QUANTITATIVE FINANCIAL ECONOMICS

STOCKS, BONDS AND FOREIGN EXCHANGE

Second Edition

KEITH CUTHBERTSON

AND

DIRK NITZSCHE

John Wiley & Sons, Ltd

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To all our students who have done well enough to be in a position to hire finance consultants

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PREFACE

Numerous emails and the fact that the first edition sold out suggests that many found it a positive NPV investment. This is encouraging, as unless these things are turned into a major motion picture, the cash rewards to the author are not great. My wife currently believes that the direct return per hour of effort is less than that of a competent plumber – and she's right. But she has yet to factor in positive externalities and real options theory – that's my counter argument anyway. Nevertheless, being risk averse and with time-varying long-term net liabilities, I do not intend giving up my day job(s).

When invited to dinner, accompanied by the finest wines someone else can buy, and asked to produce a second edition, there comes a critical point late in the evening when you invariably say, 'Yes'. This is a mistake. The reason is obvious. Were it not for electronic copies, the mass of 'stuff' in this area that has appeared over the last 10 years would fill the Albert Hall. The research and first draft were great, subsequent drafts less so and by the end it was agony – Groundhog Day. For me this could have easily been 'A Book Too Far'. But fortuitously, for the second edition, I was able to engage an excellent forensic co-author in Dirk Nitzsche.

For those of you who bought the first edition, a glance at the 'blurb' on the cover tells you that the second edition is about 65% new material and the 'old material' has also been revamped. We hope we have chosen a coherent, interesting and varied range of topics. For those who illegally photocopied the first edition and maybe also got me to sign it – the photocopy that is – I admire your dedication to the cause of involuntary personal contributions to foreign aid. But I hope you think that for this much-improved second edition, you could 'Show Me The Money'.

Who's it for?

The book is aimed at students on quantitative MSc's in finance and financial economics and should also be useful on PhD programmes. All you need as a pre-requisite is a basic undergraduate course in theory of finance (with the accompanying math) and something on modern time-series econometrics. Any finance practitioners who want to 'get a handle' on whether there is any practical value in all that academic stuff (answer, 'Yes there is – sometimes'), should also dip in at appropriate points. At least, you will then be able to spot whether the poor Emperor, as he emerges from his ivory tower and saunters into the market place, is looking sartorially challenged.

In the book, we cover the main theoretical ideas in the pricing of (spot) assets and the determination of strategic investment decisions (using discrete time analysis), as well as analysing specific trading strategies. Illustrative empirical results are provided, although these are by no means exhaustive (or we hope exhausting). The emphasis is on the intuition behind the finance and economic concepts and the math and stats we need to analyse these ideas in a rigorous fashion. We feel that the material allows 'entry' into recent research work in these areas and we hope it also encourages the reader to move on to explore derivatives pricing, financial engineering and risk management.

We hope you enjoy the book and it leads you to untold riches – who knows, maybe this could be the beginning of a beautiful friendship. Anyway, make the most of it, as after all our past efforts, our goal from now on is to understand everything and publish nothing. Whether this will increase social welfare, only time and you can tell.

Keith Cuthbertson
Dirk Nitzsche
October 2004

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Many thanks to Yvonne Doyle at Imperial College, who expertly typed numerous drafts and is now well on her way to becoming an expert in Greek.

1

BASIC CONCEPTS IN FINANCE

Aims

- To consider different methods of measuring returns for pure discount bonds, coupon-paying bonds and stocks.
- Use discounted present value techniques, DPV, to price assets.
- Show how utility functions can be used to incorporate risk aversion, and derive asset demand functions from one-period utility maximisation.
- Illustrate the optimal level of physical investment and consumption for a two-period horizon problem.

The aim of this chapter is to quickly run through some of the basic tools of analysis used in finance literature. The topics covered are not exhaustive and they are discussed at a fairly intuitive level.

