contribution also contains an extensive review of current literature on exotics driven in Lévy markets. In Chapter 6, H. Albrecher and W. Schoutens present a simple static super-hedging strategy for the Asian option, based on stop-loss transforms and comonotonic theory. A numerical implementation is given in detail and the hedging performance is illustrated for several stochastic volatility models. Real options form the main theme of Chapter 7, authored by P. Barrieu and N. Bellamy. There, the impact of market crises on investment decisions is analysed through real options under a jump-diffusion model, where the jumps characterize the crisis effects. In Chapter 8, J.M. Corcuera, D. Nualart and W. Schoutens show how moment derivatives can complete Lévy-type markets in the sense that, by allowing trade in these derivatives, any contingent claim can be perfectly hedged by a dynamic portfolio in terms of bonds, stocks and moment-derivative related products.

In the fourth part, exotics of an American nature are considered. Optimal stopping problems are central here. Chapter 9 is a contribution at the special request of the editors. This consists of T. Chan's original unpublished manuscript dating back to early 2000, in which many important features of the perpetual American put pricing problem are observed for the case of a Lévy-driven stock which has no positive jumps. G. Peskir and N. Uys work in Chapter 10 under the traditional Black–Scholes market but consider a new type of Asian option where the holder may exercise at any time up to the expiry of the option. Using recent techniques developed by Peskir concerning local time–space calculus, they are able to give an integral equation characterizing uniquely the optimal exercise boundary. Solving this integral equation numerically brings forward stability issues connected with the Hartman–Watson distribution. In Chapter 11, P. Carr and A. Hirsa give forward equations for the value of an American put in a Lévy market. A numerical scheme for the VG case for very fast pricing of an American put is given in its Appendix. In the same spirit, A. Almendral discusses the numerical valuation of American options under the CGMY model. A numerical solution scheme for the Partial-Integro-Differential Equation is provided; computations are accelerated by the Fast-Fourier Transform. Pricing American options and their early exercise boundaries can be carried out within seconds.

The final part considers game options. In Chapter 13, C. Kühn and J. Kallsen give a review of the very recent literature concerning game-type options, that is, options in which both holder and writer have the right to exercise. Game-type options are very closely related to convertible bonds and Kühn and Kallsen also bring this point forward in their contribution.

Last, but by far not least, P. Gapeev gives a concrete example of a new game-type option within the Black–Scholes market for which an explicit representation can be obtained.

We should like to thank all contributors for working hard to keep to the tempo that has allowed us to compile this text within a reasonable period of time. We would also like to heartily thank the referees, all of whom responded gracefully to the firm request to produce their reports within a shorter than normal period of time and without compromising their integrity.

This book grew out of the 2004 Workshop, *Exotic Option Pricing under Advanced Lévy Models*, hosted at EURANDOM in The Netherlands. In addition to the excellent managerial and organizational support offered by EURANDOM, it was generously supported by grants from Nederlands Organizatie voor Wetenschappelijk Onderzoek (The Dutch Organization for Scientific Research), Koninklijke Nederlandse Akademie van Wetenschappen (The Royal Dutch Academy of Science) and *The Journal of Applied Econometrics*. Special thanks goes to Jef Teugels and Lucienne Coolen. Thanks also to `wilmott.com` and `mathfinance.de` for publicizing the event.

A. E. Kyprianou, Edinburgh, UK
W. Schoutens, Leuven, Belgium
P. Wilmott, London, UK

About the Editors

Andreas E. Kyprianou
Address: School of Mathematical and Computer Sciences, Heriot Watt University, Edinburgh, EH14 4AS, Scotland, UK
E-mail: `kyprianou@ma.hw.ac.uk`
Affiliation: Heriot Watt University, Scotland, UK

Andreas Kyprianou has a degree in mathematics from Oxford University and a PhD in probability theory from Sheffield University. He has held academic positions in the Mathematics and/or Statistics Departments at The London School of Economics, Edinburgh University, Utrecht University and, currently, Heriot Watt University. He has also worked for nearly two years as a research mathematician with Shell International Exploration and Production. His research interests are focused on pure and applied probability with recent focus on Lévy processes. He has taught a range of courses on probability theory, stochastic analysis, financial stochastics and Lévy processes on the Amsterdam–Utrecht Masters programme in Stochastics and Financial Mathematics and the MSc programme in Financial Mathematics at Edinburgh University.

