PERIODICALLY CORRELATED RANDOM SEQUENCES
Each generation has its unique needs and aspirations. When Charles Wiley first opened his small printing shop in lower Manhattan in 1807, it was a generation of boundless potential searching for an identity. And we were there, helping to define a new American literary tradition. Over half a century later, in the midst of the Second Industrial Revolution, it was a generation focused on building the future. Once again, we were there, supplying the critical scientific, technical, and engineering knowledge that helped frame the world. Throughout the 20th Century, and into the new millennium, nations began to reach out beyond their own borders and a new international community was born. Wiley was there, expanding its operations around the world to enable a global exchange of ideas, opinions, and know-how.

For 200 years, Wiley has been an integral part of each generation’s journey, enabling the flow of information and understanding necessary to meet their needs and fulfill their aspirations. Today, bold new technologies are changing the way we live and learn. Wiley will be there, providing you the must-have knowledge you need to imagine new worlds, new possibilities, and new opportunities.

Generations come and go, but you can always count on Wiley to provide you the knowledge you need, when and where you need it!

William J. Pesce  Peter Booth Wiley
President and Chief Executive Officer  Chairman of the Board
PERIODICALLY CORRELATED RANDOM SEQUENCES
Spectral Theory and Practice

Harry L. Hurd
The University of North Carolina at Chapel Hill

Abolghassem Miamee
Hampton University
To Marcia, Cheryl, Robert, Olivia, Angela and to Effie, Goly, Nazy, and Ali
CONTENTS

Preface xiii
Acknowledgments xv
Glossary xvii

1 Introduction 1
1.1 Summary 6
1.2 Historical Notes 14
Problems and Supplements 16

2 Examples, Models, and Simulations 19
2.1 Examples and Models 20
2.1.1 Random Periodic Sequences 20
2.1.2 Sums of Periodic and Stationary Sequences 21
2.1.3 Products of Scalar Periodic and Stationary Sequences 21

vii
2.1.4 Time Scale Modulation of Stationary Sequences 22
2.1.5 Pulse Amplitude Modulation 23
2.1.6 A More General Example 24
2.1.7 Periodic Autoregressive Models 25
2.1.8 Periodic Moving Average Models 27
2.1.9 Periodically Perturbed Dynamical Systems 28

2.2 Simulations 29
2.2.1 Sums of Periodic and Stationary Sequences 29
2.2.2 Products of Scalar Periodic and Stationary Sequences 30
2.2.3 Time Scale Modulation of Stationary Sequences 32
2.2.4 Pulse Amplitude Modulation 33
2.2.5 Periodically Perturbed Logistic Maps 35
2.2.6 Periodic Autoregressive Models 38
2.2.7 Periodic Moving Average Models 40
Problems and Supplements 42

3 Review of Hilbert Spaces 45
3.1 Vector Spaces 45
3.2 Inner Product Spaces 47
3.3 Hilbert Spaces 49
3.4 Operators 51
3.5 Projection Operators 53
3.6 Spectral Theory of Unitary Operators 60
3.6.1 Spectral Measures 60
3.6.2 Spectral Integrals 61
3.6.3 Spectral Theorems 64
Problems and Supplements 65

4 Stationary Random Sequences 67
4.1 Univariate Spectral Theory 68
4.1.1 Unitary Shift 68
4.1.2 Spectral Representation 70
4.1.3 Mean Ergodic Theorem 72
4.1.4 Spectral Domain 74
4.2 Univariate Prediction Theory 75
4.2.1 Infinite Past, Regularity and Singularity 75
4.2.2 Wold Decomposition 76
4.2.3 Innovation Subspaces 78
4.2.4 Spectral Theory and Prediction 84
4.2.5 Finite Past Prediction 91

4.3 Multivariate Spectral Theory 99
4.3.1 Unitary Shift 100
4.3.2 Spectral Representation 101
4.3.3 Mean Ergodic Theorem 102
4.3.4 Spectral Domain 102

4.4 Multivariate Prediction Theory 107
4.4.1 Infinite Past, Regularity and Singularity 107
4.4.2 Wold Decomposition 108
4.4.3 Innovations and Rank 109
4.4.4 Regular Processes 116
4.4.5 Infinite Past Prediction 119
4.4.6 Spectral Theory and Rank 121
4.4.7 Spectral Theory and Prediction 123
4.4.8 Finite Past Prediction 125
Problems and Supplements 129