1.1 Returns on Stocks, Bonds and Real Assets

Much of the theoretical work in finance is conducted in terms of compound rates of return or interest rates, even though rates quoted in the market use 'simple interest'. For example, an interest rate of 5 percent payable every six months will be quoted as a simple interest rate of 10 percent per annum in the market. However, if an investor rolled over two six-month bills and the interest rate remained constant, he could actually earn a 'compound' or 'true' or 'effective' annual rate of $(1.05)^2 = 1.1025$ or 10.25 percent. The effective annual rate of return exceeds the simple rate because in the former case the investor earns 'interest-on-interest'.

We now examine how we calculate the terminal value of an investment when the frequency with which interest rates are compounded alters. Clearly, a quoted interest rate of 10 percent per annum when interest is calculated monthly will amount to more at the end of the year than if interest accrues only at the end of the year.

Consider an amount $\$A$ invested for n years at a rate of R per annum (where R is expressed as a decimal). If compounding takes place only at the end of the year, the future value after n years is FV_n, where

$$FV_n = \$A(1 + R)^n \tag{1}$$

However, if interest is paid m times per annum, then the terminal value at the end of n years is

$$FV_n^m = \$A(1 + R/m)^{mn} \tag{2}$$

R/m is often referred to as the periodic interest rate. As m, the frequency of compounding, increases, the rate becomes 'continuously compounded', and it may be shown that the investment accrues to

$$FV_n^c = \$Ae^{R_c n} \tag{3}$$

where $R_c =$ the continuously compounded rate per annum. For example, if the quoted (simple) interest rate is 10 percent per annum, then the value of \$100 at the end of one year ($n = 1$) for different values of m is given in Table 1. For *daily* compounding, with $R = 10\%$ p.a., the terminal value after one year using (2) is \$110.5155. Assuming $R_c = 10\%$ gives $FV_n^c = \$100e^{0.10(1)} = \$100.5171$. So, daily compounding is almost equivalent to using a continuously compounded rate (see the last two entries in Table 1).

We now consider how to switch between simple interest rates, periodic rates, effective annual rates and continuously compounded rates. Suppose an investment pays a periodic interest rate of 2 percent each quarter. This will usually be quoted in the market as 8 percent per annum, that is, as a simple annual rate. At the end of the year, $\$A = \$100$ accrues to

$$\$A(1 + R/m)^m = 100(1 + 0.08/4)^4 = \$108.24 \tag{4}$$

The effective annual rate R_e is 8.24% since $\$100(1 + R_e) = 108.24$. R_e exceeds the simple rate because of the payment of interest-on-interest. The relationship between

Table 1 Compounding frequency

Compounding Frequency	Value of \$100 at End of Year ($R = 10\%$ p.a.)
Annually ($m = 1$)	110.00
Quarterly ($m = 4$)	110.38
Weekly ($m = 52$)	110.51
Daily ($m = 365$)	110.5155
Continuous ($n = 1$)	110.5171

the quoted simple rate R with payments m times per year and the *effective annual rate* R_e is

$$(1 + R_e) = (1 + R/m)^m \tag{5}$$

We can use (5) to move from periodic interest rates to effective rates and vice versa. For example, an interest rate with quarterly payments that would produce an effective annual rate of 12 percent is given by $1.12 = (1 + R/4)^4$, and hence,

$$R = [(1.12)^{1/4} - 1]4 = 0.0287(4) = 11.48\% \tag{6}$$

So, with interest compounded quarterly, a simple interest rate of 11.48 percent per annum is equivalent to a 12 percent effective rate.

We can use a similar procedure to switch between a simple interest rate R, which applies to compounding that takes place over m periods, and an equivalent continuously compounded rate R_c. One reason for doing this calculation is that much of the advanced theory of bond pricing (and the pricing of futures and options) uses continuously compounded rates.

