REMOTE SENSING
WITH POLARIMETRIC RADAR

HAROLD MOTT
The University of Alabama

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REMOTE SENSING WITH POLARIMETRIC RADAR
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REMOTE SENSING WITH POLARIMETRIC RADAR

HAROLD MOTT
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To my sister Aileen
CONTENTS

PREFACE       xiii
ACKNOWLEDGMENTS   xv

1. ELECTROMAGNETIC WAVES       1

1.1. The Time-Invariant Maxwell Equations / 2
1.2. The Electromagnetic Traveling Wave / 3
1.3. Power Density / 6
1.4. The Polarization Ellipse / 7
1.5. Polarization Vector and Polarization Ratio / 11
1.6. Circular Wave Components / 11
1.7. Change of Polarization Basis / 12
1.8. Ellipse Characteristics in Terms of \( P \) and \( Q \) / 14
1.9. Coherency and Stokes Vectors / 15
1.10. The Poincaré Sphere / 17

References / 19
Problems / 19

2. ANTENNAS       21

2.1. Elements of the Antenna System / 21
2.2. The Vector Potentials / 22
2.3. Solutions for the Vector Potentials / 24
2.4. Far-Zone Fields / 25
2.5. Radiation Pattern / 28

vii
## CONTENTS

2.6. Gain and Directivity / 30  
2.7. The Receiving Antenna / 34  
2.8. Transmission Between Antennas / 41  
2.9. Antenna Arrays / 41  
2.10. Effective Length of an Antenna / 47  
2.11. Reception of Completely Polarized Waves / 48  
2.12. Gain, Effective Area, and Radiation Resistance / 51  
2.13. Maximum Received Power / 52  
2.14. Polarization Efficiency / 52  
2.15. The Modified Friis Transmission Equation / 54  
2.16. Alignment of Antennas / 54  

References / 57  
Problems / 57  

3. COHERENTLY SCATTERING TARGETS 59  
3.1. Radar Targets / 59  
3.2. The Jones Matrix / 61  
3.3. The Sinclair Matrix / 62  
3.4. Matrices With Relative Phase / 64  
3.5. FSA–BSA Conventions / 65  
3.6. Relationship Between Jones and Sinclair Matrices / 65  
3.7. Scattering with Circular Wave Components / 66  
3.8. Backscattering / 67  
3.9. Polarization Ratio of the Scattered Wave / 68  
3.10. Change of Polarization Basis: The Scattering Matrix / 68  
3.11. Polarizations for Maximum and Minimum Power / 70  
3.12. The Polarization Fork / 77  
3.13. Nonaligned Coordinate Systems / 81  
3.14. Determination of Scattering Parameters / 82  

References / 88  
Problems / 89  

4. AN INTRODUCTION TO RADAR 91  
4.1. Pulse Radar / 92  
4.2. CW Radar / 98  
4.3. Directional Properties of Radar Measurements / 98  
4.4. Resolution / 99  
4.5. Imaging Radar / 104  
4.6. The Traditional Radar Equation / 105  
4.7. The Polarimetric Radar Equation / 107  
4.8. A Polarimetric Radar / 108  

References / 112  
Problems / 113
4.9. Noise / 110
    References / 117
    Problems / 117

5. SYNTHETIC APERTURE RADAR / 119
   5.1. Creating a Terrain Map / 119
   5.2. Range Resolution / 124
   5.3. Azimuth Resolution / 125
   5.4. Geometric Factors / 132
   5.5. Polarimetric SAR / 133
   5.6. SAR Errors / 133
   5.7. Height Measurement / 136
   5.8. Polarimetric Interferometry / 141
   5.9. Phase Unwrapping / 142
       References / 147
       Problems / 147

6. PARTIALLY POLARIZED WAVES / 149
   6.1. Representation of the Fields / 150
   6.2. Representation of Partially Polarized Waves / 154
   6.3. Reception of Partially Polarized Waves / 164
       References / 166
       Problems / 166

7. SCATTERING BY DEPOLARIZING TARGETS / 169
   7.1. Targets / 170
   7.2. Averaging the Sinclair Matrix / 173
   7.3. The Kronecker-Product Matrices / 174
   7.4. Matrices for a Depolarizing Target: Coherent Measurement / 177
   7.5. Incoherently Measured Target Matrices / 178
   7.6. Matrix Properties and Relationships / 186
   7.7. Modified Matrices / 189
   7.8. Names / 191
   7.9. Additional Target Information / 191
   7.10. Target Covariance and Coherency Matrices / 192
   7.11. A Scattering Matrix with Circular Components / 196
   7.12. The Graves Power Density Matrix / 197
   7.13. Measurement Considerations / 199
   7.14. Degree of Polarization and Polarimetric Entropy / 200
   7.15. Variance of Power / 201
8. OPTIMAL POLARIZATIONS FOR RADAR 207