5 Harmonizable Sequences 133
5.1 Vector Measure Integration 134
5.2 Harmonizable Sequences 141
5.3 Limit of Ergodic Average 145
5.4 Linear Time Invariant Filters 146
Problems and Supplements 149

6 Fourier Theory of the Covariance 151
6.1 Fourier Series Representation of the Covariance 152
6.2 Harmonizability of PC Sequences 160
6.3 Some Properties of $B_k(\tau), F_k,$ and $F$ 168
6.4 Covariance and Spectra for Specific Cases 170
6.4.1 PC White Noise 170
6.4.2 Products of Scalar Periodic and Stationary Sequences 171
6.5 Asymptotic Stationarity 172
6.6 Lebesgue Decomposition of $F$ 173
6.7 The Spectrum of $m_t$ 174
6.8 Effects of Common Operations on PC Sequences 176
6.8.1 Linear Time Invariant Filtering 176
6.8.2 Differencing 181
6.8.3 Random Shifts 182
6.8.4 Sampling 187
6.8.5 Bandshifting 191
6.8.6 Periodically Time Varying (PTV) Filters 192
Problems and Supplements 194

7 Representations of PC Sequences 199
7.1 The Unitary Operator of a PC Sequence 200
7.2 Representations Based on the Unitary Operator 201
7.2.1 Gladyshev Representation 201
7.2.2 Another Representation of Gladyshev Type 203
7.2.3 Time-Dependent Spectral Representation 203
7.2.4 Harmonizability Again 205
7.2.5 Representation Based on Principal Components 207
7.3 Mean Ergodic Theorem 210
7.4 PC Sequences as Projections of Stationary Sequences 212
Problems and Supplements 213

8 Prediction of PC Sequences 215
8.1 Wold Decomposition 218
8.2 Innovations 220
8.3 Periodic Autoregressions of Order 1 226
8.4 Spectral Density of Regular PC Sequences 229
8.4.1 Spectral Densities for PAR(1) 231
8.5 Least Mean-Square Prediction 235
8.5.1 Prediction Based on Infinite Past 235
8.5.2 Prediction for a PAR(1) Sequence 236
8.5.3 Finite Past Prediction 237
Problems and Supplements 246
9 **Estimation of Mean and Covariance**  
9.1 Estimation of $m_t$: Theory  
9.2 Estimation of $m_t$: Practice  
\[ \text{9.2.1 Computation of } \tilde{m}_{t,N} \]  
\[ \text{9.2.2 Computation of } \tilde{m}_{k,N} \]  
9.3 Estimation of $R(t+\tau, t)$: Theory  
\[ \text{9.3.1 Estimation of } R(t+\tau, t) \]  
\[ \text{9.3.2 Estimation of } B_k(\tau) \]  
9.4 Estimation of $R(t+\tau, t)$: Practice  
\[ \text{9.4.1 Computation of } \tilde{R}_N(t+\tau, t) \]  
\[ \text{9.4.2 Computation of } \tilde{B}_{k,NT}(\tau) \]  
Problems and Supplements  

10 **Spectral Estimation**  
10.1 The Shifted Periodogram  
10.2 Consistent Estimators  
10.3 Asymptotic Normality  
10.4 Spectral Coherence  
\[ \text{10.4.1 Spectral Coherence for Known } T \]  
\[ \text{10.4.2 Spectral Coherence for Unknown } T \]  
10.5 Spectral Estimation: Practice  
\[ \text{10.5.1 Confidence Intervals} \]  
\[ \text{10.5.2 Examples} \]  
10.6 Effects of Discrete Spectral Components  
\[ \text{10.6.1 Removal of the Periodic Mean} \]  
\[ \text{10.6.2 Testing for Additive Discrete Spectral Components} \]  
\[ \text{10.6.3 Removal of Detected Components} \]  
Problems and Supplements  

11 **A Paradigm for Nonparametric Analysis of PC Time Series**  
11.1 The Period $T$ is Known  
11.2 The Period $T$ is Unknown  

References  
Index
Periodically correlated (or cyclostationary) processes are random processes that have a periodic structure, but are still very much random. Roughly speaking, if the model of a physical system contains randomness and periodicity together, then measurements made on the system (over time) will very likely have a structure that is periodically nonstationary, or in the second order case, periodically correlated. For example, meteorological systems, communication systems, systems containing rotating shafts, and economic systems all have these properties.