Wim Schoutens
Address: Katholieke Universiteit Leuven – UCS, W. De Croylaan 54, B-3001 Leuven, Belgium
E-mail: `Wim.Schoutens@wis.kuleuven.be`
Affiliation: Katholieke Universiteit Leuven, Belgium

Wim Schoutens has a degree in Computer Science and a PhD in Science (Mathematics). He is a research professor at the Department of Mathematics at the Catholic University of Leuven (Katholieke Universiteit Leuven), Belgium. He has been a consultant to the banking industry and is the author of the Wiley book *Lévy Processes in Finance – Pricing Financial Derivatives*.

His research interests are focused on financial mathematics and stochastic processes. He currently teaches several courses related to financial engineering in different Master programmes.

Paul Wilmott
Address: 'Wherever I lay my hat'
E-mail: paul@wilmott.com
Affiliation: Various

Paul Wilmott has undergraduate and DPhil degrees in mathematics. He has written over 100 articles on mathematical modeling and finance, as well as internationally acclaimed books including *Paul Wilmott on Quantitative Finance*, published by Wiley. Paul has extensive consulting experience in quantitative finance with leading US and European financial institutions. He has founded a university degree course and the popular Certificate in Quantitative Finance. Paul also manages wilmott.com.

About the Contributors

Hansjoerg Albrecher is Associate Professor of Applied Mathematics at the Graz University of Technology. He studied Mathematics in Graz, Limerick and Baltimore, receiving his doctorate in 2001. He held visiting appointments at the Katholieke Universiteit Leuven and the University of Aarhus. Research interests include ruin theory, stochastic simulation and quantitative finance.

Ariel Almendral will take up a research position at the Norwegian Computing Center, starting in August 2005. In 2004, he obtained his PhD from the University of Oslo, Norway. In his thesis he focused on numerical methods for financial derivatives in the presence of jump processes, from a differential equation perspective. Parts of his PhD research were carried out at Delft University of Technology, The Netherlands, where he held a postdoctoral position for a year.

Pauline Barrieu has been a lecturer in the Department of Statistics at the London School of Economics since 2002, after obtaining a PhD in finance (doctorate HEC, France) and a PhD in Mathematics (University of Paris 6, France). Her research interests are mainly problems at the interface of insurance and finance, in particular, optimal design of new types of derivatives and securitization. She also works on quantitative methods for assessing financial and non-financial risks, on stochastic optimization and environmental economics.

Nadine Bellamy is Associate Professor in Mathematics at the University of Evry, France. Her PhD thesis (University of Evry, 1999) deals with hedging and pricing in markets driven by discontinuous processes and her current research interests are related to optimization and real options problems.

Dr Peter Carr heads Quantitative Research at Bloomberg LP. He also directs the Masters in Mathematical Finance program at NYU's Courant Institute. Formerly, Dr Carr was a finance professor for eight years at Cornell University. Since receiving his PhD in Finance from UCLA in 1989, he has published extensively in both academic and industry-oriented journals. He has recently won awards from *Wilmott Magazine* for Cutting Edge Research and from *Risk Magazine* for Quant of the Year.

Terence Chan completed his PhD at Cambridge University UK after which he obtained his current position at Heriot-Watt University, Edinburgh. Among his research interests are Lévy processes but he only occasionally dabbles in financial mathematics to maintain the illusion that he is doing something of practical use!

José Manuel Corcuera is an associate professor since 1997 at the Faculty of Mathematics of the University of Barcelona. His main research interest is in the theoretical aspects of statistics and quantitative finance.

Ernst Eberlein is Professor of Stochastics and Mathematical Finance at the University of Freiburg. He is a co-founder of the Freiburg Center for Data Analysis and Modeling (FDM), an elected member of the International Statistical Institute and at present Executive Secretary of the Bachelier Finance Society. His current research focuses on statistical analysis and realistic modeling of financial markets, risk management, as well as pricing of derivatives.