8.1. Antenna Selection Criteria / 207
8.2. Lagrange Multipliers / 208

A. COHERENTLY SCATTERING TARGETS 209

8.3. Maximum Power / 209
8.4. Power Contrast: Backscattering / 211

B. DEPOLARIZING TARGETS 211

8.5. Iterative Procedure for Maximizing Power Contrast / 212
8.6. The Backscattering Covariance Matrix / 215
8.7. The Bistatic Covariance Matrix / 216
8.8. Maximizing Power Contrast by Matrix Decomposition / 217
8.9. Optimization with the Graves Matrix / 218

References / 222
Problems / 223

9. CLASSIFICATION OF TARGETS 225

A. CLASSIFICATION CONCEPTS 225

9.1. Representation and Classification of Targets / 226
9.2. Bayes Decision Rule / 228
9.3. The Neyman–Pearson Decision Rule / 231
9.4. Bayes Error Bounds / 232
9.5. Estimation of Parameters from Data / 232
9.6. Nonparametric Classification / 236

B. CLASSIFICATION BY MATRIX DECOMPOSITION 242

9.7. Coherent Decomposition / 243
9.8. Decomposition of Power-Type Matrices / 245

C. REMOVAL OF UNPOLARIZED SCATTERING 249

9.9. Decomposition of the D Matrix / 249
9.10. Polarized Clutter / 255
9.11. A Similar Decomposition / 255
9.12. Polarimetric Similarity Classification / 256
    References / 256
    Problems / 257

APPENDIX A. FADING AND SPECKLE 259
    Reference / 261

APPENDIX B. PROBABILITY AND RANDOM PROCESSES 263
    B.1. Probability / 263
    B.2. Random Variables / 273
    B.3. Random Vectors / 279
    B.5. Random Processes / 288
    References / 294

APPENDIX C. THE KENNAUGH MATRIX 295

APPENDIX D. BAYES ERROR BOUNDS 299
    References / 301

INDEX 303
PREFACE

The author’s purpose in writing this book was to present the principles necessary for understanding polarized radiation, transmission, scattering, and reception in communication systems and polarimetric-radar remote sensing. The book can be used as a text for an undergraduate or graduate course in these topics and as a reference text for engineers and for scientists who use remotely sensed information about the earth. Chapters 1, 2, 4, and 5 are at an introductory level and, together with Chapter 6 and selected material from Chapter 3, can be used for an undergraduate course in electrical engineering. Chapters 3 and 6–9 are at a more advanced technical and mathematical level and provide suitable material for graduate study. Student deficiencies in antennas and radar can be corrected with selections from Chapters 2, 4, and 5. Problems, ranging from straightforward in the introductory chapters to more challenging in the advanced chapters, are provided for pedagogical purposes. Scientists who can profitably study the book are agronomists, geographers, meteorologists, and others who use remotely sensed information. For those who wish to go beyond this discussion of principles to learn of the achievements of polarimetric radar remote sensing and see predictions of future technological developments, I recommend a complementary book, Principles & Applications of Imaging Radar: Manual of Remote Sensing, Volume 2, by F. M. Henderson and A. J. Lewis, Wiley, 1998. A comprehensive description of earth-survey sensors is given by H. J. Kramer, Observation of the Earth and Its Environment: Survey of Missions and Sensors, Third ed., Springer, 1996.

The material in Chapters 3 and 6 is at a higher mathematical level than the introductory chapters. That in Chapters 7–9 is still more detailed and original and will require more diligent study for a complete understanding. The reader with an understanding of calculus, vector analysis, matrices, and elementary physics can readily comprehend the material, however.
In the author’s earlier books, *Polarization in Antennas and Radar* and *Antennas for Radar and Communication: A Polarimetric Approach*, published by Wiley-Interscience in 1986 and 1992, a polarization ratio was defined for an antenna functioning as a transmitter and another for it as a receiver. After reflection, it was thought best to describe the antenna in the same way, regardless of its function, and that has been done in this text. In the earlier books, there was no distinction made between coherently and incoherently measured target matrices, but the distinction is an important part of this book.
ACKNOWLEDGMENTS

The following persons read chapters of this book and provided valuable suggestions: Dr. Ernst Lüneburg of EML Consultants, Wessling, Germany, Jerry L. Eaves of the Georgia Tech Research Institute, Professor Emeritus Ronald C. Houts of the University of West Florida, Assistant Professor John H. Mott of Purdue University, Professor Jian Yang of Tsinghua University, Beijing, Professor Robert W. Scharstein of the University of Alabama, and Dipl. Ing. Andreas Danklmayer of the Deutschen Zentrum für Luft-und Raumfahrt (DLR), Oberpfaffenhofen. Many of their suggestions were incorporated in the text, and I am most grateful for their help.