The intent of this work is to introduce the main ideas of periodically correlated processes through the simpler periodically correlated sequences. Our approach is to provide (1) motivating and illustrative examples, (2) an account of the second order theory, and (3) some basic theory and methods for practical time series analysis. Our particular view of the second order theory places emphasis on the unitary operator that propagates or shifts the sequence by one period. This view makes clear the well known connection between stationary vector sequences and periodically correlated sequences. But we do not rely completely on this connection and have sometimes chosen methods of proof that are extensible to continuous time or to almost PC
processes. As for time series analysis, we suppose that a reader is presented with a sample of a time series and asked to determine if periodic correlation is present, and if so, to say something about it, to characterize it. We present the theory, methods, and algorithms that will help the reader answer this question, within the scope of covariance and spectral estimation. The topic of periodic autoregressive moving average (or PARMA) became too large for inclusion at this time, especially when we began to consider sequences of less than full rank.

Accordingly, the book is roughly organized into three parts. Chapters 1 and 2 present basic definitions, simple mathematical models, and simulations whose intent is to motivate and give insight. In this we present a number of examples that illustrate that the usual periodogram analysis cannot be expected to reveal the presence of periodic correlation in a time series. We give a historical review of the topic that mainly emphasizes the early development but gives references to application-specific bibliographies. Chapters 3–8 give background and theoretical structure, beginning with a review of Hilbert space including the spectral theorem for unitary operators and correlation and spectral theory for multivariate stationary sequences. We present the (spectral) theory of harmonizable sequences and then the Fourier theory for the covariance of PC sequences. This is naturally followed by representations for PC sequences and here is where the unitary operator plays its part. We then treat the prediction problem for PC sequences and introduce the rank of a PC sequence.

The last three chapters (Chapters 9–11) treat issues of time series analysis for PC sequences. We first treat the nonparametric estimation of mean, correlation, and spectrum. Chapter 11 summarizes the methods into a paradigm for nonparametric time series analysis of possibly PC sequences.

MATLAB scripts used in preparing the figures and in conducting the time series analyses, as well as the data used, can be obtained from the website http://www.unc.edu/~hhurd/pc-sequences.

The material beginning with Chapter 3 would be useful as a basis for a course of study. It would be helpful for students to have a senior level background in vector spaces, probability, and random processes. The material of Chapter 2 is designed to provide motivation and insight and would probably be helpful to most students except those who may have some familiarity with the topic.

HARRY L. HURD AND ABOLOGHASSEM MIAMEE

Chapel Hill, NC and Hampton, VA
January 31, 2007
The authors gratefully acknowledge the support of ONR, USARO, NSA, and the Iranian IPM for work leading to this book. In addition, we acknowledge the encouragement, interest, and helpfulness of Stamatis Cambanis, Harry Chang, Dominique Dehay, Neil Gerr, J. C. Hardin, Christian Houdre, Gopinath Kallianpur, Timo Koski, Douglas Lake, Robert Launer, Jacek Leskow, Andrzej Makagon, P. R. Masani, Antonio Napolitano, M. Pourahmadi, M. M. Rao, H. Salehi, and A. M. Yaglom.

HLH and AGM
GLOSSARY

$X_t$  
A univariate process (or sequence).

$X_t$  
A vector (or multivariate) sequence.

$X_n$  
The T-variate sequence formed from blocking.

$m_t, m(t)$  
The mean of $X_t$; that is, $m(t) = E\{X_t\}$.

$R_X(s,t)$  
The covariance of $X_t$ evaluated at $(s,t)$.

$F$  
The matrix spectral distribution function of the T-variate vector stationary sequence arising from the blocking (lifting) of a univariate PC-T sequence.

$f$  
The matrix spectral density of the T-variate vector stationary sequence arising from the blocking (lifting) of a univariate PC-T sequence.

$\mathcal{F}$  
The matrix spectral distribution function of the T-variate vector stationary sequence \{\(Z^j_t, j = 0, 1, \ldots, T - 1, t \in \mathbb{Z}\)\} resulting from Gladyshev’s transformation.

rank (\(X\))  
The rank of the PC-T sequence $X_t$.

rank (\(A\))  
The rank of the matrix $A$.

$\mathcal{H}_X$  
Hilbert space generated by the sequence $X_t$.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{L}$</td>
<td>Generic set with a linear structure.</td>
</tr>
<tr>
<td>$\mathcal{M}$</td>
<td>Generic subspace of a Hilbert space.</td>
</tr>
<tr>
<td>NND</td>
<td>Nonnegative definite.</td>
</tr>
</tbody>
</table>
CHAPTER 1

INTRODUCTION

Periodically correlated (PC) random processes are random processes in which there exists a periodic rhythm in the structure that is generally more complicated than periodicity in the mean function. We will begin with an illustration of some meteorological data.