Suppose we wish to calculate a value for R_c when we know the m-period rate R. Since the terminal value after n years of an investment of $\$A$ must be equal when using either interest rate we have

$$Ae^{R_c n} = A(1 + R/m)^{mn} \tag{7}$$

and therefore,

$$R_c = m \ln[1 + R/m] \tag{8}$$

Also, if we are given the continuously compounded rate R_c, we can use the above equation to calculate the simple rate R, which applies when interest is calculated m times per year:

$$R = m(e^{R_c/m} - 1) \tag{9}$$

We can perhaps best summarise the above array of alternative interest rates by using one final illustrative example. Suppose an investment pays a periodic interest rate of 5 percent every six months ($m = 2$, $R/2 = 0.05$). In the market, this might be quoted as a 'simple rate' of 10 percent per annum. An investment of \$100 would yield $100[1 + (0.10/2)]^2 = \$110.25$ after one year (using equation 2). Clearly, the effective annual rate is 10.25% p.a. Suppose we wish to convert the simple annual rate of $R = 0.10$ to an equivalent continuously compounded rate. Using (8), with $m = 2$, we see that this is given by $R_c = 2\ln(1 + 0.10/2) = 0.09758$ (9.758% p.a.). Of course, if interest is continuously compounded at an annual rate of 9.758 percent, then \$100 invested today would accrue to $100\ e^{R_c \cdot n} = \$110.25$ in $n = 1$ year's time.

Arithmetic and Geometric Averages

Suppose prices in successive periods are $P_0 = 1$, $P_1 = 0.7$ and $P_2 = 1$, which correspond to (periodic) returns of $R_1 = -0.30$ (−30%) and $R_2 = 0.42857$ (42.857%). The *arithmetic average* return is $\overline{R} = (R_1 + R_2)/2 = 6.4285\%$. However, it would be incorrect to assume that if you have an initial wealth $W_0 = \$100$, then your final wealth after 2 periods will be $W_2 = (1 + \overline{R})W_0 = \$106.4285$. Looking at the price series it is clear that your wealth is unchanged between $t = 0$ and $t = 2$:

$$W_2 = W_0[(1 + R_1)(1 + R_2)] = \$100\ (0.70)(1.42857) = \$100$$

Now define the *geometric average* return as

$$(1 + \overline{R}_g)^2 = (1 + R_1)(1 + R_2) = 1$$

Here $\overline{R}_g = 0$, and it correctly indicates that the return on your 'wealth portfolio' $R_w(0 \to 2) = (W_2/W_0) - 1 = 0$ between $t = 0$ and $t = 2$. Generalising, the geometric average return is defined as

$$(1 + \overline{R}_g)^n = (1 + R_1)(1 + R_2) \cdots (1 + R_n) \tag{10}$$

and we can always write

$$W_n = W_0(1 + \overline{R}_g)^n$$

Unless (periodic) returns R_t are constant, the geometric average return is always less than the arithmetic average return. For example, using one-year returns R_t, the geometric average return on a US equity value weighted index over the period 1802–1997 is 7% p.a., considerably lower than the arithmetic average of 8.5% p.a. (Siegel 1998).

If returns are serially uncorrelated, $R_t = \mu + \varepsilon_t$ with $\varepsilon_t \sim iid(0, \sigma^2)$, then the arithmetic average is the best return forecast for any *randomly selected* future year. Over long holding periods, the best *forecast* would also use the arithmetic average return compounded, that is, $(1 + \overline{R})^n$. Unfortunately, the latter clear simple result does not apply in practice over long horizons, since stock returns are not *iid*.

In our simple example, if the sequence is repeated, returns are negatively serially correlated (i.e. −30%, +42.8%, alternating in each period). In this case, forecasting over long horizons requires the use of the geometric average return compounded, $(1 + \overline{R}_g)^n$. There is evidence that over long horizons stock returns are 'mildly' mean reverting (i.e. exhibit some negative serial correlation) so that the arithmetic average overstates *expected* future returns, and it may be better to use the geometric average as a *forecast* of future average returns.

Long Horizons

The (periodic) return is $(1 + R_1) = P_1/P_0$. In intertemporal models, we often require an expression for terminal wealth:

$$W_n = W_0(1 + R_1)(1 + R_2) \cdots (1 + R_n)$$

Alternatively, this can be expressed as

$$\begin{aligned} \ln(W_n/W_0) &= \ln(1 + R_1) + \ln(1 + R_2) + \cdots + \ln(1 + R_n) \\ &= (R_{c1} + R_{c2} + \cdots + R_{cn}) = \ln(P_n/P_0) \end{aligned}$$

where $R_{ct} \equiv \ln(1 + R_t)$ are the continuously compounded rates. Note that the term in parentheses is equal to $\ln(P_n/P_0)$. It follows that

$$W_n = W_0 \exp(R_{c1} + R_{c2} + \cdots + R_{cn}) = W_0(P_n/P_0)$$

Continuously compounded rates are additive, so we can *define* the (total continuously compounded) return over the whole period from $t = 0$ to $t = n$ as

$$R_c(0 \to n) \equiv (R_{c1} + R_{c2} + \cdots + R_{cn})$$
$$W_n = W_0 \exp[R_c(0 \to n)]$$

