Further Praise for *The Volatility Surface*

“As an experienced practitioner, Jim Gatheral succeeds admirably in combining an accessible exposition of the foundations of stochastic volatility modeling with valuable guidance on the calibration and implementation of leading volatility models in practice.”

—Eckhard Platen, Chair in Quantitative Finance, University of Technology, Sydney

“Dr. Jim Gatheral is one of Wall Street’s very best regarding the practical use and understanding of volatility modeling. *The Volatility Surface* reflects his in-depth knowledge about local volatility, stochastic volatility, jumps, the dynamic of the volatility surface and how it affects standard options, exotic options, variance and volatility swaps, and much more. If you are interested in volatility and derivatives, you need this book!

—Espen Gaarder Haug, option trader, and author to *The Complete Guide to Option Pricing Formulas*

“Anybody who is interested in going beyond Black-Scholes should read this book. And anybody who is not interested in going beyond Black-Scholes isn’t going far!”

—Mark Davis, Professor of Mathematics, Imperial College London

“This book provides a comprehensive treatment of subjects essential for anyone working in the field of option pricing. Many technical topics are presented in an elegant and intuitively clear way. It will be indispensable not only at trading desks but also for teaching courses on modern derivatives and will definitely serve as a source of inspiration for new research.”

—Anna Shepeleva, Vice President, ING Group
Founded in 1807, John Wiley & Sons is the oldest independent publishing company in the United States. With offices in North America, Europe, Australia, and Asia, Wiley is globally committed to developing and marketing print and electronic products and services for our customers’ professional and personal knowledge and understanding.

The Wiley Finance series contains books written specifically for finance and investment professionals as well as sophisticated individual investors and their financial advisors. Book topics range from portfolio management to e-commerce, risk management, financial engineering, valuation, and financial instrument analysis, as well as much more.

For a list of available titles, please visit our Web site at www.WileyFinance.com.
To Yukiko and Ayako
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Figures</td>
<td>xiii</td>
</tr>
<tr>
<td>List of Tables</td>
<td>xix</td>
</tr>
<tr>
<td>Foreword</td>
<td>xxii</td>
</tr>
<tr>
<td>Preface</td>
<td>xxxi</td>
</tr>
<tr>
<td>Acknowledgments</td>
<td>xxvii</td>
</tr>
</tbody>
</table>

## CHAPTER 1

### Stochastic Volatility and Local Volatility

- Stochastic Volatility ........................................ 1
  - Derivation of the Valuation Equation .......................... 4
- Local Volatility....................................................... 7
  - History ................................................................. 7
  - A Brief Review of Dupire’s Work ................................ 8
  - Derivation of the Dupire Equation ............................... 9
  - Local Volatility in Terms of Implied Volatility ............. 11
  - Special Case: No Skew .............................................. 13
  - Local Variance as a Conditional Expectation of Instantaneous Variance ........................................ 13

## CHAPTER 2

### The Heston Model

- The Process .......................................................... 15
- The Heston Solution for European Options ........................ 16
  - A Digression: The Complex Logarithm in the Integration (2.13) ................................................................. 19
- Derivation of the Heston Characteristic Function .............. 20
- Simulation of the Heston Process ................................ 21
  - Milstein Discretization ............................................ 22
  - Sampling from the Exact Transition Law ........................ 23
  - Why the Heston Model Is so Popular .............................. 24

vii
CHAPTER 3  
The Implied Volatility Surface 25

- Getting Implied Volatility from Local Volatilities 25
- Model Calibration 25
- Understanding Implied Volatility 26
- Local Volatility in the Heston Model 31
- Ansatz 32
- Implied Volatility in the Heston Model 33
  - The Term Structure of Black-Scholes Implied Volatility in the Heston Model 34
  - The Black-Scholes Implied Volatility Skew in the Heston Model 35
- The SPX Implied Volatility Surface 36
  - Another Digression: The SVI Parameterization 37
  - A Heston Fit to the Data 40
  - Final Remarks on SV Models and Fitting the Volatility Surface 42

CHAPTER 4  
The Heston-Nandi Model 43

- Local Variance in the Heston-Nandi Model 43
- A Numerical Example 44
  - The Heston-Nandi Density 45
  - Computation of Local Volatilities 45
  - Computation of Implied Volatilities 46
- Discussion of Results 49

