The Volatility Surface

A Practitioner's Guide

JIM GATHERAL

Foreword by Nassim Nicholas Taleb

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To Yukiko and Ayako

Contents

List of Figures xiii

List of Tables xix

Foreword xxi

Preface xxiii

Acknowledgments xxvii

CHAPTER 1
Stochastic Volatility and Local Volatility 1

Stochastic Volatility 1
Derivation of the Valuation Equation 4
Local Volatility 7
History 7
A Brief Review of Dupire's Work 8
Derivation of the Dupire Equation 9
Local Volatility in Terms of Implied Volatility 11
Special Case: No Skew 13
Local Variance as a Conditional Expectation of Instantaneous Variance 13

CHAPTER 2
The Heston Model 15

The Process 15
The Heston Solution for European Options 16
A Digression: The Complex Logarithm in the Integration (2.13) 19
Derivation of the Heston Characteristic Function 20
Simulation of the Heston Process 21
Milstein Discretization 22
Sampling from the Exact Transition Law 23
Why the Heston Model Is so Popular 24

CHAPTER 3
The Implied Volatility Surface **25**

Getting Implied Volatility from Local Volatilities 25
Model Calibration 25
Understanding Implied Volatility 26
Local Volatility in the Heston Model 31
Ansatz 32
Implied Volatility in the Heston Model 33
The Term Structure of Black-Scholes Implied Volatility in the Heston Model 34
The Black-Scholes Implied Volatility Skew in the Heston Model 35
The SPX Implied Volatility Surface 36
Another Digression: The SVI Parameterization 37
A Heston Fit to the Data 40
Final Remarks on SV Models and Fitting the Volatility Surface 42

CHAPTER 4
The Heston-Nandi Model **43**

Local Variance in the Heston-Nandi Model 43
A Numerical Example 44
The Heston-Nandi Density 45
Computation of Local Volatilities 45
Computation of Implied Volatilities 46
Discussion of Results 49

CHAPTER 5
Adding Jumps **50**

Why Jumps are Needed 50
Jump Diffusion 52
Derivation of the Valuation Equation 52
Uncertain Jump Size 54
Characteristic Function Methods 56
Lévy Processes 56
Examples of Characteristic Functions for Specific Processes 57
Computing Option Prices from the Characteristic Function 58
Proof of (5.6) 58

Computing Implied Volatility 60
Computing the At-the-Money Volatility Skew 60
How Jumps Impact the Volatility Skew 61
Stochastic Volatility Plus Jumps 65
Stochastic Volatility Plus Jumps in the Underlying Only (SVJ) 65
Some Empirical Fits to the SPX Volatility Surface 66
Stochastic Volatility with Simultaneous Jumps in Stock Price and Volatility (SVJJ) 68
SVJ Fit to the September 15, 2005, SPX Option Data 71
Why the SVJ Model Wins 73

CHAPTER 6
Modeling Default Risk 74

Merton's Model of Default 74
Intuition 75
Implications for the Volatility Skew 76
Capital Structure Arbitrage 77
Put-Call Parity 77
The Arbitrage 78
Local and Implied Volatility in the Jump-to-Ruin Model 79
The Effect of Default Risk on Option Prices 82
The CreditGrades Model 84
Model Setup 84
Survival Probability 85
Equity Volatility 86
Model Calibration 86

CHAPTER 7
Volatility Surface Asymptotics 87

Short Expirations 87
The Medvedev-Scaillet Result 89
The SABR Model 91
Including Jumps 93
Corollaries 94
Long Expirations: Fouque, Papanicolaou, and Sircar 95
Small Volatility of Volatility: Lewis 96
Extreme Strikes: Roger Lee 97
Example: Black-Scholes 99
Stochastic Volatility Models 99
Asymptotics in Summary 100