Pavel Gapeev was born in Moscow in 1976. He studied and obtained his PhD in Stochastics at Moscow State University in 2001. He is now working as a Senior Researcher at the Institute of Control Sciences, Russian Academy of Sciences in Moscow. He has held a visiting appointment at Humboldt University, Berlin (2001/2002) in addition to some short term research visits to Aarhus, Bochum, Copenhagen, Frankfurt, Helsinki, and Zurich. His main field of research is stochastic analysis and its applications into financial mathematics, optimal control, optimal stopping, and quickest detection. Apart from mathematics he is interested in arts, sports and travelling, and enjoys playing the violin.

Hélyette Geman is a Professor of Finance at the University Paris Dauphine and ESSEC Graduate Business School. She is a graduate of Ecole Normale Superieure in Mathematics, holds a Masters degree in theoretical physics and a PhD in mathematics from the University Pierre et Marie Curie and a PhD in Finance from the University Pantheon Sorbonne. Professor Geman has published more than 60 papers in major finance journals including the Journal of Finance, Mathematical Finance, Journal of Financial Economics, Journal of Banking and Finance and Journal of Business. Professor Geman's research includes asset price modelling using jump-diffusions and Lévy processes, commodity forward curve modelling and exotic option pricing for which she won the first prize of the Merrill Lynch Awards. She has written a book entitled *Commodities and Commodity Derivatives* (John Wiley & Sons Ltd, 2005).

Ali Hirsa joined Caspian Capital Management as the Head of Analytical Trading Strategy in April 2004. At CCM his responsibilities include design and testing of new trading strategies. Prior to his current position, Ali worked at Morgan Stanley for four years. Ali is also an adjunct professor at Columbia University and New York University where he teaches in the mathematics of finance program. Ali received his PhD in applied mathematics from University of Maryland at College Park under the supervision of Dilip B. Madan.

Jan Kallsen is a Professor of Mathematical Finance at Munich University of Technology. His research interests include pricing and hedging in incomplete markets and the general theory of stochastic processes.

Christoph Kühn is Junior Professor at the Frankfurt MathFinance Institute. He holds a diploma in Mathematical Economics from the University of Marburg and a PhD in mathematics from Munich University of Technology. His main research interests are pricing and hedging of derivatives in incomplete markets and the microstruture of financial markets.

Ronnie Loeffen was born in 1981 in the Netherlands and has recently received a Master's degree in Mathematics at the University Utrecht. The subject of his Master's thesis was American options on a jump-diffusion model.

Dilip B. Madan is Professor of Finance at the Robert H. Smith School of Business, University of Maryland. He is co-editor of Mathematical Finance and served as President of the Bachelier Finance Society 2002–2003. He has been a consultant to Morgan Stanley since 1996. He now also consults for Bloomberg and Caspian Capital. His primary research focus is on stochastic processes as they are applied to the management and valuation of financial risks.

David Nualart is Professor at the Faculty of Mathematics of the University of Barcelona. His research interests include a variety of topics in stochastic analysis, with emphasis on stochastic partial differential equations, Malliavin calculus and fractional Brownian motion. He is the author of the monograph *Malliavin Calculus and Related Topics*.

Antonis Papapantoleon is a research assistant at the Department for Mathematical Stochastics, University of Freiburg. He received a Diploma in Mathematics from the University of Patras (2000) and an MSc in Financial Mathematics from the University of Warwick (2001). From January to August 2002 he worked at the FX Quantitative Research group of Commerzbank in Frankfurt.

Goran Peskir is the Chair in Probability at the School of Mathematics, University of Manchester. In the period 1996–2005 he was an Associate Professor at the Department of Mathematical Sciences, University of Aarhus in Denmark. He is an internationally leading expert in the field of Optimal Stopping and author to over sixty papers dealing with various problems in the field of probability and its applications (optimal stopping, stochastic calculus, option pricing). Together with Albert Shiryaev he has co-authored the book *Optimal Stopping and Free-Boundary Problems*.

Erwin Simons works in Quantitative Modeling at ING Brussels. After 3 years of front-office experience in Equity derivatives pricing, over the last year he switched to Interest-Rate derivatives modeling. He holds a PhD in Applied Mathematics from the Catholic University Leuven, von Karman Institute for Fluid Dynamics on the subject of large-scale computing of incompressible turbulent flows.

Jurgen Tistaert joined the Credit Risk Management Department of ING Brussels at the end of 1996 where he developed several rating, exposure and risk/performance models. He moved to Financial Markets in 2001, where the team is focusing on the R&D of pricing models for a broad range of derivative products. Before joining ING, he was a research assistant at the Quantitative Methods Group of K.U. Leuven Applied Economics Faculty, where he currently is appointed as a Fellow.