Let us now 'connect' the continuously compounded returns to the geometric average return. It follows from (10) that

$$\ln(1 + \overline{R}_g)^n = (R_{c1} + R_{c2} + \cdots + R_{cn}) \equiv R_c(0 \to n)$$

Hence

$$W_n = W_0 \exp[\ln(1 + \overline{R}_g)^n] = W_0(1 + \overline{R}_g)^n$$

as we found earlier.

Nominal and Real Returns

A number of asset pricing models focus on real rather than nominal returns. The real return is the (percent) rate of return from an investment, in terms of the purchasing power over goods and services. A real return of, say, 3% p.a. implies that your initial investment allows you to purchase 3% more of a fixed basket of domestic goods (e.g. Harrod's Hamper for a UK resident) at the end of the year.

If at $t = 0$ you have a nominal wealth W_0, then your real wealth is $W_0^r = W_0/P_o^g$, where P^g = price index for goods and services. If R = nominal (proportionate) return on your wealth, then at the end of year-1 you have nominal wealth of $W_0(1 + R)$ and real wealth of

$$W_1^r \equiv \frac{W_1}{P_1^g} = \frac{(W_0^r P_o^g)(1 + R)}{P_1^g}$$

Hence, the increase in your real wealth or, equivalently, your (proportionate) real return is

$$(1 + R^r) \equiv W_1^r/W_0^r = (1 + R)/(1 + \pi) \tag{11}$$

$$R^r \equiv \frac{\Delta W_1^r}{W_0^r} = \frac{R - \pi}{1 + \pi} \approx R - \pi \tag{12}$$

where $1 + \pi \equiv (P_1^g/P_0^g)$. The proportionate change in real wealth is your real return R^r, which is *approximately* equal to the nominal return R minus the rate of goods price inflation, π. In terms of continuously compounded returns,

$$\ln(W_1^r/W_0^r) \equiv R_c^r = \ln(1 + R) - \ln(P_1^g/P_o^g) = R_c - \pi_c \tag{13}$$

where $R_c =$ (continuously compounded) nominal return and $\pi_c =$ continuously compounded rate of inflation. Using continuously compounded returns has the advantage that the log real return over a horizon $t = 0$ to $t = n$ is additive:

$$\begin{aligned} R_c^r(0 \to n) &= (R_{c1} - \pi_{c1}) + (R_{c2} - \pi_{c2}) + \cdots + (R_{cn} - \pi_{cn}) \\ &= (R_{c1}^r + R_{c2}^r + \cdots + R_{cn}^r) \end{aligned} \tag{14}$$

Using the above, if initial real wealth is W_0^r, then the level of real wealth at $t = n$ is $W_n^r = W_0^r e^{R_c^n(0 \to n)} = W_0^r e^{(R_{c1}^r + R_{c2}^r + \cdots + R_{cn}^r)}$. Alternatively, if we use proportionate changes, then

$$W_n^r = W_0^r(1 + R_1^r)(1 + R_2^r) \cdots (1 + R_n^r) \tag{15}$$

and the *annual average geometric real return* from $t = 0$ to $t = n$, denoted $\overline{R}_{r,g}$ is given by

$$(1 + \overline{R}_{r,g}) = \sqrt[n]{(1 + R_1^r)(1 + R_2^r) \cdots (1 + R_n)^r}$$