The top trace of Figure 1.1 shows a 40 day record of hourly solar radiation levels taken at meteorological station DELTA on Ellsmere Island, N.W.T., Canada.

A daily (24 hour period) rhythm may be observed in this data in two ways: in the periodic average (or mean) and in the variation about the periodic mean. Since solar radiation can be expected to have a 24 hour period, let us compute the average of the 40 measurements for each of the 24 hours. Precisely, if the time series is denoted by $X_t, t = 1, 2, ..., NT$, where $NT = 960$, then the sample periodic mean (with period $T = 24$) is computed by

$$m_N(t) = \frac{1}{N} \sum_{p=0}^{N-1} X_{t+pT}, \quad t = 1, 2, ..., T,$$  \hspace{1cm} (1.1)

Copyright © 2007 John Wiley & Sons, Inc.
and plotted in the bottom trace of Figure 1.1. For t not in the base interval, \( \bar{m}_N(t) \) is defined periodically. It is visually clear that the sample periodic mean is not constant (but properly periodic) and a simple hypothesis test for difference in mean, say, between hour 1 and hour 13, indicates a difference with much significance.

We postpone the details of testing for a proper fluctuation in the mean (i.e., for rejection of the hypothesis that the true mean \( m(t) \) is constant) to Chapter 9.

The top trace of Figure 1.2 is the deviation \( Y_t = X_t - \bar{m}_N(t) \) of \( X_t \) from the sample periodic mean \( \bar{m}_N(t) \). The bottom trace presents the sample periodic variance,

\[
S_N^2(t) = \frac{1}{N - 1} \sum_{p=0}^{N-1} Y_{t+pT}^2, \quad t = 1, 2, ..., T,
\]

and it too appears to have a significant (with the details again postponed) variation through the period. So it is not just the mean that appears to have a periodic rhythm, the variance does too, suggesting that the entire probability law may have a periodic rhythm. We will state this more precisely following some discussion of notation.

First, a stochastic (or random) process \( X(t, \omega) \) is taken to be a function \( X : \mathbb{I} \times \Omega \to \mathbb{C} \), where \( \mathbb{C} \) is the set of complex numbers, \( \mathbb{I} \) is called the index set, and \( \Omega \) is a space, on which a sigma-algebra \( \mathcal{F} \) of subsets and a probability measure \( P \) are defined. An \( \mathcal{F} \)-measurable function is called a random variable, and for a stochastic process, the function \( X(t, \cdot) \) is assumed to be a random variable for each \( t \in \mathbb{I} \). Although the focus of this book is random sequences
Figure 1.2 (Top) Deviation around sample periodic mean. (Bottom) $S_N(t)$ with 95% confidence limits determined by the chi-squared distribution with $N - 1 = 39$ degrees of freedom.

Having a periodic rhythm, extensions of the ideas to fields ($\mathbb{I} = \mathbb{Z}^2$), to processes ($\mathbb{I} = \mathbb{R}$), to multivariate sequences, and to almost periodic sequences are briefly described in the supplements to this chapter.

We will most often denote the element of the random sequence by $X_t$ so that the dependence on $\omega$ is suppressed and the index is the subscript symbol $t$, conveying time. The essential structure needed to characterize a stochastic process is its probability law, meaning the collection of finite dimensional distributions, defined as the probabilities

$$P_{t_1, t_2, \ldots, t_n} (A_1, A_2, \ldots, A_n) = P[X_{t_1} \in A_1, X_{t_2} \in A_2, \ldots, X_{t_n} \in A_n]$$

for arbitrary $n$, collection of times $t_1, t_2, \ldots, t_n$ in $\mathbb{Z}$, and Borel sets $A_1, A_2, \ldots, A_n$ of $\mathbb{C}$.

**Definition 1.1 (Strict Stationarity)** A stochastic process $X_t(\omega)$ is called (strictly) stationary if its probability law is invariant with respect to time shifts, or more precisely, if for arbitrary $n$, collection of times $t_1, t_2, \ldots, t_n$ in $\mathbb{Z}$, and Borel sets $A_1, A_2, \ldots, A_n$ of $\mathbb{C}$ we have

$$P_{t_1+1, t_2+1, \ldots, t_n+1} (A_1, A_2, \ldots, A_n) = P_{t_1, t_2, \ldots, t_n} (A_1, A_2, \ldots, A_n).$$

Now we can formalize the structure suggested by Figures 1.1 and 1.2.