The late Dr. Ernst Lüneburg encouraged me and provided assistance in the writing of the book over a period of several years. His meticulous and selfless analysis of Chapters 3, 6, 7, and 9 led to extensive changes in the presentation and mathematical developments of the chapters. He read Chapters 1 and 2, also, and exerted a strong influence over still others. Dr. Lüneburg was one of the most able electromagnetics theoreticians and mathematicians of our time and a most generous and helpful friend and colleague. It was a privilege to know and work with him.

Professor Emeritus Wolfgang-Martin Boerner of the University of Illinois, Chicago, maintained his interest in the text from its beginning and provided valuable help and encouragement at critical times. My second book was dedicated to him, and I am thankful for his continued friendship and assistance. I also wish to thank the University of Alabama for providing me with an office and assistance during the years since my retirement as Professor of Electrical Engineering.
This book is dedicated to my sister Aileen in remembrance of her help in my early years and for her love throughout my life.

Harold Mott

Tuscaloosa, Alabama
CHAPTER 1

ELECTROMAGNETIC WAVES

The Maxwell equations,

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (1.1) \]
\[ \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (1.2) \]
\[ \nabla \cdot \vec{D} = \rho \quad (1.3) \]
\[ \nabla \cdot \vec{B} = 0 \quad (1.4) \]

represent the physical laws that are the electromagnetic basis of radar remote sensing. \( \vec{E} \), \( \vec{H} \), \( \vec{D} \), \( \vec{B} \), and \( \vec{J} \) are real vectors that symbolize the space- and time-dependent physical quantities of electric field intensity, magnetic field intensity, electric flux density, magnetic flux density, and electric current density. They are in a bold typeface, as are all vectors and matrices in this book. The parameter \( \rho \) is a real scalar function of space and time representing electric charge density. The operations indicated are the curl and divergence and the partial time derivative. The rationalized meter-kilogram-second (SI) unit system is used throughout this work.

Note: Only referenced equations are numbered, and numbered equations are not more important than unnumbered ones.
Electric current density $\mathbf{J}$ has value because of the flow of electric charge and is related to the rate of change of electric charge density in a region by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \mathbf{\hat{\rho}}}{\partial t}$$

This relationship expresses the conservation of charge, and the equation is the \textit{equation of conservation of charge} or the \textit{equation of continuity}. It can be derived from the Maxwell equations, or, conversely, the divergence equations 1.3 and 1.4 can be derived from it and the curl equations 1.1 and 1.2.

\subsection*{1.1. THE TIME-ININVARIANT MAXWELL EQUATIONS}

The sources and fields vary in a sinusoidal manner in many phenomena of electromagnetics, and the Maxwell equations can be written in a more tractable form by the substitution, shown here for the electric field intensity, but applicable to all field and source terms,

$$\tilde{\mathbf{E}} = \Re (\mathbf{E} e^{j\omega t})$$

The tilde is used in this formulation to represent quantities that vary with space and time. Quantities without the tilde are functions of space only. With this substitution,

$$\nabla \times \mathbf{E} = -\mathbf{M} - j\omega \mathbf{B} \quad (1.5)$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega \mathbf{D} \quad (1.6)$$
$$\nabla \cdot \mathbf{D} = \mathbf{\rho} \quad (1.7)$$
$$\nabla \cdot \mathbf{B} = \mathbf{\rho}_M \quad (1.8)$$

In forming these equations, a magnetic charge density $\mathbf{\rho}_M$ and magnetic current density $\mathbf{M}$ were added to the equations formed directly from (1.1) and (1.4). They correspond to the electric sources $\mathbf{J}$ and $\mathbf{\rho}$ and make the Maxwell equations symmetric. Physical quantities corresponding to these additions do not exist, but it is convenient, when considering some antenna or scattering problems, to replace the actual sources by equivalent magnetic sources having properties that ensure the fields obey (1.5)–(1.8). (Elliott, 1981, p. 32). The equations 1.5–1.8 are called the \textit{time-invariant Maxwell equations} or the \textit{complex Maxwell equations}.

For linear, isotropic media the field terms are related by the constitutive equations,

$$\mathbf{D} = \mathbf{\varepsilon E}$$
$$\mathbf{B} = \mathbf{\mu H}$$
$$\mathbf{J} = \sigma \mathbf{E}$$
where the constants are respectively the permittivity, permeability, and conductivity of the medium in which the electromagnetic field exists.