and $W_n^r = W_0^r(1 + \overline{R}_{r,g})^n$

Foreign Investment

Suppose you are considering investing abroad. The nominal *return measured in terms of your domestic currency* can be shown to equal the foreign currency return (sometimes called the *local currency return*) plus the appreciation in the foreign currency. By investing abroad, you can gain (or lose) either from holding the foreign asset or from changes in the exchange rate. For example, consider a UK resident with initial nominal wealth W_0 who exchanges (the UK pound) sterling for USDs at a rate S_0 (£s per \$) and invests in the United States with a nominal (proportionate) return R^{us}. Nominal wealth in Sterling at $t = 1$ is

$$W_1 = \frac{W_0(1 + R^{us})S_1}{S_0} \tag{16}$$

Hence, using $S_1 = S_0 + \Delta S_1$, the (proportionate) nominal return to foreign investment for a UK investor is

$$R(UK \to US) \equiv (W_1/W_0) - 1 = R^{us} + \Delta S_1/S_0 + R^{us}(\Delta S_1/S_0) \approx R^{US} + R^{FX} \tag{17}$$

where $R^{FX} = \Delta S_1/S_0$ is the (proportionate) appreciation of FX rate of the USD against sterling, and we have assumed that $R^{us}(\Delta S_1/S_0)$ is negligible. The nominal return to foreign investment is obviously

Nominal return(UK resident) = local currency(US)return + appreciation of USD

In terms of continuously compound returns, the equation is exact:

$$R_c(UK \to US) \equiv \ln(W_1/W_0) = R_c^{us} + \Delta s \tag{18}$$

where $R_c^{us} \equiv \ln(1 + R^{us})$ and $\Delta s \equiv \ln(S_1/S_0)$. Now suppose you are concerned about the *real* return of your foreign investment, in terms of purchasing power *over domestic goods*. The real return to foreign investment is just the nominal return less the domestic rate of price inflation. To demonstrate this, take a UK resident investing in the United States, but ultimately using any profits to spend on UK goods. Real wealth at $t = 1$, in terms of purchasing power over UK goods is

$$W_1^r = \frac{(W_0^r P_o^g)(1 + R^{us})S_1}{P_1^g S_0} \tag{19}$$

It follows that the continuously compounded and proportionate real return to foreign investment is

$$R_c^r(UK \to US) \equiv \ln(W_1^r/W_0^r) = R_c^{us} + \Delta s - \pi_c^{uk} \tag{20}$$

$$R^r(UK \to US) \equiv \Delta W_1^r/W_0^r \approx R^{us} + R^{FX} - \pi^{uk} \tag{21}$$

where $\Delta s = \ln(S_1/S_0)$. Hence, the real return $R^r(UK \to US)$ to a UK resident in terms of UK purchasing power from a round-trip investment in US assets is

Real return (UK resident) = nominal 'local currency' return in US

+ appreciation of USD − inflation in UK

From (20) it is interesting to note that the *real* return to foreign investment for a UK resident $R_c^r(UK \to US)$ would equal the real return to a US resident investing in the US, $(R_c^{us} - \pi_c^{us})$ if

$$\pi_c^{uk} - \pi_c^{us} = \Delta s \tag{22}$$

As we shall see in Chapter 24, equation (22) is the relative purchasing power parity (PPP) condition. Hence, if relative PPP holds, the *real* return to foreign investment is equal to the real local currency return $R_c^{us} - \pi_c^{us}$, and the change in the exchange rate is immaterial. This is because, under relative PPP, the exchange rate alters to just offset the differential inflation rate between the two countries. As relative PPP holds only over horizons of 5–10 years, the real return to foreign investment over shorter horizons will depend on exchange rate changes.

1.2 Discounted Present Value, DPV

Let the quoted annual rate of interest on a completely safe investment over n years be denoted as r_n. The future value of $\$A$ in n years' time with interest calculated annually is

$$FV_n = \$A(1 + r_n)^n \tag{23}$$

It follows that if you were given the opportunity to receive with certainty $\$FV_n$ in n years' time, then you would be willing to give up $\$A$ today. The value *today* of a certain payment of FV_n in n years' time is \$$A$. In a more technical language, the *discounted present value* DPV of FV_n is

$$DPV = FV_n/(1 + r_n)^n \tag{24}$$

We now make the assumption that the safe interest rate applicable to $1, 2, 3, \ldots, n$ year horizons is *constant* and equal to r. We are assuming that the term structure of interest rates is flat. The DPV of a *stream* of receipts FV_i ($i = 1$ to n) that carry no default risk is then given by