**Definition 1.2 (Periodically Stationarity)** A stochastic sequence $X_t(\omega)$ is called (strictly) periodically stationary with period $T$ if, for every $n$, any collection of times $t_1, t_2, \ldots, t_n$ in $\mathbb{Z}$, and Borel sets $A_1, A_2, \ldots, A_n$ of $\mathbb{C}$,

$$P_{t_1+T, t_2+T, \ldots, t_n+T} (A_1, A_2, \ldots, A_n) = P_{t_1, t_2, \ldots, t_n} (A_1, A_2, \ldots, A_n),$$

for arbitrary $n$, collection of times $t_1, t_2, \ldots, t_n$ in $\mathbb{Z}$, and Borel sets $A_1, A_2, \ldots, A_n$ of $\mathbb{C}$. 

The essential structure needed to characterize a stochastic process is its probability law, meaning the collection of finite dimensional distributions, defined as the probabilities.
and there are no smaller values of $T > 0$ for which (1.5) holds.

Synonyms for periodically stationary include periodically nonstationary, cyclo-stationary (think of cyclically stationary), processes with periodic structure, and a few others. For a little more on this nomenclature, see the historical notes (Section 1.2) at the end of this chapter.

If (1.5) holds for $T = 1$, then the process (or sequence) is stationary and it is clear that if $X_t$ is periodically stationary with period $T$, then it is also for period $kT, k \in \mathbb{Z}$. And so we say that a sequence is properly periodically stationary if the least $T$ for which (1.5) holds exceeds 1. Most often we will be considering second order random sequences, so that

$$E\{|X_t|^2\} = \int_{\Omega} |X_t(\omega)|^2 P(d\omega) < \infty, \quad \text{for all } t \in \mathbb{Z}.$$  

We will sometimes just write that $X_t \in L^2$. The mean exists for second order sequences

$$m(t) := \int_{\Omega} X_t(\omega) P(d\omega), \quad \text{for all } t \in \mathbb{Z}$$

and we define the covariance of the pair $(X_s, X_t)$ to be

$$R(s, t) := \text{Cov}(X_s, X_t) = E\{|X_s - m_s||X_t - m_t|\}.$$  

If there is no ambiguity, we will write $m(t)$ and $R(s, t)$ for the mean and covariance of $X_t$. Sometimes, in order to conserve space, we will write variables as subscripts rather than in parentheses, such as $m_t$ for $m(t)$ and $R_{s,t}$ for $R(s, t)$.

Since, for a zero mean sequence $X_t$, the covariance

$$\text{Cov}(X_s, X_t) = E\{X_sX_t\}$$

is clearly the $L^2$ inner product, our conclusions about zero mean second order random sequences can be interpreted for sequences of vectors in a Hilbert space. For some topics (e.g., those involving shift operators) it will be more natural to think of $X_t$ in this manner.

The notion of stationarity for second order sequences is expressed in terms of the first two moments.

**Definition 1.3 (Weak Stationarity)** A second order random process $X_t \in L^2(\Omega, \mathcal{F}, P)$ with $t \in \mathbb{Z}$ is called (weakly) stationary if for every $s, t \in \mathbb{Z}$

$$m(t) \equiv m \quad \text{and} \quad R(s, t) \equiv R(s - t).$$

If $X_t$ is of second order, periodic stationarity induces a rhythmic structure in the mean and covariance.
Definition 1.4 (Periodically Correlated) A second order process $X_t \in L^2(\Omega, \mathcal{F}, P)$ is called periodically correlated with period $T$ (PC-$T$) if for every $s, t \in \mathbb{Z}$

$$m(t) = m(t + T)$$

and

$$R(s, t) = R(s + T, t + T)$$

and there are no smaller values of $T > 0$ for which (1.6) and (1.7) hold.

It is clear that if the period is $T$, then (1.6) and (1.7) also hold when $T$ is replaced by $kT$, for any integer $k$. If $X_t$ is PC-1 then it is stationary (weakly) because then $R(s, t)$ is a function only of $s - t$. Clearly a stationary sequence is PC with every period.

We will write an indexed collection $\{X^j_t, j = 1, 2, \ldots, q\}$ of random sequences as the vector sequence $X_t = [X^1_t, X^2_t, \ldots, X^q_t]'$.

