THE PHYSICS OF VIBRATIONS AND WAVES

Sixth Edition

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Imperial College of Science and Technology, London, UK

John Wiley & Sons, Ltd

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Contents

Introduction to First Edition

The opening session of the physics degree course at Imperial College includes an introduction to vibrations and waves where the stress is laid on the underlying unity of concepts which are studied separately and in more detail at later stages. The origin of this short textbook lies in that lecture course which the author has given for a number of years. Sections on Fourier transforms and non-linear oscillations have been added to extend the range of interest and application.

At the beginning no more than school-leaving mathematics is assumed and more advanced techniques are outlined as they arise. This involves explaining the use of exponential series, the notation of complex numbers and partial differentiation and putting trial solutions into differential equations. Only plane waves are considered and, with two exceptions, Cartesian coordinates are used throughout. Vector methods are avoided except for the scalar product and, on one occasion, the vector product.

Opinion canvassed amongst many undergraduates has argued for a 'working' as much as for a 'reading' book; the result is a concise text amplified by many problems over a wide range of content and sophistication. Hints for solution are freely given on the principle that an undergraduates gains more from being guided to a result of physical significance than from carrying out a limited arithmetical exercise.

The main theme of the book is that a medium through which energy is transmitted via wave propagation behaves essentially as a continuum of coupled oscillators. A simple oscillator is characterized by three parameters, two of which are capable of storing and exchanging energy, whilst the third is energy dissipating. This is equally true of any medium.

The product of the energy storing parameters determines the velocity of wave propagation through the medium and, in the absence of the third parameter, their ratio governs the impedance which the medium presents to the waves. The energy dissipating parameter introduces a loss term into the impedance; energy is absorbed from the wave system and it attenuates.

This viewpoint allows a discussion of simple harmonic, damped, forced and coupled oscillators which leads naturally to the behaviour of transverse waves on a string, longitudinal waves in a gas and a solid, voltage and current waves on a transmission line and electromagnetic waves in a dielectric and a conductor. All are amenable to this common treatment, and it is the wide validity of relatively few physical principles which this book seeks to demonstrate.

H. J. PAIN
May 1968

Introduction to Second Edition

The main theme of the book remains unchanged but an extra chapter on Wave Mechanics illustrates the application of classical principles to modern physics.

Any revision has been towards a simpler approach especially in the early chapters and additional problems. Reference to a problem in the course of a chapter indicates its relevance to the preceding text. Each chapter ends with a summary of its important results.

Constructive criticism of the first edition has come from many quarters, not least from successive generations of physics and engineering students who have used the book; a second edition which incorporates so much of this advice is the best acknowledgement of its value.

H. J. PAIN
June 1976

Introduction to Third Edition

Since this book was first published the physics of optical systems has been a major area of growth and this development is reflected in the present edition. Chapter 10 has been rewritten to form the basis of an introductory course in optics and there are further applications in Chapters 7 and 8.

The level of this book remains unchanged.

H. J. PAIN
January 1983

Introduction to Fourth Edition

Interest in non-linear dynamics has grown in recent years through the application of chaos theory to problems in engineering, economics, physiology, ecology, meteorology and astronomy as well as in physics, biology and fluid dynamics. The chapter on non-linear oscillations has been revised to include topics from several of these disciplines at a level appropriate to this book. This has required an introduction to the concept of phase space which combines with that of normal modes from earlier chapters to explain how energy is distributed in statistical physics. The book ends with an appendix on this subject.

H. J. PAIN
September 1992

Introduction to Fifth Edition

In this edition, three of the longer chapters of earlier versions have been split in two: Simple Harmonic Motion is now the first chapter and Damped Simple Harmonic Motion the second. Chapter 10 on waves in optical systems now becomes Chapters 11 and 12, Waves in Optical Systems, and Interference and Diffraction respectively through a reordering of topics. A final chapter on non-linear waves, shocks and solitons now follows that on non-linear oscillations and chaos.

New material includes matrix applications to coupled oscillations, optical systems and multilayer dielectric films. There are now sections on e.m. waves in the ionosphere and other plasmas, on the laser cavity and on optical wave guides. An extended treatment of solitons includes their role in optical transmission lines, in collisionless shocks in space, in non-periodic lattices and their connection with Schrödinger's equation.