$$DPV = \sum_{i=1}^{n} FV_i/(1 + r)^i \tag{25}$$

Annuities

If the future payments are *constant* in each year ($FV_i = \$C$) and the first payment is at the end of the first year, then we have an *ordinary annuity*. The DPV of these payments is

$$DPV = C \sum_{i=1}^{n} 1/(1 + r)^i \tag{26}$$

Using the formula for the sum of a geometric progression, we can write the DPV of an ordinary annuity as

$$DPV = C \cdot A_{n,r} \quad \text{where } A_{n,r} = (1/r)[1 - 1/(1 + r)^n] \tag{27}$$

and $DPV = C/r$ as $n \to \infty$

The term $A_{n,r}$ is called the *annuity factor*, and its numerical value is given in annuity tables for various values of n and r. A special case of the annuity formula is when n approaches infinity, then $A_{n,r} = 1/r$ and $DPV = C/r$. This formula is used to price a bond called a perpetuity or console, which pays a coupon \$$C$ (but is never redeemed by the issuers). The annuity formula can be used in calculations involving constant payments such as mortgages, pensions and for pricing a coupon-paying bond (see below).

Physical Investment Project

Consider a physical investment project such as building a new factory, which has a set of prospective net receipts (profits) of FV_i. Suppose the capital cost of the project which we assume all accrues today (i.e. at time $t = 0$) is \$$KC$. Then the entrepreneur should invest in the project if

$$DPV \geq KC \tag{28}$$

or, equivalently, if the net present value NPV satisfies

$$NPV = DPV - KC \geq 0 \tag{29}$$

If $NPV = 0$, then it can be shown that the net receipts (profits) from the investment project are just sufficient to pay back both the principal (\$$KC$) and the interest on the

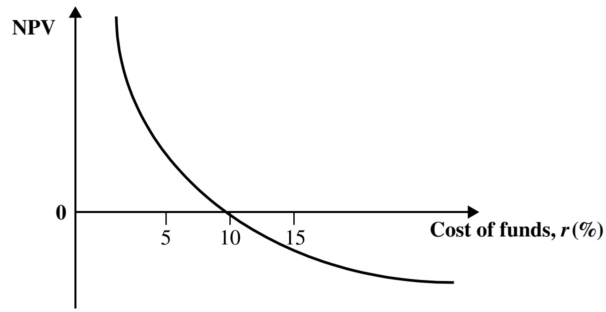

Figure 1 NPV and the discount rate

loan, which was taken out to finance the project. If $NPV > 0$, then there are surplus funds available even after these loan repayments.

As the cost of funds r increases, then the NPV falls for any given stream of profits FV_i from the project (Figure 1). There is a value of r (= 10% in Figure 1) for which the $NPV = 0$. This value of r is known as the internal rate of return IRR of the investment project. Given a stream of net receipts FV_i and the capital cost KC for a project, one can always calculate a project's IRR. It is that constant value of y for which

$$KC = \sum_{i=1}^{n} FV_i/(1+y)^i \tag{30}$$

An equivalent investment rule to the NPV condition (28) is to invest in the project if

$$IRR(=y) \geq \text{cost of borrowing } (=r) \tag{31}$$

There are some technical problems with IRR (which luckily are often not problematic in practice). First, a meaningful solution for IRR assumes all the $FV_i > 0$, and hence do not alternate in sign, because otherwise there may be more than one solution for the IRR. Second, the IRR should not be used to compare two projects as it may not give the same decision rule as NPV (see Cuthbertson and Nitzsche 2001a).