Nadia Uys completed her Bachelors in Economic Science, majoring in Mathematical Statistics and Actuarial Science, at the University of the Witwatersrand in 2000, followed with Honours in Advanced Mathematics of Finance in 2001. Her MSc dissertation entitled 'Optimal Stopping Problems and American Options' was completed under the supervision of Professor G. Peskir (University of Aarhus) and Mr H. Hulley (Sydney Polytechnic) and received a distinction in 2005. She is currently teaching in the Programme in Advanced Mathematic of Finance at the University of the Witwatersrand and engaging in research toward a PhD under the supervision of Professor F. Lombard (University of Johannesburg).

Nick Webber is Director of the Financial Options Research Centre, University of Warwick. Formerly Professor of Computational Finance at Cass Business School, he is interested not only in theoretical financial mathematics, but also in methods for the fast evaluation of options prices under a variety of assumptions for returns distributions. As well as work with Lévy processes and numerical methods he has also worked on copulas, credit models and interest rates.

1
Lévy Processes in Finance Distinguished by their Coarse and Fine Path Properties

Andreas E. Kyprianou
Heriot Watt University, Edinburgh, UK

and

R. Loeffen
University of Utrecht, The Netherlands

Abstract

We give a brief introduction to Lévy processes and indicate the diversity of this class of stochastic processes by quoting a number of complete characterizations of coarse and fine path properties. The theory is exemplified by distinguishing such properties for Lévy processes which are currently used extensively in financial models. Specifically, we treat jump-diffusion models (including Merton and Kou models), spectrally one-sided processes, truncated stable processes (including CGMY and Variance Gamma models), Meixner processes and generalized hyperbolic processes (including hyperbolic and normal inverse Gaussian processes).

1.1 INTRODUCTION

The main purpose of this text is to provide an *entrée* to the compilation *Exotic Options and Advanced Lévy Models*. Since path fluctuations of Lévy processes play an inevitable role in the computations which lead to the pricing of exotic options, we have chosen to give a review of what subtleties may be encountered there. In addition to giving a brief introduction to the general structure of Lévy processes, path variation and its manifestation in the Lévy–Khintchine formula, we shall introduce classifications of drifting and oscillation, regularity of the half line, the ability to visit fixed points and creeping. The theory is exemplified by distinguishing such properties for Lévy processes which are currently used extensively in financial models. Specifically, we treat jump-diffusion models (including Merton and Kou models), spectrally one-sided processes, truncated stable processes (including CGMY and variance gamma models), Meixner processes and generalized hyperbolic processes (including hyperbolic and normal inverse Gaussian processes).

To support the presentation of more advanced path properties and for the sake of completeness, a number of known facts and properties concerning these processes are reproduced from the literature. We have relied heavily upon the texts by Schoutens (2003) and Cont and Tankov (2004) for inspiration. Another useful text in this respect is that of Boyarchenko and Levendorskii (2002).

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The job of exhibiting the more theoretical facts concerning path properties have been greatly eased by the existence of the two indispensable monographs on Lévy processes, namely Bertoin (1996) and Sato (1999); see, in addition, the more recent monograph of Applebaum (2004) which also contains a section on mathematical finance. In the course of this text, we shall also briefly indicate the relevance of the path properties considered to a number of exotic options. In some cases, the links to exotics is rather vague due to the fact that the understanding of pricing exotics and advanced Lévy models is still a 'developing market', so to speak. Nonetheless, we believe that these issues will in due course become of significance as research progresses.

1.2 LÉVY PROCESSES

We start with the definition of a real valued Lévy process followed by the Lévy–Khintchine characterization.

Definition 1 *A Lévy process $X = \{X_t : t \geq 0\}$ is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ which satisfies the following properties:*

(i) The paths of X are right continuous with left limits almost surely.
(ii) $X_0 = 0$ almost surely.
(iii) X has independent increments; for $0 \leq s \leq t$, $X_t - X_s$ is independent of $\sigma(X_u : u \leq s)$.
(iv) X has stationary increments; for $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

It turns out that there is an intimate relationship between Lévy processes and a class of distributions known as *infinitely divisible* distributions which gives a precise impression of how varied the class of Lévy processes really is. To this end, let us devote a little time to discussing infinitely divisible distributions.