CHAPTER 8
Dynamics of the Volatility Surface **101**

Dynamics of the Volatility Skew under Stochastic Volatility 101
Dynamics of the Volatility Skew under Local Volatility 102
Stochastic Implied Volatility Models 103
Digital Options and Digital Cliquets 103
Valuing Digital Options 104
Digital Cliquets 104

CHAPTER 9
Barrier Options **107**

Definitions 107
Limiting Cases 108
Limit Orders 108
European Capped Calls 109
The Reflection Principle 109
The Lookback Hedging Argument 112
One-Touch Options Again 113
Put-Call Symmetry 113
QuasiStatic Hedging and Qualitative Valuation 114
Out-of-the-Money Barrier Options 114
One-Touch Options 115
Live-Out Options 116
Lookback Options 117
Adjusting for Discrete Monitoring 117
Discretely Monitored Lookback Options 119
Parisian Options 120
Some Applications of Barrier Options 120
Ladders 120
Ranges 120
Conclusion 121

CHAPTER 10
Exotic Cliquets **122**

Locally Capped Globally Floored Cliquet 122
Valuation under Heston and Local Volatility Assumptions 123
Performance 124
Reverse Cliquet 125

Valuation under Heston and Local Volatility Assumptions 126
Performance 127
Napoleon 127
Valuation under Heston and Local Volatility Assumptions 128
Performance 130
Investor Motivation 130
More on Napoleons 131

CHAPTER 11
Volatility Derivatives 133

Spanning Generalized European Payoffs 133
Example: European Options 134
Example: Amortizing Options 135
The Log Contract 135
Variance and Volatility Swaps 136
Variance Swaps 137
Variance Swaps in the Heston Model 138
Dependence on Skew and Curvature 138
The Effect of Jumps 140
Volatility Swaps 143
Convexity Adjustment in the Heston Model 144
Valuing Volatility Derivatives 146
Fair Value of the Power Payoff 146
The Laplace Transform of Quadratic Variation under Zero Correlation 147
The Fair Value of Volatility under Zero Correlation 149
A Simple Lognormal Model 151
Options on Volatility: More on Model Independence 154
Listed Quadratic-Variation Based Securities 156
The VIX Index 156
VXB Futures 158
Knock-on Benefits 160
Summary 161

Postscript 162

Bibliography 163

Index 169

Figures

1.1 SPX daily log returns from December 31, 1984, to December 31, 2004. Note the −22.9% return on October 19, 1987! 2

1.2 Frequency distribution of (77 years of) SPX daily log returns compared with the normal distribution. Although the −22.9% return on October 19, 1987, is not directly visible, the x-axis has been extended to the left to accommodate it! 3

1.3 Q-Q plot of SPX daily log returns compared with the normal distribution. Note the extreme tails. 3

3.1 Graph of the pdf of x_t conditional on $x_T = \log(K)$ for a 1-year European option, strike 1.3 with current stock price = 1 and 20% volatility. 31

3.2 Graph of the SPX-implied volatility surface as of the close on September 15, 2005, the day before triple witching. 36

3.3 Plots of the SVI fits to SPX implied volatilities for each of the eight listed expirations as of the close on September 15, 2005. Strikes are on the x-axes and implied volatilities on the y-axes. The black and grey diamonds represent bid and offer volatilities respectively and the solid line is the SVI fit. 38

3.4 Graph of SPX ATM skew versus time to expiry. The solid line is a fit of the approximate skew formula (3.21) to all empirical skew points except the first; the dashed fit excludes the first three data points. 39

3.5 Graph of SPX ATM variance versus time to expiry. The solid line is a fit of the approximate ATM variance formula (3.18) to the empirical data. 40

3.6 Comparison of the empirical SPX implied volatility surface with the Heston fit as of September 15, 2005. From the two views presented here, we can see that the Heston fit is pretty good for longer expirations but really not close for short expirations. The paler upper surface is the empirical SPX volatility surface and the darker lower one the Heston fit. The Heston fit surface has been shifted down by five volatility points for ease of visual comparison. 41