### 1.2. THE ELECTROMAGNETIC TRAVELING WAVE

The nature of solutions to the Maxwell equations is brought out more completely in Chapter 2, but a simple development that illustrates the characteristics of certain solutions of importance in remote sensing is shown here.

In a lossless region, with current and charge densities zero, the time-invariant Maxwell curl equations are

\[ \nabla \times \mathbf{E} = -j \omega \mu_0 \mathbf{H} \quad (1.9) \]
\[ \nabla \times \mathbf{H} = j \omega \varepsilon_0 \mathbf{E} \quad (1.10) \]

\( \mathbf{E} \) and \( \mathbf{H} \) are functions of \( \mathbf{r} \), the vector distance from an origin to the point at which the fields are determined, but an important class of electromagnetic fields is that for which they depend, locally, only on the scalar distance from a point. Figure 1.1 shows the coordinates.

In the vicinity of point \( P \) we set

\[ E, \ H = E(r), \ H(r) \quad (1.11) \]

where we assume that the variation of \( E \) and \( H \) with \( \theta \) and \( \phi \) is negligible compared to the variation with \( r \). This functional variation of the fields accurately describes configurations with sources or reflecting objects in the vicinity of the origin if point \( P \) is far from sources or scatterers. It is incorrect in the vicinity of sources or reflectors.

---

**Fig. 1.1.** Coordinate system for the traveling wave.
If the curl equations 1.9 and 1.10 are expanded while treating the field components as functions only of $r$, they become

$$\nabla \times \mathbf{E} = -\frac{1}{r} \frac{d}{dr} (r E_\theta) \mathbf{u}_\theta + \frac{1}{r} \frac{d}{dr} (r E_\phi) \mathbf{u}_\phi = -j \omega \mu_0 (H_r \mathbf{u}_r + H_\theta \mathbf{u}_\theta + H_\phi \mathbf{u}_\phi)$$

$$\nabla \times \mathbf{H} = -\frac{1}{r} \frac{d}{dr} (r H_\phi) \mathbf{u}_\theta + \frac{1}{r} \frac{d}{dr} (r H_\theta) \mathbf{u}_\phi = j \omega \varepsilon_0 (E_r \mathbf{u}_r + E_\theta \mathbf{u}_\theta + E_\phi \mathbf{u}_\phi)$$

where $\mathbf{u}_r$, $\mathbf{u}_\theta$, and $\mathbf{u}_\phi$ are real unit vectors, shown in Fig. 1.1. Equating coefficients of like unit vectors in the first equation, differentiating the resulting equations, and substituting the coefficients of $\mathbf{u}_\theta$ and $\mathbf{u}_\phi$ from the curl of $\mathbf{H}$ gives

$$\frac{d^2 (r E_\theta, \phi)}{dr^2} + k^2 r E_\theta, \phi = 0$$

where $k^2 = \omega^2 \mu_0 \varepsilon_0$. Solutions to these equations are

$$E_{\theta, \phi} = \frac{C_{\theta, \phi}}{r} e^{\pm jkr}$$

$\mathbf{H}$ can be found similarly, and its components are

$$H_\theta = \pm \frac{C_\phi / Z_0}{r} e^{\pm jkr}$$

$$H_\phi = \mp \frac{C_\theta / Z_0}{r} e^{\pm jkr}$$

where $Z_0 = \sqrt{\mu_0 / \varepsilon_0}$.

If we take one of the field components, say $E_\theta$, and write its corresponding time-varying form, we obtain

$$\tilde{E}_\theta = \text{Re} \left( E_\theta e^{j\omega t} \right) = \frac{|C_\theta|}{r} \cos(\omega t \pm k r + \Phi)$$

where $\Phi$ is some phase angle. This equation represents a component of a traveling electromagnetic wave that appears to move in the increasing $r$ direction for the negative sign in the cosine function and in the decreasing $r$ direction for the positive sign. We reject the wave traveling toward the origin for physical reasons and retain the negative sign in the cosine wave.

When the Maxwell curl equations were expanded in spherical coordinates, with the assumption that the fields vary only with $r$, the curl of $\mathbf{E}$ had no radial component. It follows from (1.9) that $\mathbf{H}$ does not have a radial component. Likewise, since the curl of $\mathbf{H}$ has no radial component, $\mathbf{E}$ does not have a radial component.
Now, we can write the field vectors whose components vary only with radial distance \( r \). The solutions represent a specialized electromagnetic field, but one of great importance in remote sensing. The fields are

\[
\mathbf{E} = \frac{1}{r} (C_\theta \mathbf{u}_\theta + C_\phi \mathbf{u}_\phi) e^{-jkr} \quad (1.12)
\]

\[
\mathbf{H} = \frac{1}{r Z_0} (-C_\phi \mathbf{u}_\theta + C_\theta \mathbf{u}_\phi) e^{-jkr} \quad (1.13)
\]

It is apparent from these equations that \( \mathbf{E} \) and \( \mathbf{H} \) are perpendicular to \( \mathbf{u}_r \), the direction of wave travel. Further, the scalar product of \( \mathbf{E} \) and \( \mathbf{H} \), neglecting the phase variation with \( r \), is zero, showing that \( \mathbf{E} \) and \( \mathbf{H} \) are perpendicular to each other. The phenomenon described by these equations is a spherical wave traveling outward from a coordinate origin.