Definition 2 *We say that a real valued random variable Θ has an infinitely divisible distribution if for each $n = 1, 2, \ldots$. there exists a sequence of iid random variables $\Theta_1, \ldots, \Theta_n$ such that*

$$\Theta \overset{d}{=} \Theta_{1,n} + \cdots + \Theta_{n,n}$$

where $\overset{d}{=}$ is equality in distribution. Alternatively, we could have expressed this relation in terms of probability laws. That is to say, the law μ of a real valued random variable is infinitely divisible if for each $n = 1, 2, \ldots$ there exists another law μ_n of a real valued random variable such that $\mu = \mu_n^{*n}$, the n-fold convolution of μ_n.

The full extent to which we may characterize infinitely divisible distributions is carried out via their characteristic function (or Fourier transform of their law) and an expression known as the Lévy–Khintchine formula.

Theorem 3 (Lévy–Khintchine formula) *A probability law μ of a real valued random variable is infinitely divisible with characteristic exponent Ψ,*

$$\int_{\mathbb{R}} e^{iux} \mu(dx) = e^{-\Psi(u)} \text{ for } u \in \mathbb{R},$$

if and only if there exists a triple (γ, σ, Π), *where* $\gamma \in \mathbb{R}$, $\sigma \geq 0$ *and* Π *is a measure supported on* $\mathbb{R}\backslash\{0\}$ *satisfying* $\int_{\mathbb{R}} \left(1 \wedge x^2\right) \Pi(dx) < \infty$, *such that*

$$\Psi(u) = i\gamma u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (1 - e^{iux} + iux\mathbf{1}_{(|x|<1)})\Pi(dx)$$

for every $u \in \mathbb{R}$.

Definition 4 *The measure* Π *is called the Lévy (characteristic) measure and the triple* (γ, σ, Π) *are called the Lévy triple.*

Note that the requirement that $\int_{\mathbb{R}}(1 \wedge x^2)\Pi(dx) < \infty$ necessarily implies that the tails of Π are finite. On the other hand, should Π be an infinite measure due to unbounded mass in the neighbourhood of the origin, then it must at least integrate locally against x^2 for small values of x.

Let us now make firm the relationship between Lévy processes and infinitely divisible distributions. From the definition of a Lévy process we see that for any $t > 0$, X_t is a random variable whose law belongs to the class of infinitely divisible distributions. This follows from the fact that for any $n = 1, 2, \ldots$

$$X_t = X_{t/n} + (X_{2t/n} - X_{t/n}) + \cdots + (X_t - X_{(n-1)t/n}) \tag{1.1}$$

together with the fact that X has stationary independent increments. Suppose now that we define for all $u \in \mathbb{R}$, $t \geq 0$

$$\Psi_t(u) = -\log \mathbb{E}\left(e^{iuX_t}\right)$$

then by using equation (1.1) twice we have for any two positive integers m, n that

$$m\Psi_1(u) = \Psi_m(u) = n\Psi_{m/n}(u)$$

and hence for any rational $t > 0$

$$\Psi_t(u) = t\Psi_1(u). \tag{1.2}$$

If t is an irrational number, then we can choose a decreasing sequence of rationals $\{t_n : n \geq 1\}$ such that $t_n \downarrow t$ as n tends to infinity. Almost sure right continuity of X implies right continuity of $\exp\{-\Psi_t(u)\}$ (by dominated convergence) and hence equation (1.2) holds for all $t \geq 0$.

In conclusion, any Lévy process has the property that

$$\mathbb{E}\left(e^{iuX_t}\right) = e^{-t\Psi(u)}$$

where $\Psi(u) := \Psi_1(u)$ is the characteristic exponent of X_1 which has an infinitely divisible distribution.

Definition 5 *In the sequel we shall also refer to* $\Psi(u)$ *as the characteristic exponent of the Lévy process.*

Note that the law of a Lévy process is uniquely determined by its characteristic exponent. This is because the latter characterizes uniquely all one-dimensional distributions of X. From the property of stationary independent increments, it thus follows that the characteristic exponent characterizes uniquely all finite dimensional distributions which themselves uniquely characterize the law of X.

