The Handbook of Portfolio Mathematics

Formulas for Optimal Allocation & Leverage

RALPH VINCE

BICENTENNIAL 1807 WILEY 2007 BICENTENNIAL

John Wiley & Sons, Inc.

The Handbook of Portfolio Mathematics

Founded in 1807, John Wiley & Sons is the oldest independent publishing company in the United States. With offices in North America, Europe, Australia, and Asia, Wiley is globally committed to developing and marketing print and electronic products and services for our customers' professional and personal knowledge and understanding.

The Wiley Trading series features books by traders who have survived the market's ever-changing temperament and have prospered—some by reinventing systems, others by getting back to basics. Whether a novice trader, professional or somewhere in between, these books will provide the advice and strategies needed to prosper today and well into the future.

For a list of available titles, visit our Web site at www.WileyFinance.com.

The Handbook of Portfolio Mathematics

Formulas for Optimal Allocation & Leverage

RALPH VINCE

BICENTENNIAL 1807 WILEY 2007 BICENTENNIAL

John Wiley & Sons, Inc.

Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Chapters 1–10 contain revised material from three of the author's previous books, *Portfolio Management Formulas: Mathematical Trading Methods for the Futures, Options, and Stock Markets* (1990), *The Mathematics of Money Management: Risk Analysis Techniques for Traders* (1992), and *The New Money Management: A Framework for Asset Allocation* (1995), all published by John Wiley & Sons, Inc.

Wiley Bicentennial Logo: Richard J. Pacifico

For general information on our other products and services or for technical support, please contact our Customer Care Department within the United States at (800) 762-2974, outside the United States at (317) 572-3993 or fax (317) 572-4002.

Wiley also publishes its books in a variety of electronic formats. Some content that appears in print may not be available in electronic format. For more information about Wiley products, visit our web site at www.wiley.com

Library of Congress Cataloging-in-Publication Data:

Vince, Ralph, 1958–
The handbook of portfolio mathematics : formulas for optimal allocation & leverage / Ralph Vince:
p. cm.
ISBN-13: 978-0-471-75768-9 (cloth)
ISBN-10: 0-471-75768-3 (cloth)
1. Portfolio management–Mathematical models. 2. Investments–Mathematical models. I. Title.
HG4529.5.V555 2007
332.601′51 – dc22 2006052577

Printed in the United States of America

10 9 8 7 6 5 4 3 2 1

"You must not be extending your empire while you are at war or run into unnecessary dangers. I am more afraid of our own mistakes than our enemies' designs."

—Pericles, in a speech to the Athenians during the Peloponnesian War, as represented by Thucydides

Contents

Preface

It's always back there, bubbling away. It seems I cannot shut off my mind from it. Every conversation I ever have, with programmers and traders, engineers and gamblers, Northfield Park Railbirds and Warrensville Workhouse jailbirds—those equations that describe these very things are cast in this book.

Let me say I am averse to gambling. I am averse to the notion of creating risk where none need exist, averse to the idea of attempting to be rewarded in the absence of creating or contributing something (or worse yet, taxing a man's labor!). Additionally, I find amorality in charging or collecting interest, and the absence of this innate sense in others riles me.

This book starts out as a compilation, cleanup, and in some cases, reformulation of the previous books I have written on this subject. I'm standing on big shoulders here. The germ of the idea of those previous books can trace its lineage to my good friend and past employer, Larry Williams. In the dust cloud of his voracious research, was the study of the Kelly Criterion, and how that might be applied to trading. What followed over the coming years then was something of an explosion in that vein, culminating in a better portfolio model than the one which is still currently practiced.

For years now I have been away from the markets—intentionally. In a peculiar irony, it has sharpened my bird's-eye view on the entire industry. People still constantly seek me out, bend my ears, try to pick my hollow, rancid pumpkin about the markets. It has all given me a truly gigantic field of view, a dizzying phantasmagoria, on who is doing what, and how.

I'd like to share some of that with you here.