The assumption (1.11) is valid only in a local sense; that is, in the vicinity of a selected point, \( P \). At a different point, the field coefficients will differ from the values at \( P \) and the direction of wave travel will be different. At \( P \), radial distance \( r \) is large compared to the dimensions of the region in which we require our assumptions about the behavior of the fields to be valid. Then a surface of constant \( r \) is almost a plane surface. It is common to describe the electromagnetic wave as a plane wave, one with amplitudes and phases constant over a plane, rather than over a spherical surface, and to construct at \( P \) a rectangular coordinate system with an axis pointing in the direction of wave travel. If the wave travels in the \( z \) direction, the fields can be written as

\[
\mathbf{E} = (E_x \mathbf{u}_x + E_y \mathbf{u}_y) e^{-jkz} \quad (1.14)
\]

\[
\mathbf{H} = \frac{1}{Z_0} (-E_y \mathbf{u}_x + E_x \mathbf{u}_y) e^{-jkz} \quad (1.15)
\]

where \( \mathbf{u}_x \) and \( \mathbf{u}_y \) are unit vectors. In the newly constructed coordinate system, the phase referenced to the original origin may be discarded, and the coefficient amplitudes vary so slowly with large \( r \) that the variation is neglected. Note that the fields of (1.14) and (1.15) satisfy the free-space wave equation to be discussed in Chapter 2 and have significance other than as an approximation to (1.12) and (1.13).

Two other wave descriptors are commonly given. From \( \cos(\omega t - kr) \), we note that at a constant position \( r_0 \), the phase of the wave changes by \( 2\pi \) radians when time changes by \( 2\pi/\omega \). The time interval

\[ T = \frac{2\pi}{\omega} = \frac{1}{f} \]

is the wave period. At constant time \( t_0 \), the phase changes by \( 2\pi \) radians when the radial distance changes by \( 2\pi/k \). The increment in \( r \) corresponding to this
$2\pi$ phase change is the wavelength. It is

$$\lambda = \Delta r = \frac{2\pi}{k}$$

If one envisions the wave in space at a constant time, the wavelength is the length of one complete sine wave cycle. In a lossless region with constants those of a vacuum, or approximately those of air,

$$\lambda = \frac{2\pi}{2\pi f \sqrt{\mu_0 \varepsilon_0}} = \frac{c}{f}$$

where $c$ is the velocity of light in a vacuum.

### 1.3. POWER DENSITY

An electromagnetic field stores energy. If the field varies with time, the energy storage is dynamic and there is a relationship between the rate of change of the stored energy and the flow of energy. Poynting's theorem states that the rate of energy flow across surface $S$ is given by

$$\int \int_S (\tilde{E} \times \tilde{H}) \cdot n \, da$$

where $n$ is a surface normal vector. A time average of this integral is of interest. It represents the power density in W/m$^2$ in an electromagnetic wave. It is straightforward to show that

$$\overline{\tilde{E} \times \tilde{H}} = \frac{1}{2} \text{Re}(\tilde{E} \times \tilde{H}^*)$$

where the overline denotes the time average.

We define a complex Poynting vector by

$$\mathcal{P}_c = \frac{1}{2} \tilde{E} \times \tilde{H}^*$$

(1.16)

The real power flow is given by

$$\mathcal{P} = \text{Re}(\mathcal{P}_c)$$

(1.17)

The Poynting vector gives both the direction of wave travel and the power density.