CHAPTER 5  
Adding Jumps 50

- Why Jumps are Needed 50
- Jump Diffusion 52
  - Derivation of the Valuation Equation 52
  - Uncertain Jump Size 54
- Characteristic Function Methods 56
  - Lévy Processes 56
  - Examples of Characteristic Functions for Specific Processes 57
  - Computing Option Prices from the Characteristic Function 58
- Proof of (5.6) 58
CHAPTER 6
Modeling Default Risk

Merton’s Model of Default 74
Intuition 75
Implications for the Volatility Skew 76
Capital Structure Arbitrage 77
Put-Call Parity 77
The Arbitrage 78
Local and Implied Volatility in the Jump-to-Ruin Model 79
The Effect of Default Risk on Option Prices 82
The CreditGrades Model 84
Model Setup 84
Survival Probability 85
Equity Volatility 86
Model Calibration 86

CHAPTER 7
Volatility Surface Asymptotics

Short Expirations 87
The Medvedev-Scaillet Result 89
The SABR Model 91
Including Jumps 93
Corollaries 94
Long Expirations: Fouque, Papanicolaou, and Sircar 95
Small Volatility of Volatility: Lewis 96
Extreme Strikes: Roger Lee 97
Example: Black-Scholes 99
Stochastic Volatility Models 99
Asymptotics in Summary 100
CHAPTER 8
Dynamics of the Volatility Surface

Dynamics of the Volatility Skew under Stochastic Volatility 101
Dynamics of the Volatility Skew under Local Volatility 102
Stochastic Implied Volatility Models 103
Digital Options and Digital Cliquets 104
Valuing Digital Options 104
Digital Cliquets 104

CHAPTER 9
Barrier Options

Definitions 107
Limiting Cases 108
Limit Orders 108
European Capped Calls 109
The Reflection Principle 109
The Lookback Hedging Argument 112
One-Touch Options Again 113
Put-Call Symmetry 113
QuasiStatic Hedging and Qualitative Valuation 114
Out-of-the-Money Barrier Options 114
One-Touch Options 115
Live-Out Options 116
Lookback Options 117
Adjusting for Discrete Monitoring 117
Discretely Monitored Lookback Options 119
Parisian Options 120
Some Applications of Barrier Options 120
Ladders 120
Ranges 120
Conclusion 121

CHAPTER 10
Exotic Cliquets

Locally Capped Globally Floored Cliquet 122
Valuation under Heston and Local Volatility Assumptions 123
Performance 124
Reverse Cliquet 125


### Contents

Valuation under Heston and Local Volatility Assumptions 126  
Performance 127  
Napoleon 127  
Valuation under Heston and Local Volatility Assumptions 128  
Performance 130  
Investor Motivation 130  
More on Napoleons 131

### CHAPTER 11

**Volatility Derivatives** 133

Spanning Generalized European Payoffs 133  
Example: European Options 134  
Example: Amortizing Options 135  
The Log Contract 135  
Variance and Volatility Swaps 136  
Variance Swaps 137  
Variance Swaps in the Heston Model 138  
Dependence on Skew and Curvature 138  
The Effect of Jumps 140  
Volatility Swaps 143  
Convexity Adjustment in the Heston Model 144  
Valuing Volatility Derivatives 146  
Fair Value of the Power Payoff 146  
The Laplace Transform of Quadratic Variation under Zero Correlation 147  
The Fair Value of Volatility under Zero Correlation 149  
A Simple Lognormal Model 151  
Options on Volatility: More on Model Independence 154  
Listed Quadratic-Variation Based Securities 156  
The VIX Index 156  
VXB Futures 158  
Knock-on Benefits 160  
Summary 161

**Postscript** 162

**Bibliography** 163

**Index** 169
1.1 SPX daily log returns from December 31, 1984, to December 31, 2004. Note the −22.9% return on October 19, 1987!

1.2 Frequency distribution of (77 years of) SPX daily log returns compared with the normal distribution. Although the −22.9% return on October 19, 1987, is not directly visible, the x-axis has been extended to the left to accommodate it!

1.3 Q-Q plot of SPX daily log returns compared with the normal distribution. Note the extreme tails.

3.1 Graph of the pdf of $x_t$ conditional on $x_T = \log(K)$ for a 1-year European option, strike 1.3 with current stock price = 1 and 20% volatility.