H. J. PAIN
March 1998

Acknowledgement

The author is most grateful to Professor L. D. Roelofs of the Physics Department, Haverford College, Haverford, PA, USA. After using the last edition he provided an informed, extended and valuable critique that has led to many improvements in the text and questions of this book. Any faults remain the author's responsibility.

Introduction to Sixth Edition

This edition includes new material on electron waves in solids using the Kronig – Penney model to show how their allowed energies are limited to Brillouin zones. The role of phonons is also discussed. Convolutions are introduced and applied to optical problems via the Array Theorem in Young's experiment and the Optical Transfer Function. In the last two chapters the sections on Chaos and Solutions have been reduced but their essential contents remain.

I am grateful to my colleague Professor Robin Smith of Imperial College for his advice on the Optical Transfer Function. I would like to thank my wife for typing the manuscript of every edition except the first.

H. J. PAIN
January 2005, Oxford

Chapter Synopses

Chapter 1 Simple Harmonic Motion

Chapter 2 Damped Simple Harmonic Motion

Chapter 3 The Forced Oscillatior

Chapter 4 Coupled Oscillations

Chapter 5 Transverse Wave Motion

Chapter 6 Longitudinal Waves

Chapter 7 Waves on Transmission Lines

Chapter 8 Electromagnetic Waves

Chapter 13 Wave Mechanics

Chapter 14 Non-linear Oscillations and Chaos

Chapter 15 Non-linear waves, Shocks and Solitons

Appendix 1 Normal Modes, Phase Space and Statistical Physics

1

Simple Harmonic Motion

At first sight the eight physical systems in Figure 1.1 appear to have little in common.

1.1(a) is a simple pendulum, a mass m swinging at the end of a light rigid rod of length l.

1.1(b) is a flat disc supported by a rigid wire through its centre and oscillating through small angles in the plane of its circumference.

1.1(c) is a mass fixed to a wall via a spring of stiffness s sliding to and fro in the x direction on a frictionless plane.

1.1(d) is a mass m at the centre of a light string of length $2l$ fixed at both ends under a constant tension T. The mass vibrates in the plane of the paper.

1.1(e) is a frictionless U-tube of constant cross-sectional area containing a length l of liquid, density ρ, oscillating about its equilibrium position of equal levels in each limb.

1.1(f) is an open flask of volume V and a neck of length l and constant cross-sectional area A in which the air of density ρ vibrates as sound passes across the neck.

1.1(g) is a hydrometer, a body of mass m floating in a liquid of density ρ with a neck of constant cross-sectional area cutting the liquid surface. When depressed slightly from its equilibrium position it performs small vertical oscillations.

1.1(h) is an electrical circuit, an inductance L connected across a capacitance C carrying a charge q.

All of these systems are simple harmonic oscillators which, when slightly disturbed from their equilibrium or rest postion, will oscillate with simple harmonic motion. This is the most fundamental vibration of a single particle or one-dimensional system. A small displacement x from its equilibrium position sets up a restoring force which is proportional to x acting in a direction towards the equilibrium position.

Thus, this restoring force F may be written

$$F = -sx$$

where s, the constant of proportionality, is called the stiffness and the negative sign shows that the force is acting against the direction of increasing displacement and back towards