From the fields of a plane wave, (1.14) and (1.15), the complex Poynting vector of the wave is found to be

$$\mathcal{P}_c = \frac{1}{2} \tilde{E} \times \tilde{H}^* = \frac{|E_x|^2 + |E_y|^2}{2Z_0^*} \mathbf{u}_z = \frac{1}{2Z_0^*} |\mathbf{E}|^2 \mathbf{u}_z$$

(1.18)
1.4. THE POLARIZATION ELLIPSE

The tip of the electric field intensity vector of a single-frequency wave traces an ellipse at a fixed position in space as time increases (Mott, 1992, p. 117). Such a wave is said to be elliptically polarized. This property is shown here for a plane wave. A plane wave traveling in the $z$ direction has two complex components and may be written as

$$
E = (E_x u_x + E_y u_y) e^{-jkz} = (u_x |E_x| e^{j\Phi_x} + u_y |E_y| e^{j\Phi_y}) e^{-jkz}
$$

The corresponding time-varying field is

$$
\tilde{E} = u_x |E_x| \cos(\beta + \Phi_x) + u_y |E_y| \cos(\beta + \Phi_y)
$$

where

$$
\beta = \omega t - kz
$$

The components can be combined to give

$$
\frac{\tilde{E}_x^2}{|E_x|^2} - 2 \frac{\tilde{E}_x \tilde{E}_y}{|E_x| |E_y|} \cos \Phi + \frac{\tilde{E}_y^2}{|E_y|^2} = \sin^2 \Phi
$$

where $\Phi = \Phi_y - \Phi_x$. This is the equation of an ellipse whose major axis is tilted at angle $\tau$ to the $\tilde{E}_x$ axis. It is shown in Fig. 1.2. With increasing time at fixed position $z$, the tip of the electric field vector traces the ellipse. Tilt angle $\tau$ is defined over the range $-\pi/2 \leq \tau \leq \pi/2$.

Define rotated coordinates $\tilde{E}_\xi$ and $\tilde{E}_\eta$ to coincide with the ellipse axes. The fields $\tilde{E}_\xi$ and $\tilde{E}_\eta$ are related to $\tilde{E}_x$ and $\tilde{E}_y$ by

$$
\begin{bmatrix}
\tilde{E}_\xi \\
\tilde{E}_\eta
\end{bmatrix} =
\begin{bmatrix}
\cos \tau & \sin \tau \\
-\sin \tau & \cos \tau
\end{bmatrix}
\begin{bmatrix}
\tilde{E}_x \\
\tilde{E}_y
\end{bmatrix}
$$

The fields can be written as

$$
\tilde{E}_\xi = m \cos(\beta + \Phi_0) \quad (1.22)
\tilde{E}_\eta = n \cos(\beta + \Phi_0 \pm \pi/2) = \pm n \sin(\beta + \Phi_0) \quad (1.23)
$$

where $\Phi_0$ is a phase angle that need not be determined, and $m$ and $n$ are positive real. If we require that $m \geq n$, $m$ is the semimajor axis of the ellipse and $n$ the semiminor.

If the positive sign is used before $\pi/2$ in (1.23), the electric vector rotates with one sense as time increases; if the negative sign is used, the rotation has the opposite sense.
We equate (1.21) and (1.22)–(1.23) and use (1.19) for $\tilde{E}_x$ and $\tilde{E}_y$. This gives

\[ m \cos(\beta + \Phi_0) = |E_x| \cos(\beta + \Phi_x) \cos \tau + |E_y| \cos(\beta + \Phi_y) \sin \tau \]
\[ \pm n \sin(\beta + \Phi_0) = -|E_x| \cos(\beta + \Phi_x) \sin \tau + |E_y| \cos(\beta + \Phi_y) \cos \tau \]

If the coefficients of $\cos \beta$ are equated and also the coefficients of $\sin \beta$, the following relationships are obtained:

\[ m^2 + n^2 = |E_x|^2 + |E_y|^2 \tag{1.24} \]
\[ \pm mn = -|E_x||E_y| \sin \Phi \tag{1.25} \]
\[ \pm \frac{n}{m} = \frac{|E_x| \sin \Phi_x \sin \tau - |E_y| \sin \Phi_y \cos \tau}{|E_x| \cos \Phi_x \cos \tau + |E_y| \cos \Phi_y \sin \tau} \]
\[ = \frac{-|E_x| \cos \Phi_x \sin \tau + |E_y| \cos \Phi_y \cos \tau}{|E_x| \sin \Phi_x \cos \tau + |E_y| \sin \Phi_y \sin \tau} \tag{1.26} \]

Cross-multiplying and collecting terms in the last equation of this set gives

\[ (|E_x|^2 - |E_y|^2) \sin 2\tau = 2|E_x||E_y| \cos 2\tau \cos \Phi \]

Tilt angle $\tau$ may be found from

\[ \tan 2\tau = \frac{2|E_x||E_y|}{|E_x|^2 - |E_y|^2} \cos \Phi \tag{1.27} \]
The ellipse shape can be specified by the axial ratio \( m/n \) or by the ellipticity angle \( \epsilon \) shown in Fig. 1.2. Both positive and negative values of \( \epsilon \), with the same magnitude, are shown. It is desirable for a graphic representation of wave polarization to use the negative value of \( \epsilon \) if positive signs are used in (1.25) and (1.26). We therefore define

\[
\tan \epsilon = \mp \frac{n}{m} - \frac{\pi}{4} \leq \epsilon \leq \frac{\pi}{4}
\]