3.2 Graph of the SPX-implied volatility surface as of the close on September 15, 2005, the day before triple witching.

3.3 Plots of the SVI fits to SPX implied volatilities for each of the eight listed expirations as of the close on September 15, 2005. Strikes are on the x-axes and implied volatilities on the y-axes. The black and grey diamonds represent bid and offer volatilities respectively and the solid line is the SVI fit.

3.4 Graph of SPX ATM skew versus time to expiry. The solid line is a fit of the approximate skew formula (3.21) to all empirical skew points except the first; the dashed fit excludes the first three data points.

3.5 Graph of SPX ATM variance versus time to expiry. The solid line is a fit of the approximate ATM variance formula (3.18) to the empirical data.

3.6 Comparison of the empirical SPX implied volatility surface with the Heston fit as of September 15, 2005. From the two views presented here, we can see that the Heston fit is pretty good.
for longer expirations but really not close for short expirations.
The paler upper surface is the empirical SPX volatility surface
and the darker lower one the Heston fit. The Heston fit surface
has been shifted down by five volatility points for ease of visual
comparison.

4.1 The probability density for the Heston-Nandi model with our
parameters and expiration $T = 0.1$.

4.2 Comparison of approximate formulas with direct numerical
computation of Heston local variance. For each expiration $T$,
the solid line is the numerical computation and the dashed line
is the approximate formula.

4.3 Comparison of European implied volatilities from application of
the Heston formula (2.13) and from a numerical PDE computa-
tion using the local volatilities given by the approximate formula
(4.1). For each expiration $T$, the solid line is the numerical
computation and the dashed line is the approximate formula.

5.1 Graph of the September 16, 2005, expiration volatility smile as
of the close on September 15, 2005. SPX is trading at 1227.73.
Triangles represent bids and offers. The solid line is a nonlinear
(SVI) fit to the data. The dashed line represents the Heston skew
with Sep05 SPX parameters.

5.2 The 3-month volatility smile for various choices of jump diffu-
sion parameters.

5.3 The term structure of ATM variance skew for various choices of
jump diffusion parameters.

5.4 As time to expiration increases, the return distribution looks
more and more normal. The solid line is the jump diffusion pdf
and for comparison, the dashed line is the normal density with
the same mean and standard deviation. With the parameters
used to generate these plots, the characteristic time $T^* = 0.67$.

5.5 The solid line is a graph of the at-the-money variance skew
in the SVJ model with BCC parameters vs. time to expiration.
The dashed line represents the sum of at-the-money Heston and
jump diffusion skews with the same parameters.

5.6 The solid line is a graph of the at-the-money variance skew in
the SVJ model with BCC parameters versus time to expiration.
The dashed line represents the at-the-money Heston skew with
the same parameters.
5.7 The solid line is a graph of the at-the-money variance skew in the SVJJ model with BCC parameters versus time to expiration. The short-dashed and long-dashed lines are SVJ and Heston skew graphs respectively with the same parameters.

5.8 This graph is a short-expiration detailed view of the graph shown in Figure 5.7.

5.9 Comparison of the empirical SPX implied volatility surface with the SVJ fit as of September 15, 2005. From the two views presented here, we can see that in contrast to the Heston case, the major features of the empirical surface are replicated by the SVJ model. The paler upper surface is the empirical SPX volatility surface and the darker lower one the SVJ fit. The SVJ fit surface has again been shifted down by five volatility points for ease of visual comparison.

6.1 Three-month implied volatilities from the Merton model assuming a stock volatility of 20% and credit spreads of 100 bp (solid), 200 bp (dashed) and 300 bp (long-dashed).

6.2 Payoff of the $1 \times 2$ put spread combination: buy one put with strike 1.0 and sell two puts with strike 0.5.

6.3 Local variance plot with $\lambda = 0.05$ and $\sigma = 0.2$.

6.4 The triangles represent bid and offer volatilities and the solid line is the Merton model fit.

7.1 For short expirations, the most probable path is approximately a straight line from spot on the valuation date to the strike at expiration. It follows that $\sigma_{BS}^2(k, T) \approx \left[ v_{loc}(0, 0) + v_{loc}(k, T) \right] / 2$ and the implied variance skew is roughly one half of the local variance skew.