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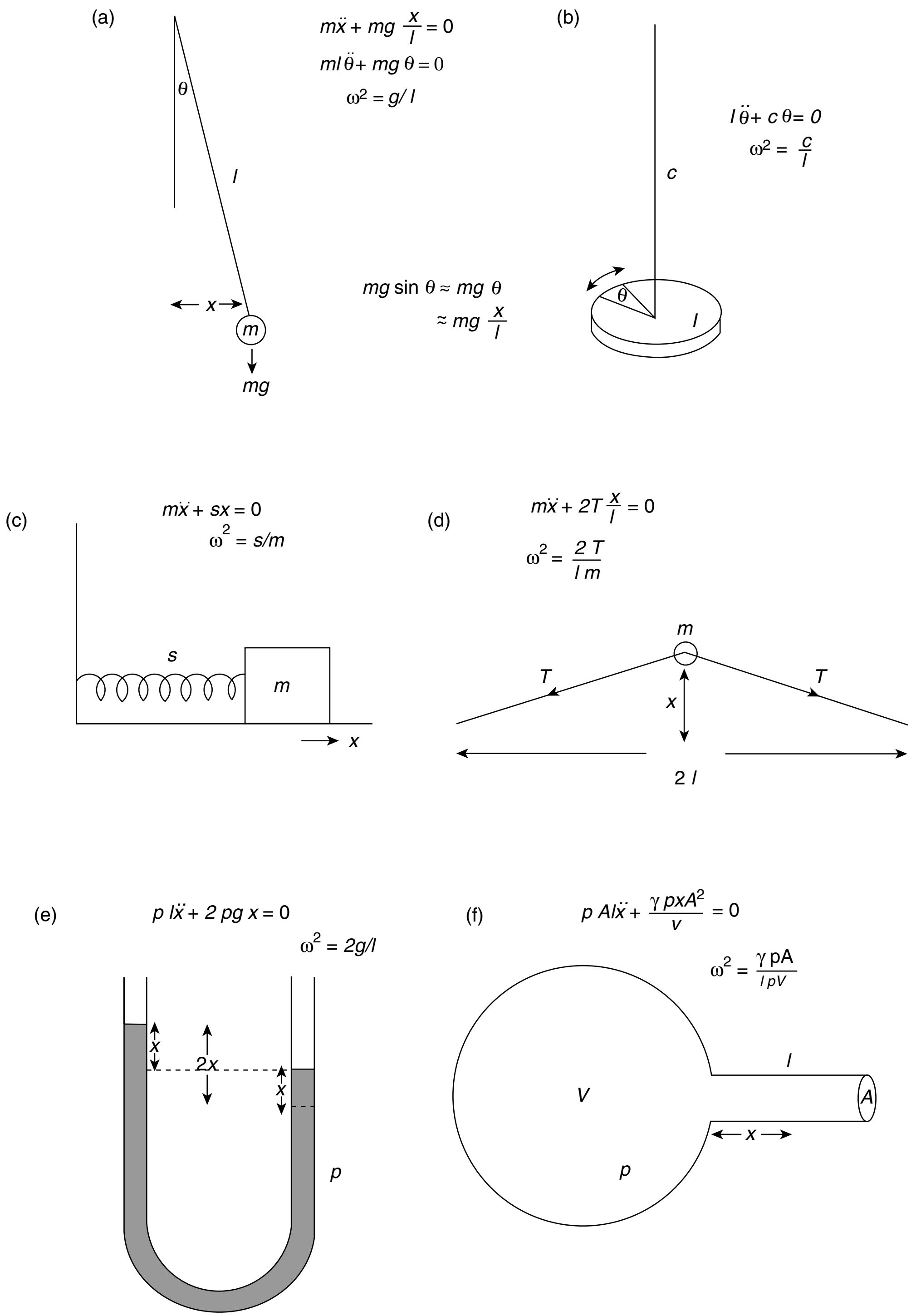
(a)
$m\ddot{x} + mg\frac{x}{l} = 0$
$ml\ddot{\theta} + mg\theta = 0$
$\omega^2 = g/l$
θ
l
x
m
mg
$mg \sin\theta \approx mg\theta$
$\approx mg\frac{x}{l}$
(b)
$I\ddot{\theta} + c\theta = 0$
$\omega^2 = \frac{c}{I}$
c
θ
I
(c)
$m\ddot{x} + sx = 0$
$\omega^2 = s/m$
s
m
x
(d)
$m\ddot{x} + 2T\frac{x}{l} = 0$
$\omega^2 = \frac{2T}{lm}$
m
T
x
T
$2l$
(e)
$\rho l\ddot{x} + 2\rho gx = 0$
$\omega^2 = 2g/l$
x
$2x$
x
ρ
(f)
$\rho Al\ddot{x} + \frac{\gamma pxA^2}{V} = 0$
$\omega^2 = \frac{\gamma pA}{l\rho V}$
l
V
A
x
p