We will use these investment rules throughout the book, beginning in this chapter, with the derivation of the yield on bills and bonds and the optimal scale of physical investment projects for the economy. Note that in the calculation of the DPV, we assumed that the interest rate used for discounting the future receipts FV_i was constant for all horizons. Suppose that 'one-year money' carries an interest rate of r_1, two-year money costs r_2, and so on, then the DPV is given by

$$DPV = FV_1/(1+r_1) + FV_2/(1+r_2)^2 + \cdots + FV_n/(1+r_n)^n = \sum \delta_i FV_i \tag{32}$$

where $\delta_i = 1/(1+r_i)^i$. The r_i are known as *spot rates* of interest since they are the rates that apply to money that you lend over the periods $r_1 = 0$ to 1 year, $r_2 = 0$ to 2 years, and so on (expressed as annual compound rates). At any point in time, the relationship between the spot rates, r_i, on default-free assets and their maturity is known as the yield curve. For example, if $r_1 < r_2 < r_3$ and so on, then the yield curve is said to be upward sloping. The relationship between changes in short rates over time and changes in long rates is the subject of the term structure of interest rates.

The DPV formula can also be expressed in real terms. In this case, future receipts FV_i are deflated by the aggregate goods price index and the discount factors are calculated using real rates of interest.

In general, physical investment projects are not riskless since the future receipts are uncertain. There are a number of alternative methods of dealing with uncertainty in the DPV calculation. Perhaps, the simplest method, and the one we shall adopt, has the discount rate δ_i consisting of the risk-free spot rate r_i plus a risk premium rp_i.

$$\delta_i = (1 + r_i + rp_i)^{-1} \tag{33}$$

Equation (33) is an identity and is not operational until we have a model of the risk premium. We examine alternative models for risk premia in Chapter 3.

Stocks

The difficulty with direct application of the DPV concept to stocks is that future dividends are uncertain and the discount factor may be time varying. It can be shown (see Chapter 4) that the *fundamental value* V_t is the expected DPV of future dividends:

$$V_t = E_t\left[\frac{D_{t+1}}{(1+q_1)} + \frac{D_{t+2}}{(1+q_1)(1+q_2)} + \cdots\right] \tag{34}$$

where q_i is the one-period return between time period $t + i - 1$ and $t + i$. If there are to be no systematic profitable opportunities to be made from buying and selling shares between well-informed rational traders, then the actual market price of the stock P_t must equal the fundamental value V_i. For example, if $P_t < V_t$, then investors should purchase the undervalued stock and hence make a capital gain as P_t rises towards V_t. In an efficient market, such profitable opportunities should be immediately eliminated.

Clearly, one cannot directly calculate V_t to see if it does equal P_t because expected dividends (and discount rates) are unobservable. However, in later chapters, we discuss methods for overcoming this problem and examine whether the stock market is efficient in the sense that $P_t = V_t$. If we add some simplifying assumptions to the DPV formula (e.g. future dividends are expected to grow at a constant rate g and the discount rate $q = R$ is constant each period), then (34) becomes

$$V_0 = D_o(1 + g)/(R - g) \tag{35}$$

which is known as the *Gordon Growth Model*. Using this equation, we can calculate the 'fair value' of the stock and compare it to the quoted market price P_0 to see whether the share is over- or undervalued. These models are usually referred to as dividend valuation models and are dealt with in Chapter 10.

Pure Discount Bonds and Spot Yields

Instead of a physical investment project, consider investing in a pure discount bond (zero coupon bond). In the market, these are usually referred to as 'zeros'. A pure discount bond has a fixed redemption price M, a known maturity period and pays no coupons. The yield on the bond if held to maturity is determined by the fact that it is purchased at a market price P_t below its redemption price M. For a one-year bond, it seems sensible to calculate the yield or interest rate as

$$r_{1t} = (M_1 - P_{1t})/P_{1t} \tag{36}$$

where r_{1t} is measured as a proportion. However, when viewing the problem in terms of DPV, we see that the one-year bond promises a future payment of M_1 at the end of the year in exchange for a capital cost of P_{1t} paid out today. Hence the IRR, y_{1t}, of the bond can be calculated from

$$P_{1t} = M_1/(1 + y_{1t}) \tag{37}$$

But on rearrangement, we have $y_{1t} = (M_1 - P_{1t})/P_{1t}$, and hence the one-year spot yield r_{1t} is simply the IRR of the bill. Applying the above principle to a two-year bill with redemption price M_2, the annual (compound) interest rate r_{2t} on the bill is the solution to

$$P_{2t} = M_2/(1 + r_{2t})^2 \tag{38}$$

which implies

$$r_{2t} = (M_2/P_{2t})^{1/2} - 1 \tag{39}$$

If spot rates are continuously compounded, then

$$P_{nt} = M_n e^{-r_{nt}n} \tag{40}$$

where r_{nt} is now the continuously compounded rate for a bond of maturity n at time t. We now see how we can, in principle, calculate a set of (compound) spot rates at t for different maturities from the market prices at time t of pure discount bonds (bills).