We are not going to violate anyone's secrets here, realizing that most of these folks work very hard to obtain what they know. What I will speak of is generalizations and commonalities in what people are doing, so that we can analyze, distinguish, compare, and, I hope, arrive at some well-founded conclusions.

But I am not in the markets' trenches anymore. My time has been spent on software for parametric geometry generation of industrial componentry and "smart" robots that understand natural language and can go out and do things like perform research for me, come back, draw inferences, and discuss their findings with me. These are wonderful endeavors for me, allowing me to extend my litany of failures.

Speaking of which, in the final section of this text, we step into the near-silent, blue-lit morgue of failure itself, dissecting it both in a mathematical and abstract sense, as well as the real-world one. In this final chapter, the two are indistinguishable.

When we speak of the *real world*, some may get the mistaken impression that the material is easy. It is not. That has not been a criterion of mine here. What has been a criterion is to address the real-world application of the previous three books that this book incorporates. That means looking at the previous material with regard to failure, with regard to drawdown. Money managers and personal traders alike tend to have utility preference curves that are incongruent with maximizing their returns. Further, I am aware of no one, nor have I ever encountered any trader, fund manager, or institution, who could even tell you what his or her utility preference function was. This is a prime example of the chasm—the disconnect—between theory and real-world application.

Historically, risk has been defined in theoretical terms as the variance (or semivariance) in returns. This, too, is rarely (though in certain situations) a desired proxy for risk. Risk is the chance of getting your head handed to you. It is not, except in rare cases, variance in returns. It is not semivariance in returns; it is not determined by a utility preference function. Risk is the probability of being ruined. Ruin is touching or penetrating a lower barrier on your equity. So we can say to most traders, fund managers, and institutions that risk is the probability of touching a lower barrier on equity, such that it would constitute ruin to someone. Even in the rare cases where variance in returns is a concern, risk is still primarily a drawdown to a lower absorbing barrier.

So what has been needed, and something I have had bubbling away for the past decade or so, is a way to apply the optimal *f* framework within the real-world constraints of this universally regarded definition of risk. That is, how do we apply optimal *f* with regard to risk of ruin and its more familiar and real-world-applicable-cousin, risk of drawdown?

Of course, the concepts are seemingly complicated—we're seeking to maximize return for a given level of drawdown, not merely juxtapose returns and variance in returns. Do you want to maximize growth for a given level of drawdown, or do you want to do something easier?

So this book is more than just a repackaging of previous books on this subject. It incorporates new material, including a study of correlations between pairwise components in a portfolio (and *why* that is such a bad idea). Chapter 11 examines what portfolio managers have (not) been doing with regards to the concepts presented in this book, and Chapter 12 takes the new Leverage Space Portfolio Model and juxtaposes it to the probability of a given drawdown to provide a now-superior portfolio model, based on the previous chapters in this book, and applicable to the real world.

I beg the reader to look at everything in this text—as merely my articulation of something, and not an autocratic dictation. Not only am I not infallible, but also my real aim here is to engage you in the study of something I find fascinating, and I want to share that very raw joy with you. Because, you see, as I started out saying, it's always back there, bubbling away—my attraction to those equations on the markets, pertaining to allocation and leverage. It's not a preoccupation with the markets, though—to me it could be the weather or any other dynamic system. It is the allure of nailing masses and motions and relationships with an equation.

Rapture!

That is my motivation, and that is why I can never shut it off. It is that very rapture that I seek to share, which augments that very rapture I find in it. As stated earlier, I stand on big shoulders. My hope is that my shoulders can support those who wish to go further with these concepts.

This book covers my thinking on these subjects for more than two and a half decades. There are a lot of people to thank. I won't mention them, either—they know who they are, and I feel uneasy mentioning the names of others here in one way or another, or others in the industry who wish to remain nameless. I don't know how they might take it.

There is one guilty party, however, whom I *will* mention—Rejeanne. This one, finally, is for you.

Ralph Vince

Chagrin Falls, Ohio
August 2006

Introduction

This is a book in two distinct parts. Originally, my task was to distill the previous three books on this subject into one book. In effect, Part I comprises that text.