It is now clear that each Lévy process can be associated with an infinitely divisible distribution. What is not clear is whether given an infinitely divisible distribution, one may construct a Lévy process such that X_1 has that distribution. This latter issue is resolved by the following theorem which gives the Lévy–Khintchine formula for Lévy processes.

Theorem 6 *Suppose that* $\gamma \in \mathbb{R}$, $\sigma \geq 0$ *and* Π *is a measure on* $\mathbb{R}\backslash\{0\}$ *such that* $\int_{\mathbb{R}}(1 \wedge |x|^2)\Pi(dx) < \infty$. *From this triple define for each* $u \in \mathbb{R}$

$$\Psi(u) = i\gamma u + \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} [1 - e^{iux} + iux\mathbf{1}_{(|x|<1)}]\Pi(dx). \tag{1.3}$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *on which a Lévy process is defined having characteristic exponent* Ψ.

It is clear from the Lévy–Khintchine formula that a general Lévy process must necessarily take the form

$$-\gamma t + \sigma B_t + J_t, t \geq 0$$

where $B := \{B_t : t \geq 0\}$ is a standard Brownian motion (and thus has normally distributed increments contributing the term $\sigma^2 u^2/2$ to Ψ) and $J := \{J_t : t \geq 0\}$ is a process independent of B. It is the process J which essentially is responsible for the huge diversity in the class of Lévy processes and also for the discontinuities or jumps in the path of X which are typically present.

The proof of Theorem 6 is rather complicated but nonetheless very rewarding as it also reveals much more about the general structure of the process J. In Section 1.4.1 we shall give a brief outline of the main points of the proof and in particular how one additionally gets a precise classification of the path variation from it. We move first, however, to some examples of Lévy processes, in particular those which have become quite popular in financial modelling.

1.3 EXAMPLES OF LÉVY PROCESSES IN FINANCE

Appealing to the idea of stochastically perturbed multiplicative growth the classic Black–Scholes model proposes that the value of a risky asset should be modeled by an exponential Brownian motion with drift. It has long been known that this assumption drastically fails to match the reality of observed data. Cont (2001) exemplifies some of the more outstanding issues. The main problem being that log returns on real data exhibit (semi) heavy tails while log returns in the Black–Scholes model are normally distributed and hence light tailed. Among the many suggestions which were proposed to address this particular problem was the simple idea to replace the use of a Brownian motion with drift by a Lévy processes. That is to say, a risky asset is modeled by the process

$$se^{X_t}, t \geq 0$$

where $s > 0$ is the initial value of the asset and X is a Lévy process.

There are essentially four main classes of Lévy processes which feature heavily in current mainstream literature on market modeling with pure Lévy processes (we exclude from the discussion stochastic volatility models such as those of Barndorff–Nielsen and Shephard (2001)). These are the jump-diffusion processes (consisting of a Brownian motion with drift plus an independent compound Poisson process), the generalized tempered stable processes (which include more specific examples such as Variance Gamma processes and CGMY), Generalized Hyperbolic processes and Meixner processes. There is also a small minority of papers which have proposed to work with the arguably less realistic case of spectrally one-sided Lévy processes. Below, we shall give more details on all of the above key processes and their insertion into the literature.

1.3.1 Compound Poisson processes and jump-diffusions

Compound Poisson processes form the simplest class of Lévy processes in the sense of understanding their paths. Suppose that ξ is a random variable with honest distribution F supported on $\mathbb{R}$ but with no atom at 0. Let

$$X_t := \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0$$

where $\{\xi_i : i \geq 1\}$ are independent copies of ξ and $N := \{N_t : t \geq 0\}$ is an independent Poisson process with rate $\lambda > 0$. Then, $X = \{X_t : t \geq 0\}$ is a compound Poisson process. The fact that X is a Lévy process can easily be verified by computing the joint characteristic of the variables $X_t - X_s$ and $X_v - X_u$ for $0 \leq v \leq u \leq s \leq t < \infty$ and showing that it factorizes. Indeed, standard facts concerning the characteristic function of the Poisson distribution leads to the following expression for the characteristic exponent of X,

$$\Psi(u) = \lambda(1 - \widehat{F}(u)) = \int_{\mathbb{R}} (1 - e^{iux})\lambda F(dx)$$

where $\widehat{F}(u) = E(e^{iu\xi})$. Consequently, we can easily identify the Lévy triple via $\sigma = 0$ and $\gamma = -\int_R x\lambda F(dx)$ and $\Pi(dx) = \lambda F(dx)$. Note that Π has finite total mass. It is not difficult to reason that any Lévy process whose Lévy triple has this property must necessarily be a compound Poisson process. Since the jumps of the process X are spaced out by independent exponential distributions, the same is true of X and hence X is pathwise piecewise constant. Up to adding a linear drift, compound Poisson processes are the only Lévy processes which are piecewise linear.