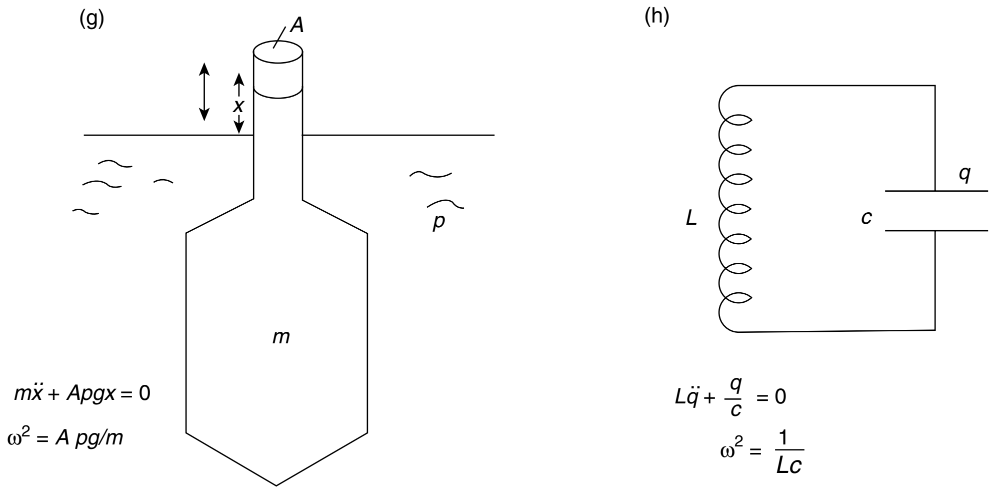

Figure 1.1 Simple harmonic oscillators with their equations of motion and angular frequencies ω of oscillation. (a) A simple pendulum. (b) A torsional pendulum. (c) A mass on a frictionless plane connected by a spring to a wall. (d) A mass at the centre of a string under constant tension *T*. (e) A fixed length of non-viscous liquid in a U-tube of constant cross-section. (f) An acoustic Helmholtz resonator. (g) A hydrometer mass *m* in a liquid of density ρ. (h) An electrical *L C* resonant circuit

the equilibrium position. A constant value of the stiffness restricts the displacement x to small values (this is Hooke's Law of Elasticity). The stiffness s is obviously the restoring force per unit distance (or displacement) and has the dimensions

$$\frac{\text{force}}{\text{distance}} \equiv \frac{MLT^{-2}}{L}$$

The equation of motion of such a disturbed system is given by the dynamic balance between the forces acting on the system, which by Newton's Law is

$$\text{mass times acceleration} = \text{restoring force}$$

or

$$m\ddot{x} = -sx$$

where the acceleration

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2}$$

This gives

$$m\ddot{x} + sx = 0$$

or

$$\ddot{x} + \frac{s}{m}x = 0$$

where the dimensions of

$$\frac{s}{m} \text{ are } \frac{MLT^{-2}}{ML} = T^{-2} = \nu^2$$

Here T is a time, or period of oscillation, the reciprocal of ν which is the frequency with which the system oscillates.

However, when we solve the equation of motion we shall find that the behaviour of x with time has a sinusoidal or cosinusoidal dependence, and it will prove more appropriate to consider, not ν, but the angular frequency $\omega = 2\pi\nu$ so that the period

$$T = \frac{1}{\nu} = 2\pi\sqrt{\frac{m}{s}}$$

where s/m is now written as ω^2. Thus the equation of simple harmonic motion

$$\ddot{x} + \frac{s}{m}x = 0$$

becomes

$$\boxed{\ddot{x} + \omega^2 x = 0} \tag{1.1}$$

(Problem 1.1)