It's been reorganized, rehashed, and reworked to resemble the original texts while creating a contiguous path of reasoning, which takes us from the basic gambling theory and statistics, through the introduction of the Kelly criterion, optimal f, and finally onto the Leverage Space Portfolio Model for multiple-simultaneous positions.

The Leverage Space Portfolio Model addresses allocations and leverage. Often these are two distinct facets, but herein they refer to the same thing. *Allocation* is the *relative* leverage between multiple portfolio components. Thus, when we speak of *leverage*, we are also speaking of *allocation*, and vice versa.

Likewise, *money management* and *portfolio construction*, as practiced, don't necessarily refer to the same exercise, yet in this text, they do. Collectively, whatever the endeavor of risk, be it a bond portfolio, a commodities fund, or a team of blackjack players invading a casino, the collective exercise will be herein referred to as *allocation*.

I have tried to keep the geometric perspective on these concepts, and keep those notions about them intact. The first section is necessarily heavy on math. The first section is purely conceptual. It is about allocation and leverage to maximize returns without respect to anything else.

Everything in Part I was conjured up more than a decade or two ago. I was younger then.

Since that time, I have repeatedly been approached with the question, "How do you apply it?" I used to be baffled by this; the obvious (to me) answer being, "As is."

As used herein, a ln utility preference curve is one that is characteristic of someone who acts so as to maximize the ratio of his or her returns to the risk assumed to do so.

The notion that someone's *utility preference function* could be anything other than ln was evidence of both the person's insanity and weakness.

I saw it as a means for risk takers to enjoy the rush of their compulsive gambling under the ruse of the academic justification of *utility preference*.

I'm older now (seemingly not tempered with age—you see, I still know the guy who wrote those previous books), but I have been able to at least accept the exercise—the rapture—of working to solve the dilemma of optimal allocations and leverage under the constraint of a utility preference curve that is *not* ln.

By the definition of a ln utility preference curve, given a few paragraphs ago, a sane[1] person is therefore one who is levered up to the optimal f level in a game favorable to him or minimizes his number of plays in a game unfavorable to him. Anyone who goes to a casino and plunks down all he is willing to lose on that trip in one play is not a compulsive gambler. But who does that? Who has that self-control? Who has a utility preference curve that *is* ln?

That takes us to Part II of the book, the part I call the *real-world application* of the concepts illuminated in Part I, because people's utility preference curves are not ln.

So Part II attempts to tackle the mathematical puzzle posed by attempting to employ the concepts of Part I, given the weakness and insanity of human beings. What could be more fun?

* * *

Many of the people who have approached me with the question of "How do you apply it?" over the years have been professionals in the industry. Since, ultimately, their clients are the very individuals whose utility preference curves are not ln, I have found that these entities have utility preference functions that mirror those of their clients (or they don't have clients for long).

Many of these entities have been successful for many years. Naturally, their procedures pertaining to allocation, leverage, and trading implementation were of great interest to me.

Part II goes into this, into what these entities typically do. The best of them, I find, have not employed the concepts of the last chapter except in very rudimentary and primitive ways. There is a long way to go.

Often, I have been criticized as being "all theory—no practice." Well, Part I is indeed all theory, but it *is* exhaustive in that sense—not on portfolio construction in general and all the multitude of ways of performing that, but rather, on portfolio construction in terms of optimal position sizes (i.e., in the vein of an optimal f approach). Further, I did not want Part I to be

[1]Academics prefer the nomenclature "rational," versus "sane." The subtle difference between the two is germane to this discussion.

a mere republishing, almost verbatim, of the previous books. Therefore, I have incorporated some new material into Part I. This is material that has become evident to me in the years since the original material was published.

Part II is entirely new. I have been fortunate in that my first exposure to the industry was as a margin clerk. I had an opportunity to observe a sizable universe of ways people go about doing things in this business. Later, thanks to my programming abilities, from which the other books germinated, I had exposure to many professionals in the industry, and was often privy to how they practiced things, or was in a position where I could reverse-engineer it. I have had the good fortune of being on a course that has afforded me a bird's-eye view of the way people practice their allocation, leverage, and trading implementations in this business. Part II is derived from that high-altitude bird's-eye view, and the desire to provide a real-world implementation of the concepts of Part I—that is, to make them applicable to those people whose utility preference functions are not ln.