Definition 1.5 (Multivariate Stationarity) A second order q-variate random sequence $X_t$ with $t \in \mathbb{Z}$ is called (weakly) stationary if

$$E\{X^j_t\} \equiv m^j$$

and

$$R^{jk}(s, t) = \text{Cov}(X^j_s, X^k_t) = R^{jk}(s - t)$$

for all $s, t \in \mathbb{Z}$ and $j, k \in \{1, 2, \ldots, q\}$. If this is the case, we denote

$$m = [m^1, m^2, \ldots, m^q]' \text{ and } R(\tau) = [R^{jk}(\tau)]_{j,k=1}^q,$$

Multivariate (or vector) sequences obtained from the blocking of univariate (or scalar) sequences will be indexed by $n$ and thus denoted as $X_n$. That is, the univariate sequence $X_t$ is related by $T$-blocking to the T-variate sequence $X_n$ by

$$[X_n]^j = X_{j+nT}, \quad n \in \mathbb{Z}, \quad j = 0, 1, \ldots, T - 1. \quad (1.10)$$

The following proposition is a simple matter of following the indices.

Proposition 1.1 (Gladyshev) A second order random sequence $\{X_t : t \in \mathbb{Z}\}$ is PC with period $T$ if and only if the $T$ is the smallest integer for which the $T$-variate blocked sequence $X_n$ (1.10) is stationary.

Proof. Considering the covariance $\text{Cov}([X_n]^j, [X_m]^k) = \text{Cov}(X_{j+nT}, X_{k+mT})$, then stationarity of $X_n$ implies

$$\text{Cov}([X_n]^j, [X_m]^k) = R^{jk}(n - m) = \text{Cov}(X_{j+nT}, X_{k+mT}),$$
which implies (1.7) holds for $X_t$, and conversely. The same argument applies to the mean.

Periodically correlated sequences are generally nonstationary but yet they are nonstationary in a very simple way that, when the period $T$ is known, makes them equivalent to vector valued stationary processes.

The term periodically correlated was introduced by E. G. Gladyshev [77], but the same property was introduced by W. R. Bennett [12] who called them cyclostationary.

Since PC sequences are so closely related to stationary vector sequences, which are rather well understood, then one can legitimately ask: why go to the effort to study the structure of these processes? There are several answers. First, the value of $T$, required to transform a PC sequence to a vector stationary sequence, sometimes is not known prior to the analysis of an observed time series. Thus studying the time and spectral structure of the process using its natural time organization can provide clues to help us develop tests for PC structure and estimators for the period $T$. Second, the issues concerning innovation rank are more easily understood for PC sequences than for multivariate sequences because the natural time order eliminates some ambiguity. Third, the methods developed here for sequences naturally carry over to continuous time and to the almost periodic case; and in those cases it is not generally possible to block the process into a stationary sequence of finite dimensional vectors.

We will often assume that $E\{X_t\} = 0$ as it is the covariance (or quadratic) structure that is of most interest. However, we shall carefully discuss the issue of the additive periodic terms of a PC sequence, and how they can be conceptually viewed, and how they can be treated in the analysis of time series.

There are several ways in which two sequences can be considered equal. For example, two random processes $X_t$ and $Y_t$ can be called equal if for each $\omega \in \Omega$ their respective sample paths $X_t(\omega)$ and $Y_t(\omega)$ are the same. However, throughout this book, unless otherwise specified, we take two processes $X_t$ and $Y_t$ to be equal if

$$E \mid X_t - Y_t \mid^2 = 0, \quad \text{for every } t \in \mathbb{I}.$$ 

1.1 SUMMARY

This summary provides a little more detail about the contents with enough precision to make our direction clear, but not with the same care we will give subsequently. And it also provides further discussion of notation.
**Chapter 1: Introduction.** Gives an introductory empirical example to motivate the definitions, and then this summary followed by a historical development of the study of these processes. In this we do not attempt a complete bibliography but concentrate on the beginnings of the topic and give additional references that contain more complete bibliographies.

**Chapter 2: Examples, Models, and Simulations.** Presents simple models for constructing PC sequences, usually by combining randomness (usually through stationary sequences) with periodicity. Some important examples are sums and products of periodic sequences and stationary sequences, time scale modulation of stationary sequences, pulse amplitude modulation, periodic autoregressions, periodic moving averages, and periodically perturbed dynamical systems.

For most of these examples, results of simulations are presented to show the extent to which some sort of periodic rhythm is visually perceptible in the time series. These also illustrate that the usual periodogram typically does not reveal the presence of the periodic structure in PC sequences, and the periodogram of the squares sometimes can reveal the periodic structure, but not always.