If this equation is combined with (1.24) and (1.25), the result is

\[
\sin 2\epsilon = \frac{2|E_x||E_y|}{|E_x|^2 + |E_y|^2} \sin \Phi \tag{1.28}
\]

The time-varying angle of \( \tilde{E} \), measured from the \( x \) toward the \( y \) axis, is

\[
\Psi = \tan^{-1} \left( \frac{\tilde{E}_y}{\tilde{E}_x} \right) = \tan^{-1} \left( \frac{|E_y| \cos(\beta + \Phi_y)}{|E_x| \cos(\beta + \Phi_x)} \right) \tag{1.29}
\]

If the derivative of this angle with respect to \( \beta \) is examined, it will be seen that

\[
\frac{\partial \Psi}{\partial \beta} < 0, \quad 0 < \Phi < \pi
\]

If we look in the direction of wave propagation, a positive value of \( \frac{\partial \Psi}{\partial \beta} \) corresponds to clockwise rotation of \( \tilde{E} \) as \( \beta \) (or time) increases. By definition, this is right-handed rotation of the vector. A negative value of the derivative corresponds to counterclockwise or left-handed rotation. It can be seen from (1.28) and the ranges of \( \Phi \) that the corresponding ellipticity angle ranges are

\[
\epsilon \begin{cases} 
< 0, \quad \text{right-handed rotation} \\
> 0, \quad \text{left-handed rotation}
\end{cases}
\]

If the defining equation for the ellipticity angle is used in (1.22) and (1.23), the field components can be written as

\[
\tilde{E}_\xi = m \cos(\beta + \Phi_0) \\
\tilde{E}_\eta = -m \tan \epsilon \cos(\beta + \Phi_0 - \pi/2)
\]

If the variation with time and distance is suppressed, the field is

\[
E(\xi, \eta) = m \left[ \frac{1}{j \tan \epsilon} \right] e^{j\Phi_0} = \frac{m}{\cos \epsilon} \left[ \frac{\cos \epsilon}{j \sin \epsilon} \right] e^{j\Phi_0}
\]
The electric field intensities in the two coordinate systems are related by

\[ E(x, y) = \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix} E(\xi, \eta) \]

Combining the last two equations allows the electric field of a single-frequency plane wave to be written in terms of tilt and ellipticity angles by

\[ E(x, y) = \frac{m}{\cos \epsilon} \begin{bmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{bmatrix} \begin{bmatrix} \cos \epsilon \\ j \sin \epsilon \end{bmatrix} e^{j\Phi_0} \]  
\[ \text{(1.30)} \]

The common phase term is normally of little significance and can be omitted.

*Note:* In the preceding three equations and in similar equations throughout the text, the symbols in parentheses denote vector components and not a functional relationship.

**Linear and Circular Polarization**

In the special cases of \(|E_x| = 0\), or \(|E_y| = 0\), or \(\Phi = 0\), the polarization ellipse degenerates to a straight line, and the wave is *linearly polarized*. The axial ratio is infinite, the tilt angle can be found by the standard equations, and rotation sense is meaningless.

If \(|E_x| = |E_y|\) and \(\Phi = \pi/2\), the axial ratio is equal to 1, the polarization ellipse degenerates to a circle, and the wave is *circularly polarized*, right circular if \(\Phi = -\pi/2\) and left circular if \(\Phi = \pi/2\).

**Rotation of \(\tilde{E}\) with Distance**

The rotation rates of the electric field vector with time \(t\) and distance \(z\), from (1.20), are

\[ \frac{\partial \Psi}{\partial t} = \omega \frac{\partial \Psi}{\partial \beta} \]
\[ \frac{\partial \Psi}{\partial z} = -k \frac{\partial \Psi}{\partial \beta} \]

where \(\Psi\) is the angle of the time-varying electric field vector measured from the \(x\)-axis. We see from these equations that a wave which appears to rotate in a clockwise sense as we look in the direction of wave travel at some fixed position as \(t\) increases, corresponding to our definition of a right-handed wave, appears to rotate counterclockwise with an increase in \(z\) at a fixed time. A right-handed circular wave drawn in space at some constant time looks like a left-handed screw. Conversely, a left-handed circular wave appears in space to be a right-handed screw. With increasing time, the left-handed screw representing a right-handed wave rotates in a clockwise direction as we look in the direction of wave motion.
It can be seen from (1.20) and (1.29) that the distance between two points of the wave having parallel field vectors at constant time is

$$\Delta z = \frac{2\pi}{k} = \lambda$$

The wave appears to rotate once in space in a distance of one wavelength.