4.1 The probability density for the Heston-Nandi model with our parameters and expiration $T = 0.1$. 45

4.2 Comparison of approximate formulas with direct numerical computation of Heston local variance. For each expiration T, the solid line is the numerical computation and the dashed line is the approximate formula. 47

4.3 Comparison of European implied volatilities from application of the Heston formula (2.13) and from a numerical PDE computation using the local volatilities given by the approximate formula (4.1). For each expiration T, the solid line is the numerical computation and the dashed line is the approximate formula. 48

5.1 Graph of the September 16, 2005, expiration volatility smile as of the close on September 15, 2005. SPX is trading at 1227.73. Triangles represent bids and offers. The solid line is a nonlinear (SVI) fit to the data. The dashed line represents the Heston skew with Sep05 SPX parameters. 52

5.2 The 3-month volatility smile for various choices of jump diffusion parameters. 63

5.3 The term structure of ATM variance skew for various choices of jump diffusion parameters. 64

5.4 As time to expiration increases, the return distribution looks more and more normal. The solid line is the jump diffusion pdf and for comparison, the dashed line is the normal density with the same mean and standard deviation. With the parameters used to generate these plots, the characteristic time $T^* = 0.67$. 65

5.5 The solid line is a graph of the at-the-money variance skew in the SVJ model with BCC parameters vs. time to expiration. The dashed line represents the sum of at-the-money Heston and jump diffusion skews with the same parameters. 67

5.6 The solid line is a graph of the at-the-money variance skew in the SVJ model with BCC parameters versus time to expiration. The dashed line represents the at-the-money Heston skew with the same parameters. 67

5.7 The solid line is a graph of the at-the-money variance skew in the SVJJ model with BCC parameters versus time to expiration. The short-dashed and long-dashed lines are SVJ and Heston skew graphs respectively with the same parameters. 70

5.8 This graph is a short-expiration detailed view of the graph shown in Figure 5.7. 71

5.9 Comparison of the empirical SPX implied volatility surface with the SVJ fit as of September 15, 2005. From the two views presented here, we can see that in contrast to the Heston case, the major features of the empirical surface are replicated by the SVJ model. The paler upper surface is the empirical SPX volatility surface and the darker lower one the SVJ fit. The SVJ fit surface has again been shifted down by five volatility points for ease of visual comparison. 72

6.1 Three-month implied volatilities from the Merton model assuming a stock volatility of 20% and credit spreads of 100 bp (solid), 200 bp (dashed) and 300 bp (long-dashed). 76

6.2 Payoff of the 1×2 put spread combination: buy one put with strike 1.0 and sell two puts with strike 0.5. 79

6.3 Local variance plot with $\lambda = 0.05$ and $\sigma = 0.2$. 81

6.4 The triangles represent bid and offer volatilities and the solid line is the Merton model fit. 83

7.1 For short expirations, the most probable path is approximately a straight line from spot on the valuation date to the strike at expiration. It follows that $\sigma^2_{BS}\left(k,T\right) \approx \left[v_{loc}(0,0) + v_{loc}(k,T)\right]/2$ and the implied variance skew is roughly one half of the local variance skew. 89

8.1 Illustration of a cliquet payoff. This hypothetical SPX cliquet resets at-the-money every year on October 31. The thick solid lines represent nonzero cliquet payoffs. The payoff of a 5-year European option struck at the October 31, 2000, SPX level of 1429.40 would have been zero. 105

9.1 A realization of the zero log-drift stochastic process and the reflected path. 110

9.2 The ratio of the value of a one-touch call to the value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility. 111

9.3 The value of a European binary call under stochastic volatility and local volatility assumptions as a function of strike. The solid line is stochastic volatility and the dashed line is local volatility. The two lines are almost indistinguishable. 111

9.4 The value of a one-touch call under stochastic volatility and local volatility assumptions as a function of barrier level. The solid line is stochastic volatility and the dashed line is local volatility. 112