* * *

Things I have written of in the past have received a good deal of criticism over the years. I welcome it, and a chance to address it. To me, it says people are thinking about these ideas, trying to mold them further, or remold those areas where I may have been wrong (I'm not so much interested in being "right" about any of this as I am about "this"). Though I have not consciously intended that, this book, in many ways, answers some of those criticisms.

The main criticism was that it was too theoretical with no real-world application. The criticism is well founded in the sense that drawdown was all but ignored. For better or worse, people and institutions never seem to have utility functions that are ln. Yet, nearly all utility functions of people and institutions are ln within a drawdown constraint. That is, they seek to maximize the ratio of returns to risk (drawdown) within a certain drawdown. That disconnect between what I have written in the past has now, more than a decade later, been resolved.

A second major criticism is that trading at optimal f is too wild for any mere human. I know of no professional funds that have traded at the optimal f levels. I have known people who have traded at optimal f, usually for short periods of time, in only a single market, before panicking in a drawdown. There it is again: drawdown. You see, it wasn't so much this construct of their utility preference curve (talk about too theoretical!) as it was their drawdown that was incongruent with their trading at the optimal f level.

If you are getting the notion that we will be looking into the nature of drawdown later on in this book, when we discuss what I have been doing in terms of working on this material for the past decade-plus, you're right. We're going to look at drawdown herein beyond what anyone has.

Which takes us to the third major criticism, being that optimal f or the Leverage Space Model allocates without respect to drawdown. This, too, has now been addressed directly in Chapter 12. However, as we will see in that chapter, drawdown is, in a sequence of independent trials, but one permutation of many permutations. Thus, to address drawdown, one must address it in those terms.

The last major criticism has been that regarding the complexity of calculation. People desire a simple solution, a heuristic, something they could perform by hand if need be.

Unfortunately, that was not the case, and that desire of others is now something even more remote. In the final chapter, we can see that one must perform millions of calculations (as a sample to billions of calculations!) in order to derive certain answers.

However, such seemingly complex tasks can be made simple by packaging them up as black-box computer applications. Once someone understands what calculations are performed and why, the machine can do the heavy lifting. Ultimately, that is even simpler than performing a simple calculation by hand.

If one can put in the scenarios, their outcomes, and probability of occurrence—their joint probabilities of occurrence with other scenarios in other scenario spectrums—one can feed the machine and derive that number which satisfies the ideal composition, the optimal allocations and leverage among portfolio components to satisfy that ln utility preference function within a certain drawdown constraint.

To be applicable to the real world, a book like this should, it would seem, be about trading. This is *not* a book on how to trade the markets. (This makes the real-world application section difficult.) It is about how very basic, mathematical laws are working on us—you and me—when we engage in a stream of risk-related outcomes wherein we don't have control over those outcomes. Rather, we have control only over the relative impacts on us. In that sense, the mathematics applies to us in trading.

I don't want to pretend to know a thing about trading, really. Just as I am not an academic, I am also not a trader. I've been around and worked for some amazing traders—but that doesn't mean I am one.

That's *your* domain—and why you are reading this book: To augment the knowledge you have about trading vis-à-vis cross-pollination with these *outside* formulas. And if they are too cumbersome, or too complicated, please don't blame me. I wish they were simply along the lines of $2 + 2$. But they are not.

This is not by my design. When you trade, you are somewhat trying to intuitively carve your way along the paths of these equations, yet you are oblivious to what the equations are. You are, for instance, trying to maximize your returns within a certain probability of a given drawdown over the next period.

But you don't really have the equations to do so. Now you do. Don't blame me if you find them to be too cumbersome. These formulas are what we seek to know—and somehow use—as they apply to us in trading, whether we acknowledge that or not. I have heard ample criticism about the difficulties in applications. In this text, I will attempt to show you what others are doing *compared* to using these formulas. However, these formulas are at work on everyone when they trade. It is in the disparity between the two that your past criticisms of me lie; it is in that very disparity that my criticisms of you lie.