The first model for risky assets in finance which had jumps was proposed by Merton (1976) and consisted of the log-price following an independent sum of a compound Poisson process, together with a Brownian motion with drift. That is,

$$X_t = -\gamma t + \sigma B_t + \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0$$

where $\gamma \in \mathbb{R}$, $\{B_t : t \geq 0\}$ is a Brownian motion and $\{\xi_i : i \geq 0\}$ are normally distributed. Kou (2002) assumed the above structure, the so called jump-diffusion model, but chose the jump distribution to be that of a two-sided exponential distribution. Kou's choice of jump distribution was heavily influenced by the fact that analysis of first passage problems become analytically tractable which itself is important for the valuation of American put options (see Chapter 11 below). Building on this idea, Asmussen *et al.* (2004) introduce a jump-diffusion model with two-sided phasetype distributed jumps. The latter form a class of distributions which generalize the two-sided exponential distribution and like Kou's model, have the desired property that first passage problems are analytically tractable.

1.3.2 Spectrally one-sided processes

Quite simply, spectrally one-sided processes are characterized by the property that the support of the Lévy measure is restricted to the upper or the lower half line. In the latter case, that is $\Pi(0, \infty) = 0$, one talks of spectrally negative Lévy processes. Without loss of generality we can and shall restrict our discussion to this case unless otherwise stated in the sequel.

Spectrally negative Lévy processes have not yet proved to be a convincing tool for modeling the evolution of a risky asset. The fact that the support of the Lévy measure is restricted to the lower half line does not necessarily imply that the distribution of the Lévy process itself is also restricted to the lower half line. Indeed, there are many examples of spectrally negative processes whose finite time distributions are supported on $\mathbb{R}$. One example, which has had its case argued for in a financial context by Carr and Wu (2003) and Cartea and Howison (2005), is a spectrally negative stable process of index $\alpha \in (1, 2)$. To be more precise, this is a process whose Lévy measure takes the form

$$\Pi(dx) = \mathbf{1}_{(x<0)} c|x|^{-1-\alpha} dx$$

for some constant $c > 0$ and whose parameter σ is identically zero. A lengthy calculation reveals that this process has the Lévy–Khintchine exponent

$$\Psi(u) = c|u|^{\alpha} \left(1 + i \tan \frac{\pi\alpha}{2} \operatorname{sign} u\right).$$

Chan (2000, 2004), Mordecki (1999, 2002) and Avram *et al*. (2002, 2004), have also worked with a general spectrally negative Lévy process for the purpose of pricing American put and Russian options. In their case, the choice of model was based purely on a degree of analytical tractability centred around the fact that when the path of a spectrally negative process passes from one point to another above it, it visits all other points between them.

1.3.3 Meixner processes

The Meixner process is defined through the Meixner distribution which has a density function given by

$$f_{\text{Meixner}}(x; \alpha, \beta, \delta, \mu) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x-\mu)}{\alpha}\right) \left|\Gamma\left(\delta + \frac{\mathrm{i}(x-\mu)}{\alpha}\right)\right|^2$$

where $\alpha > 0, -\pi < \beta < \pi, \delta > 0, m \in \mathbb{R}$. The Meixner distribution is infinitely divisible with a characteristic exponent

$$\Psi_{\text{Meixner}}(u) = -\log\left(\left(\frac{\cos(\beta/2)}{\cosh(\alpha u - \mathrm{i}\beta)/2}\right)^{2\delta}\right) - \mathrm{i}\mu u,$$

and therefore there exists a Lévy process with the above characteristic exponent. The Lévy triplet (γ, σ, Π) is given by

$$\gamma = -\alpha\delta \tan(\beta/2) + 2\delta \int_1^{\infty} \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}\, \mathrm{d}x - \mu,$$