9.5 Values of knock-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility. 115

9.6 Values of knock-out call options struck at 0.9 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility. 116

9.7 Values of live-out call options struck at 1 as a function of barrier level. The solid line is stochastic volatility; the dashed line is local volatility. 117

9.8 Values of lookback call options as a function of strike. The solid line is stochastic volatility; the dashed line is local volatility. 118

10.1 Value of the "Mediobanca Bond Protection 2002–2005" locally capped and globally floored cliquet (minus guaranteed redemption) as a function of *MinCoupon*. The solid line is stochastic volatility; the dashed line is local volatility. 124

10.2 Historical performance of the "Mediobanca Bond Protection 2002–2005" locally capped and globally floored cliquet. The dashed vertical lines represent reset dates, the solid lines coupon setting dates and the solid horizontal lines represent fixings. 125

10.3 Value of the Mediobanca reverse cliquet (minus guaranteed redemption) as a function of *MaxCoupon*. The solid line is stochastic volatility; the dashed line is local volatility. 127

10.4 Historical performance of the "Mediobanca 2000–2005 Reverse Cliquet Telecommunicazioni" reverse cliquet. The vertical lines represent reset dates, the solid horizontal lines represent fixings and the vertical grey bars represent negative contributions to the cliquet payoff. 128

10.5 Value of (risk-neutral) expected Napoleon coupon as a function of *MaxCoupon*. The solid line is stochastic volatility; the dashed line is local volatility. 129

10.6 Historical performance of the STOXX 50 component of the "Mediobanca 2002–2005 World Indices Euro Note Serie 46" Napoleon. The light vertical lines represent reset dates, the heavy vertical lines coupon setting dates, the solid horizontal lines represent fixings and the thick grey bars represent the minimum monthly return of each coupon period. 130

11.1 Payoff of a variance swap (dashed line) and volatility swap (solid line) as a function of realized volatility Σ_T. Both swaps are struck at 30% volatility. 143

11.2 Annualized Heston convexity adjustment as a function of T with Heston-Nandi parameters. 145

11.3 Annualized Heston convexity adjustment as a function of T with Bakshi, Cao, and Chen parameters. 145

11.4 Value of 1-year variance call versus variance strike K with the BCC parameters. The solid line is a numerical Heston solution; the dashed line comes from our lognormal approximation. 153

11.5 The pdf of the log of 1-year quadratic variation with BCC parameters. The solid line comes from an exact numerical Heston computation; the dashed line comes from our lognormal approximation. 154

11.6 Annualized Heston VXB convexity adjustment as a function of t with Heston parameters from December 8, 2004, SPX fit. 160

Tables

3.1 At-the-money SPX variance levels and skews as of the close on September 15, 2005, the day before expiration. 39

3.2 Heston fit to the SPX surface as of the close on September 15, 2005. 40

5.1 September 2005 expiration option prices as of the close on September 15, 2005. Triple witching is the following day. SPX is trading at 1227.73. 51

5.2 Parameters used to generate Figures 5.2 and 5.3. 63

5.3 Interpreting Figures 5.2 and 5.3. 64

5.4 Various fits of jump diffusion style models to SPX data. JD means *Jump Diffusion* and SVJ means *Stochastic Volatility plus Jumps*. 69

5.5 SVJ fit to the SPX surface as of the close on September 15, 2005. 71

6.1 Upper and lower arbitrage bounds for one-year 0.5 strike options for various credit spreads (at-the-money volatility is 20%). 79

6.2 Implied volatilities for January 2005 options on GT as of October 20, 2004 (GT was trading at 9.40). Merton vols are volatilities generated from the Merton model with fitted parameters. 82

10.1 Estimated "Mediobanca Bond Protection 2002–2005" coupons. 125

10.2 Worst monthly returns and estimated Napoleon coupons. Recall that the coupon is computed as 10% plus the worst monthly return averaged over the three underlying indices. 131

11.1 Empirical VXB convexity adjustments as of December 8, 2004. 159