**Chapter 3: Review of Hilbert Spaces.** Presents the basic facts about Hilbert space that will be needed. After definitions of vector space, inner product, and Hilbert space, general properties of (linear) operators are discussed. Of particular interest are projection operators, which have an important use in prediction, and unitary operators, which have a fundamental role in stationary and PC sequences. Finally, we review the spectral theory for unitary operators, including spectral measures, integrals, and the representation

$$U = \int_0^{2\pi} e^{i\lambda} E(d\lambda).$$  \hspace{1cm} (1.11)

This spectral representation plays a critical role in the spectral theory for stationary and PC sequences.

**Chapter 4: Stationary Random Sequences.** Emphasizes the role of the unitary operator and its spectral representation as we believe this helps to give a clear view of PC sequences. The core result is that if $X_j^t, j = 1, 2, \ldots, q$ are jointly (weakly) stationary and $\mathcal{H} = \overline{sp}\{X_j^t : j = 1, 2, \ldots, q, t \in \mathbb{Z}\}$, the stationary covariance structure allows one to prove quite easily that there exists a unitary operator $U : \mathcal{H} \rightarrow \mathcal{H}$ for which

$$X_{t+1}^j = UX_t^j$$  \hspace{1cm} (1.12)

for every $j = 1, 2, \ldots, q$ and $t \in \mathbb{Z}$. Iterating (1.12) gives $X_t^j = U^tX_0^j$ for all $t$, and by applying the spectral representation (1.11) we obtain the spectral
representation of the sequence

$$X_t^j = \int_0^{2\pi} e^{i\lambda t} \xi^j(d\lambda)$$

(1.13)

where $\xi^j$ is orthogonally scattered.

We then discuss the main topics connected with prediction, regularity and singularity, the Wold decomposition, innovations, the predictor expressed by innovations, the connection between spectral theory and prediction, and finally, finite past prediction. We also discuss the issue of rank in connection with innovations and spectral theory.

Chapter 5: Harmonizable Sequences. Presents the main facts about harmonizable random sequences with emphasis on what is important to PC sequences. As a generalization of the spectral representation for stationary sequences (and also for continuous time), M. Loève [138], who also wrote about (strongly) harmonizable processes in the first edition of Probability Theory [139], defined a sequence to be harmonizable if it has a spectral representation

$$X_t = \int_0^{2\pi} e^{i\lambda t} \xi(d\lambda),$$

(1.14)

where $\xi(\cdot)$ is an $L^2(\Omega, F, P)$ valued measure but no longer has orthogonally scattered (or uncorrelated) increments as it does in the stationary case. In order to convey the precise meaning of (1.14), we discuss vector valued measures and integration with respect to such measures. Then we discuss weakly and strongly harmonizable sequences, their connection to projections of stationary sequences, and spectral representation

$$R(s, t) = \int_0^{2\pi} \int_0^{2\pi} e^{i\lambda_1 s - i\lambda_2 t} F(d\lambda_1, d\lambda_2)$$

(1.15)

of the covariance, where the sense of integration depends on whether $X_t$ is weakly or strongly harmonizable.

Finally, we show how time invariant linear filtering affects the spectral representation of a harmonizable sequence (and of its covariance).

Chapter 6: Fourier Theory of the Covariance. This is a topic introduced and mainly completed by Gladyshev [77]. The bijection between PC-T sequences and T-variate stationary vector sequences makes it no surprise that the Fourier theory for the covariance for PC sequences is very much related to the Fourier theory for the covariance of stationary vector sequences.

The PC structure in the covariance (1.7) implies easily that

$$R(t + \tau, t) = \sum_{k=0}^{T-1} B_k(\tau) e^{i2\pi kt/T},$$

(1.16)
where $B_k(\tau) = T^{-1} \sum_{t=0}^{T-1} e^{-i2\pi kt/T} R(t + \tau, t)$. Using the connection to stationary vector sequences, Gladyshev argued that the coefficient functions $\{B_k(\tau) : k = 0, 1, \ldots, T-1\}$ are Fourier transforms

$$B_k(\tau) = \int_0^{2\pi} e^{i\lambda \tau} dF_k(\lambda). \quad (1.17)$$

We show it by use of a characterization of Fourier transforms based on a theorem of Riesz.