Displacement in Simple Harmonic Motion

The behaviour of a simple harmonic oscillator is expressed in terms of its displacement x from equilibrium, its velocity $\dot{x}$, and its acceleration $\ddot{x}$ at any given time. If we try the solution

$$x = A\cos\omega t$$

where A is a constant with the same dimensions as x, we shall find that it satisfies the equation of motion

$$\ddot{x} + \omega^2 x = 0$$

for

$$\dot{x} = -A\omega\sin\omega t$$

and

$$\ddot{x} = -A\omega^2\cos\omega t = -\omega^2 x$$

Another solution

$$x = B \sin \omega t$$

is equally valid, where B has the same dimensions as A, for then

$$\dot{x} = B\omega \cos \omega t$$

and

$$\ddot{x} = -B\omega^2 \sin \omega t = -\omega^2 \mathrm{x}$$

The complete or general solution of equation (1.1) is given by the addition or superposition of both values for x so we have

$$x = A \cos \omega t + B \sin \omega t \tag{1.2}$$

with

$$\ddot{x} = -\omega^2 (A \cos \omega t + B \sin \omega t) = -\omega^2 x$$

where A and B are determined by the values of x and $\dot{x}$ at a specified time. If we rewrite the constants as

$$A = a \sin \phi \quad \text{and} \quad B = a \cos \phi$$

where ϕ is a constant angle, then

$$A^2 + B^2 = a^2 (\sin^2\phi + \cos^2\phi) = a^2$$

so that

$$a = \sqrt{A^2 + B^2}$$

and

$$\begin{aligned} x &= a \sin \phi \cos \omega t + a \cos \phi \sin \omega t \\ &= a \sin (\omega t + \phi) \end{aligned}$$

The maximum value of sin $(\omega t + \phi)$ is unity so the constant a is the maximum value of x, known as the amplitude of displacement. The limiting values of $\sin (\omega t + \phi)$ are ± 1 so the system will oscillate between the values of $x = \pm a$ and we shall see that the magnitude of a is determined by the total energy of the oscillator.

The angle ϕ is called the ‘phase constant’ for the following reason. Simple harmonic motion is often introduced by reference to ‘circular motion’ because each possible value of the displacement x can be represented by the projection of a radius vector of constant length a on the diameter of the circle traced by the tip of the vector as it rotates in a positive

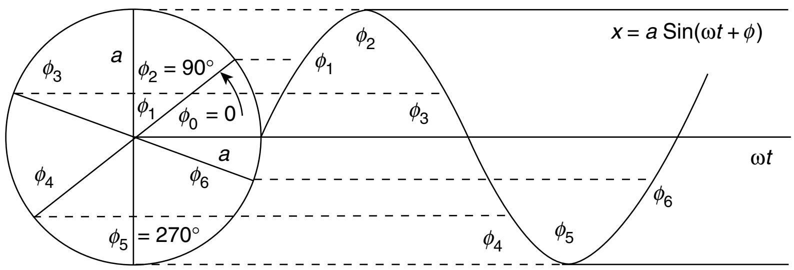

Figure 1.2 Sinusoidal displacement of simple harmonic oscillator with time, showing variation of starting point in cycle in terms of phase angle ϕ

anticlockwise direction with a constant angular velocity ω. Each rotation, as the radius vector sweeps through a phase angle of 2π rad, therefore corresponds to a complete vibration of the oscillator. In the solution

$$x = a \sin(\omega t + \phi)$$

the phase constant ϕ, measured in radians, defines the position in the cycle of oscillation at the time $t = 0$, so that the position in the cycle from which the oscillator started to move is

$$x = a \sin\phi$$

The solution

$$x = a \sin\omega t$$

defines the displacement only of that system which starts from the origin $x = 0$ at time $t = 0$ but the inclusion of ϕ in the solution

$$x = a \sin(\omega t + \phi)$$

where ϕ may take all values between zero and 2π allows the motion to be defined from any starting point in the cycle. This is illustrated in Figure 1.2 for various values of ϕ.

(Problems 1.2, 1.3, 1.4, 1.5)

Velocity and Acceleration in Simple Harmonic Motion

The values of the velocity and acceleration in simple harmonic motion for

$$x = a \sin(\omega t + \phi)$$

are given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = a\omega \cos(\omega t + \phi)$$

and

$$\frac{d^2x}{dt^2} = \ddot{x} = -a\omega^2 \sin(\omega t + \phi)$$

The maximum value of the velocity $a\omega$ is called the velocity *amplitude* and the *acceleration amplitude* is given by $a\omega^2$.