When you step up to the service line and line up to serve to my backhand, say, the fact that gravity operates with an acceleration of 9.8 meters per second squared applies to you. It applies to your serve landing in the box or not (among other things), whether you acknowledge this or not. It is an explanation of how things work more so than how to work things. You are trying to operate within a world defined by certain formulas. It does not mean you can implement them in your work, or that, because you cannot, they are therefore invalid. Perhaps you can implement them in your work. Clearly, if you could, without expense to the other aspects of "your work," wouldn't it be safe to say, then, that you certainly wouldn't be worse off?

And so with the equations in the book. Perhaps you can implement them—and if you can, without expense to the other aspects of your game, then won't you be better off? And if not, does it invalidate their truths any more than a tennis pro who dishes up a first serve, oblivious to the 9.8 m/s^2 at work?

* * *

This is, in its totality, what I know about allocations and leverage in trading. It is the sum of all I have written of it in the past, and what I have savored over the past decade-plus. As with many things, I truly love this stuff. I hope my passion for it rings contagiously herein. However, it sits as dead and cold as any inanimate abstraction. It is only your working with these concepts, your application and your critiques of them, your volley back over the net, that give them life.

PART I

Theory

CHAPTER 1

The Random Process and Gambling Theory

We will start with the simple coin-toss case. When you toss a coin in the air there is no way to tell for certain whether it will land heads or tails. Yet over many tosses the outcome can be reasonably predicted.

This, then, is where we begin our discussion.

Certain axioms will be developed as we discuss the random process. The first of these is that *the outcome of an individual event in a random process cannot be predicted. However, we can reduce the possible outcomes to a probability statement.*

Pierre Simon Laplace (1749–1827) defined the probability of an event as the ratio of the number of ways in which the event can happen to the total possible number of events. Therefore, when a coin is tossed, the probability of getting tails is 1 (the number of tails on a coin) divided by 2 (the number of possible events), for a probability of .5. In our coin-toss example, we do not know whether the result will be heads or tails, but we do know that the probability that it will be heads is .5 and the probability it will be tails is .5. So, *a probability statement is a number between 0 (there is no chance of the event in question occurring) and 1 (the occurrence of the event is certain).*

Often you will have to convert from a probability statement to odds and vice versa. The two are interchangeable, as the odds imply a probability, and a probability likewise implies the odds. These conversions are given now. The formula to convert to a probability statement, when you know the given odds is:

$$\text{Probability} = \text{odds for}/(\text{odds for} + \text{odds against}) \qquad (1.01)$$

If the odds on a horse, for example, are 4 to 1 (4:1), then the probability of that horse winning, as implied by the odds, is:

$$\begin{aligned}\text{Probability} &= 1/(1+4)\\ &= 1/5\\ &= .2\end{aligned}$$

So a horse that is 4:1 can also be said to have a probability of winning of .2. What if the odds were 5 to 2 (5:2)? In such a case the probability is:

$$\begin{aligned}\text{Probability} &= 2/(2+5)\\ &= 2/7\\ &= .2857142857\end{aligned}$$

The formula to convert from probability to odds is:

$$\text{Odds (against, to one)} = 1/\text{probability} - 1 \tag{1.02}$$

So, for our coin-toss example, when there is a .5 probability of the coin's coming up heads, the odds on its coming up heads are given as:

$$\begin{aligned}\text{Odds} &= 1/.5 - 1\\ &= 2 - 1\\ &= 1\end{aligned}$$

This formula always gives you the odds "to one." In this example, we would say the odds on a coin's coming up heads are 1 to 1.

How about our previous example, where we converted from odds of 5:2 to a probability of .2857142857? Let's work the probability statement back to the odds and see if it works out.

$$\begin{aligned}\text{Odds} &= 1/.2857142857 - 1\\ &= 3.5 - 1\\ &= 2.5\end{aligned}$$

Here we can say that the odds in this case are 2.5 to 1, which is the same as saying that the odds are 5 to 2. So when someone speaks of odds, they are speaking of a probability statement as well.