The plausibility that $R(s, t)$ given by (1.16) can be put into the form (1.15), which would make the covariance strongly harmonizable, turns out to be a fact, so every PC sequence is strongly harmonizable. The defining rhythm (1.7) associated with a PC sequence constrains the support set of the spectral measure $F$ appearing in (1.15) to the $2T - 1$ diagonal lines

$$S_T = \{(\lambda_1, \lambda_2) \in [0, 2\pi)^2 : \lambda_2 = \lambda_1 - 2\pi k/T, k = -(T-1), \ldots, T-1\}, \quad (1.18)$$

as illustrated in Figure 1.3. The support lines of $F$ may be identified with the sequence $\{F_k(\cdot) : k = 0, \ldots, T-1\}$ of complex measures whose Fourier transforms are $B_k(\tau)$.

We discuss the Lebesgue decomposition of $F$ and the issue of point masses in the random spectral measure $\xi(\cdot)$, some of which are produced by the mean $m(t)$. The effects of time invariant and periodic filtering, sampling, and random time shifting of PC sequences are examined. We also give the mapping between the spectral measure $F$ and the matrix valued spectral measure $F$ of the (blocked) vector stationary sequence $X_n$. 

![Figure 1.3](image-url)
Chapter 7: Representations of PC Sequences. Addresses various representations of PC sequences, with an emphasis on the connection to the unitary operator of a PC sequence. The basic covariance structure (1.7) implies that on the Hilbert space $\mathcal{H} = \mathbb{R}p\{X_t : t \in \mathbb{Z}\}$ there exists a unitary operator $U: \mathcal{H} \rightarrow \mathcal{H}$ for which

$$X_{t+T} = UX_t$$

for every $t \in \mathbb{Z}$. Thus $U$ is a shift operator for $X$ but only for shifts of length $T$. The most basic consequence of (1.19) is that we can find (derived from $U$) another unitary operator $V: \mathcal{H} \rightarrow \mathcal{H}$ and a periodic function $P_t$ taking values in $\mathcal{H}$ for which

$$X_t = V^t P_t, \text{ for every } t \in \mathbb{Z}.$$ 

(1.20)

Using the spectral theorem for the unitary $V$ leads to a spectral representation $X_t = \int_0^{2\pi} e^{i\lambda t} \xi_1(d\lambda, t)$, where $\xi_1(\cdot, t)$ is orthogonally scattered for all $t$ whereas harmonizability implies that PC sequences also have a spectral decomposition (1.14) with respect to a time invariant random spectral measure $\xi$ that is not orthogonally scattered. With the aid of (1.20) we explicitly construct the time independent random measure $\xi$. By expanding $P_t$ in a Fourier series we obtain the Gladyshev representation $X_t = \sum_{k=0}^{T-1} Z_t^k e^{i2\pi kt/T}$ as a Fourier series having jointly stationary coefficients $\{Z_t^k : k = 0, 1, \ldots, T - 1\}$. We show (see [160]) how to explicitly construct a dilated sequence $Y_t$ such that $X_t$ can be recovered by projection, $X_t = PY_t$.

Chapter 8: Prediction of PC Sequences. Treats the prediction problem for PC sequences, again with the help of the unitary operator $U$. We discuss regularity and singularity, the Wold decomposition and innovations, where we find that, at any $t$, the dimension $d_t$ of the innovation space is either 0 or 1, and $d_t = d_{t+T}$. It follows that a regular PC-$T$ sequence has an infinite moving average representation with respect to the orthonormal sequence $\{\xi_t : t \in D^+\}$,

$$X_t = \sum_{j \geq 0 : t-j \in D^+} a_t^j \xi_{t-j},$$

where the $\ell^2$ sequence of coefficients $A_t = \{a_t^j : j \geq 0\}$ is periodic $A_t = A_{t+T}$ and $D^+ = \{ t : d_t > 0 \}$ is the set of times where nontrivial innovation occurs. The number $r = \sum_{j=1}^{T-1} d_{t+j}$ is a constant (independent of $t$) and is defined to be the rank of a PC-$T$ sequence. A PC-$T$ sequence is of full rank whenever $r = T$. For a simply constructed PC sequence of less than full rank, let $\{\xi_t, t \in \mathbb{Z}\}$ be an orthonormal sequence; the sequence $\{\ldots, \xi_{-1}, \xi_{-1}, \xi_0, \xi_0, \xi_1, \xi_1, \ldots\}$ is PC-2 but of rank 1. We discuss the prediction problem for infinite and finite sets of predictors and give some illustrative results for periodic autoregressions of order 1, which, although simple, may also be of less than full