From Figure 1.2 we see that a positive phase angle of $\pi/2$ rad converts a sine into a cosine curve. Thus the velocity

$$\dot{x} = a\omega \cos(\omega t + \phi)$$

leads the displacement

$$x = a \sin(\omega t + \phi)$$

by a phase angle of $\pi/2$ rad and its maxima and minima are always a quarter of a cycle ahead of those of the displacement; the velocity is a maximum when the displacement is zero and is zero at maximum displacement. The acceleration is 'anti-phase' (π rad) with respect to the displacement, being maximum positive when the displacement is maximum negative and vice versa. These features are shown in Figure 1.3.

Often, the relative displacement or motion between two oscillators having the same frequency and amplitude may be considered in terms of their phase difference $\phi_1 - \phi_2$ which can have any value because one system may have started several cycles before the other and each complete cycle of vibration represents a change in the phase angle of $\phi = 2\pi$. When the motions of the two systems are diametrically opposed; that is, one has

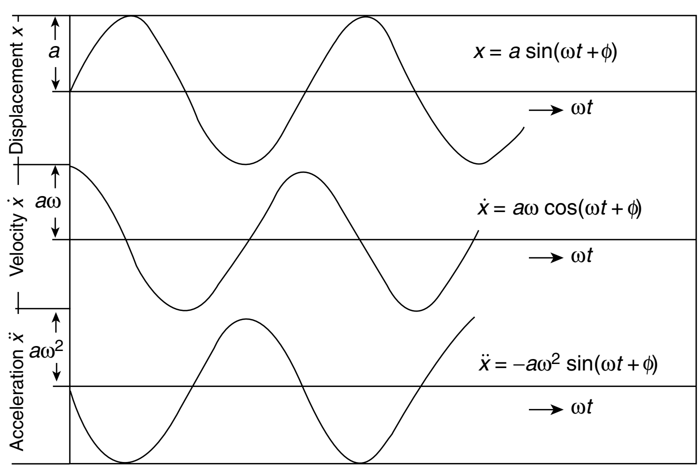

Figure 1.3 Variation with time of displacement, velocity and acceleration in simple harmonic motion. Displacement lags velocity by $\pi/2$ rad and is π rad out of phase with the acceleration. The initial phase constant ϕ is taken as zero

$x = +a$ whilst the other is at $x = -a$, the systems are 'anti-phase' and the total phase difference

$$\phi_1 - \phi_2 = n\pi \text{ rad}$$

where n is an *odd* integer. Identical systems 'in phase' have

$$\phi_1 - \phi_2 = 2n\pi \text{ rad}$$

where n is any integer. They have exactly equal values of displacement, velocity and acceleration at any instant.

(Problems 1.6, 1.7, 1.8, 1.9)

Non-linearity

If the stiffness s is constant, then the restoring force $F = -sx$, when plotted versus x, will produce a straight line and the system is said to be linear. The displacement of a linear simple harmonic motion system follows a sine or cosine behaviour. Non-linearity results when the stiffness s is not constant but varies with displacement x (see the beginning of Chapter 14).

Energy of a Simple Harmonic Oscillator

The fact that the velocity is zero at maximum displacement in simple harmonic motion and is a maximum at zero displacement illustrates the important concept of an exchange between kinetic and potential energy. In an ideal case the total energy remains constant but this is never realized in practice. If no energy is dissipated then all the potential energy becomes kinetic energy and vice versa, so that the values of (a) the total energy at any time, (b) the maximum potential energy and (c) the maximum kinetic energy will all be equal; that is

$$E_{\text{total}} = \text{KE} + \text{PE} = \text{KE}_{\text{max}} = \text{PE}_{\text{max}}$$

The solution $x = a \sin(\omega t + \phi)$ implies that the total energy remains constant because the amplitude of displacement $x = \pm a$ is regained every half cycle at the position of maximum potential energy; when energy is lost the amplitude gradually decays as we shall see later in Chapter 2. The potential energy is found by summing all the small elements of work $sx.\, dx$ (force sx times distance dx) *done by the system against the restoring force* over the range zero to x where $x = 0$ gives zero potential energy.

Thus the potential energy =

$$\int_0^x sx \cdot \mathrm{d}x = \tfrac{1}{2} sx^2$$

The kinetic energy is given by $\frac{1}{2} m\dot{x}^2$ so that the total energy

$$\boxed{E = \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} sx^2}$$