Coupon-Paying Bonds

A *level coupon* (non-callable) bond pays a fixed coupon \$$C$ at known fixed intervals (which we take to be every year) and has a fixed redemption price M_n payable when the bond matures in year n. For a bond with n years left to maturity, the current market price is P_{nt}. The question is how do we measure the return on the bond if it is held to maturity?

The bond is analogous to our physical investment project with the capital outlay today being P_{nt} and the future receipts being \$$C$ each year (plus the redemption price). The internal rate of return on the bond, which is called the yield to maturity y_t, can be calculated from

$$P_{nt} = C/(1 + y_t) + C/(1 + y_t)^2 + \cdots + (C + M_n)/(1 + y_t)^n \tag{41}$$

The yield to maturity is that *constant* rate of discount that at a point in time equates the DPV of future payments with the current market price. Since P_{nt}, M_n and C are the known values in the market, (41) has to be solved to give the quoted rate for the yield to maturity y_t. There is a subscript ‘t’ on y_t because as the market price falls, the yield to maturity rises (and vice versa) as a matter of 'actuarial arithmetic'. Although widely used in the market and in the financial press, there are some theoretical/conceptual problems in using the yield to maturity as an unambiguous measure of the return on a bond even when it is held to maturity. We deal with some of these issues in Part III.

In the market, coupon payments C are usually paid every six months and the interest rate from (41) is then the periodic six-month rate. If this periodic yield to maturity is calculated as, say, 6 percent, then in the market the quoted yield to maturity will be the simple annual rate of 12 percent per annum (known as the bond-equivalent yield in the United States).

A *perpetuity* is a level coupon bond that is never redeemed by the primary issuer (i.e. $n \to \infty$). If the coupon is $\$C$ per annum and the current market price of the bond is $P_{\infty,t}$, then from (41) the yield to maturity on a perpetuity is

$$y_{\infty,t} = C/P_{\infty,t} \tag{42}$$

It is immediately obvious from (42) that for small changes, the percentage change in the price of a perpetuity equals the percentage change in the yield to maturity. The flat yield or interest yield or running yield $y_{rt} = (C/P_{nt})100$ and is quoted in the financial press, but it is not a particularly theoretically useful concept in analysing the pricing and return on bonds.

Although compound rates of interest (or yields) are quoted in the markets, we often find it more convenient to express bond prices in terms of continuously compounded spot interest rates/yields. If the continuously compounded spot yield is r_{nt}, then a coupon-paying bond may be considered as a portfolio of 'zeros', and the price is (see Cuthbertson and Nitzsche 2001a)

$$P_{nt} = \sum_{k=1}^{n} C_k e^{-r_{kt}k} + M_n e^{-r_{nt}n} = \sum_{k=1}^{n} P_{kt}^* + P_{nt}^* \tag{43}$$

where $P_k^* = C_k e^{-r_k k}$ and P_n^* are the prices of zero coupon bonds paying C_k at time $t+k$ and M_n at time $t+n$, respectively.

Holding Period Return

Much empirical work on stocks deals with the one-period holding period return H_{t+1}, which is defined as

$$H_{t+1} = \frac{P_{t+1} - P_t}{P_t} + \frac{D_{t+1}}{P_t} \tag{44}$$

The first term is the proportionate capital gain or loss (over one period) and the second term is the (proportionate) dividend yield. H_{t+1} can be calculated *ex-post* but, of course, viewed from time t, P_{t+1} and (perhaps) D_{t+1} are uncertain, and investors can only try and forecast these elements. It also follows that

$$1 + H_{t+i+1} = [(P_{t+i+1} + D_{t+i+1})/P_{t+i}] \tag{45}$$