Most people can't handle the uncertainty of a probability statement; it just doesn't sit well with them. We live in a world of exact sciences, and human beings have an innate tendency to believe they do not understand an event if it can only be reduced to a probability statement. The domain of physics seemed to be a solid one prior to the emergence of quantum physics. We had equations to account for most processes we had observed. These equations were real and provable. They repeated themselves over and over and the outcome could be exactly calculated before the event took place. With the emergence of quantum physics, suddenly a theretofore exact science could only reduce a physical phenomenon to a probability statement. Understandably, this disturbed many people.

I am not espousing the random walk concept of price action nor am I asking you to accept anything about the markets as random. Not yet, anyway. Like quantum physics, the idea that there is or is not randomness in the markets is an emotional one. At this stage, let us simply concentrate on the random process as it pertains to something we are certain is random, such as coin tossing or most casino gambling. In so doing, we can understand the process first, and later look at its applications. Whether the random process is applicable to other areas such as the markets is an issue that can be developed later.

Logically, the question must arise, "When does a random sequence begin and when does it end?" It really doesn't end. The blackjack table continues running even after you leave it. As you move from table to table in a casino, the random process can be said to follow you around. If you take a day off from the tables, the random process may be interrupted, but it continues upon your return. So, when we speak of a random process of X events in length we are arbitrarily choosing some finite length in order to study the process.

INDEPENDENT VERSUS DEPENDENT TRIALS PROCESSES

We can subdivide the random process into two categories. First are those events for which the probability statement is constant from one event to the next. These we will call independent trials processes or sampling with replacement. A coin toss is an example of just such a process. Each toss has a 50/50 probability regardless of the outcome of the prior toss. Even if the last five flips of a coin were heads, the probability of this flip being heads is unaffected, and remains .5.

Naturally, the other type of random process is one where the outcome of prior events *does* affect the probability statement and, naturally, the probability statement is not constant from one event to the next. These types of events are called dependent trials processes or sampling without replacement. Blackjack is an example of just such a process. Once a card is played, the composition of the deck for the next draw of a card is different from what it was for the previous draw. Suppose a new deck is shuffled and a card removed. Say it was the ace of diamonds. Prior to removing this card the probability of drawing an ace was 4/52 or .07692307692. Now that an ace has been drawn from the deck, and not replaced, the probability of drawing an ace on the next draw is 3/51 or .05882352941.

Some people argue that dependent trials processes such as this are really not random events. For the purposes of our discussion, though, we will assume they are—since the outcome still cannot be known beforehand. The best that can be done is to reduce the outcome to a probability statement. Try to think of the difference between independent and dependent trials processes as simply whether the probability statement is *fixed* (independent trials) or *variable* (dependent trials) from one event to the next based on prior outcomes. This is in fact the only difference.

Everything can be reduced to a probability statement. Events where the outcomes can be known prior to the fact differ from random events mathematically only in that their probability statements equal 1. For example, suppose that 51 cards have been removed from a deck of 52 cards and you know what the cards are. Therefore, you know what the one remaining card is with a probability of 1 (certainty). For the time being, we will deal with the independent trials process, particularly the simple coin toss.

MATHEMATICAL EXPECTATION

At this point it is necessary to understand the concept of mathematical expectation, sometimes known as the player's edge (if positive to the player) or the house's advantage (if negative to the player):

$$\text{Mathematical Expectation} = (1 + \text{A}) * \text{P} - 1 \tag{1.03}$$

where: P = Probability of winning.
A = Amount you can win/Amount you can lose.

So, if you are going to flip a coin and you will win \$2 if it comes up heads, but you will lose \$1 if it comes up tails, the mathematical expectation per flip is:

$$\begin{aligned}\text{Mathematical Expectation} &= (1 + 2) * .5 - 1 \\ &= 3 * .5 - 1 \\ &= 1.5 - 1 \\ &= .5\end{aligned}$$

In other words, you would expect to make 50 cents on average each flip.