8.1 Illustration of a cliquet payoff. This hypothetical SPX cliquet resets at-the-money every year on October 31. The thick solid lines represent nonzero cliquet payoffs. The payoff of a 5-year European option struck at the October 31, 2000, SPX level of 1429.40 would have been zero.

9.1 A realization of the zero log-drift stochastic process and the reflected path.

9.2 The ratio of the value of a one-touch call to the value of a European binary call under stochastic volatility and local
volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility.

9.3 The value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility. The two lines are almost indistinguishable.

9.4 The value of a one-touch call under stochastic volatility and local volatility assumptions as a function of barrier level. The solid line is stochastic volatility and the dashed line is local volatility.

9.5 Values of knock-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

9.6 Values of knock-out call options struck at 0.9 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

9.7 Values of live-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility.

9.8 Values of lookback call options as a function of strike. The solid line is stochastic volatility; the dashed line is local volatility.

10.1 Value of the “Mediobanca Bond Protection 2002–2005” locally capped and globally floored cliquet (minus guaranteed redemption) as a function of MinCoupon. The solid line is stochastic volatility; the dashed line is local volatility.

10.2 Historical performance of the “Mediobanca Bond Protection 2002–2005” locally capped and globally floored cliquet. The dashed vertical lines represent reset dates, the solid lines coupon setting dates and the solid horizontal lines represent fixings.

10.3 Value of the Mediobanca reverse cliquet (minus guaranteed redemption) as a function of MaxCoupon. The solid line is stochastic volatility; the dashed line is local volatility.

10.4 Historical performance of the “Mediobanca 2000–2005 Reverse Cliquet Telecommunicazioni” reverse cliquet. The vertical lines represent reset dates, the solid horizontal lines represent fixings and the vertical grey bars represent negative contributions to the cliquet payoff.

10.5 Value of (risk-neutral) expected Napoleon coupon as a function of MaxCoupon. The solid line is stochastic volatility; the dashed line is local volatility.
10.6 Historical performance of the STOXX 50 component of the “Mediobanca 2002–2005 World Indices Euro Note Serie 46” Napoleon. The light vertical lines represent reset dates, the heavy vertical lines coupon setting dates, the solid horizontal lines represent fixings and the thick grey bars represent the minimum monthly return of each coupon period.

11.1 Payoff of a variance swap (dashed line) and volatility swap (solid line) as a function of realized volatility $\Sigma_T$. Both swaps are struck at 30\% volatility.

11.2 Annualized Heston convexity adjustment as a function of $T$ with Heston-Nandi parameters.

11.3 Annualized Heston convexity adjustment as a function of $T$ with Bakshi, Cao, and Chen parameters.

11.4 Value of 1-year variance call versus variance strike $K$ with the BCC parameters. The solid line is a numerical Heston solution; the dashed line comes from our lognormal approximation.

11.5 The pdf of the log of 1-year quadratic variation with BCC parameters. The solid line comes from an exact numerical Heston computation; the dashed line comes from our lognormal approximation.

11.6 Annualized Heston VXB convexity adjustment as a function of $t$ with Heston parameters from December 8, 2004, SPX fit.
Tables

3.1 At-the-money SPX variance levels and skews as of the close on September 15, 2005, the day before expiration. 39
3.2 Heston fit to the SPX surface as of the close on September 15, 2005. 40

5.1 September 2005 expiration option prices as of the close on September 15, 2005. Triple witching is the following day. SPX is trading at 1227.73. 51
5.2 Parameters used to generate Figures 5.2 and 5.3. 63
5.3 Interpreting Figures 5.2 and 5.3. 64
5.4 Various fits of jump diffusion style models to SPX data. JD means Jump Diffusion and SVJ means Stochastic Volatility plus Jumps. 69
5.5 SVJ fit to the SPX surface as of the close on September 15, 2005. 71

6.1 Upper and lower arbitrage bounds for one-year 0.5 strike options for various credit spreads (at-the-money volatility is 20%). 79
6.2 Implied volatilities for January 2005 options on GT as of October 20, 2004 (GT was trading at 9.40). Merton vols are volatilities generated from the Merton model with fitted parameters. 82

10.2 Worst monthly returns and estimated Napoleon coupons. Recall that the coupon is computed as 10% plus the worst monthly return averaged over the three underlying indices. 131

11.1 Empirical VXB convexity adjustments as of December 8, 2004. 159