### 1.5. POLARIZATION VECTOR AND POLARIZATION RATIO

A description of an elliptically polarized wave in terms of tilt angle, axial ratio, and rotation sense leads to a physical understanding of the wave, but a more convenient mathematical description of the wave is needed. The time-invariant electric field itself contains all information about the wave polarization, and so does $p$, the polarization vector or the polarization state of the wave. This is the electric field normalized by its magnitude. Another useful descriptor is the complex polarization ratio, $P$. The electric field can be written as

$$E = E_x u_x + E_y u_y = E_x (u_x + Pu_y) e^{-jkz}$$

where $P$ is the polarization ratio,

$$P = \frac{E_y}{E_x} = \frac{|E_y|}{|E_x|} e^{j\Phi} \quad (1.31)$$

The value of $E_x$ does not affect the wave polarization, and it can be neglected unless power density or received power is required. Values of the polarization ratio are $\infty$, $0$, $j$, $-j$ for linear vertical ($y$-directed), linear horizontal ($x$-directed), and left- and right-circular waves, respectively.

### 1.6. CIRCULAR WAVE COMPONENTS

To this point, a plane electromagnetic wave has been considered the sum of two linearly polarized plane waves perpendicular to each other and to the direction of wave travel. It may also be considered the sum of left- and right-circular plane waves, and the common use of such a description justifies the formulation.

The orthonormal vectors

$$u_L = \frac{1}{\sqrt{2}}(u_x + ju_y) = \frac{1}{\sqrt{2}}u_x + \frac{1}{\sqrt{2}}e^{j\pi/2}u_y \quad (1.32)$$

$$u_R = \frac{1}{\sqrt{2}}(u_x - ju_y) = \frac{1}{\sqrt{2}}u_x + \frac{1}{\sqrt{2}}e^{-j\pi/2}u_y \quad (1.33)$$
together with \( \mathbf{u}_z \), which is orthogonal to both, are a vector triplet that defines a coordinate system that we will call a circular-coordinate system. A time-invariant electric field expressed in terms of an \( xyz \) rectangular system can be converted to the circular-coordinate system using (1.32) and (1.33),

\[
E = E_x \mathbf{u}_x + E_y \mathbf{u}_y = E_L \mathbf{u}_L + E_R \mathbf{u}_R = |E_L|e^{j\Theta_L} \mathbf{u}_L + |E_R|e^{j\Theta_R} \mathbf{u}_R
\]

The field components are related by

\[
\begin{bmatrix}
E_L \\
E_R
\end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -j \\
j & 1
\end{bmatrix} \begin{bmatrix}
E_x \\
E_y
\end{bmatrix}
\] (1.34)

Let us examine the real time-varying field components associated with \( \mathbf{u}_L \) and \( \mathbf{u}_R \). They are

\[
\begin{align*}
\text{Re} \left( E_L \mathbf{u}_L e^{j(\omega t - kz)} \right) & = \frac{|E_L|}{\sqrt{2}} \mathbf{u}_x \cos(\omega t - kz + \Theta_L) \\
& + \frac{|E_L|}{\sqrt{2}} \mathbf{u}_y \cos(\omega t - kz + \Theta_L + \pi/2) \\
\text{Re} \left( E_R \mathbf{u}_R e^{j(\omega t - kz)} \right) & = \frac{|E_R|}{\sqrt{2}} \mathbf{u}_x \cos(\omega t - kz + \Theta_R) \\
& + \frac{|E_R|}{\sqrt{2}} \mathbf{u}_y \cos(\omega t - kz + \Theta_R - \pi/2)
\end{align*}
\]

These can be recognized as real time-varying vectors of constant amplitude rotating, respectively, in a left- and right-handed sense. The vector \( \mathbf{u}_L \) represents a left-circular wave and \( \mathbf{u}_R \) a right-circular, and \( E_L \) and \( E_R \) are circular wave components. The linear polarization ratio of a wave was defined previously as the ratio of linear components of the wave. We define the **circular polarization ratio** as the ratio of circular wave components,

\[
Q = \frac{E_R}{E_L} = \frac{|E_R|}{|E_L|} e^{j\Theta}
\] (1.35)

where \( \Theta = \Theta_R - \Theta_L \).

### 1.7. CHANGE OF POLARIZATION BASIS

A plane wave traveling in the \( z \) direction can be considered the sum of two orthogonally polarized waves, neither linear nor circular, but elliptical. Let

\[
E = E_1 + E_2 = E_1 \mathbf{u}_1 + E_2 \mathbf{u}_2
\] (1.36)

where \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \) are complex orthonormal unit vectors.