Advanced Stochastic Models, Risk Assessment, and Portfolio Optimization

The Ideal Risk, Uncertainty, and Performance Measures

SVETLOZAR T. RACHEV
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STR
To my children, Boryana and Vladimir

SVS
To my parents, Veselin and Evgeniya Kolevi, and my brother, Pavel Stoyanov

FJF
To the memory of my parents,
Josephine and Alfonso Fabozzi

Contents

CHAPTER 4
Ideal Probability Metrics 103

CHAPTER 5
Choice under Uncertainty 139

Preface

Modern portfolio theory, as pioneered in the 1950s by Harry Markowitz, is well adopted by the financial community. In spite of the fundamental shortcomings of mean-variance analysis, it remains a basic tool in the industry.

Since the 1990s, significant progress has been made in developing the concept of a risk measure from both a theoretical and a practical viewpoint. This notion has evolved into a materially different form from the original idea behind mean-variance analysis. As a consequence, the distinction between risk and uncertainty, which translates into a distinction between a risk measure and a dispersion measure, offers a new way of looking at the problem of optimal portfolio selection.

As concepts develop, other tools become appropriate to exploring evolved ideas than existing techniques. In applied finance, these tools are being imported from mathematics. That said, we believe that probability metrics, which is a field in probability theory, will turn out to be well-positioned for the study and further development of the quantitative aspects of risk and uncertainty. Going one step further, we make a parallel. In the theory of probability metrics, there exists a concept known as an *ideal probability metric*. This is a quantity best suited for the study of a given approximation problem in probability or stochastic processes. We believe that the ideas behind this concept can be borrowed and applied in the field of asset management to construct an *ideal risk measure* that would be ideal for a given optimal portfolio selection problem.

The development of probability metrics as a branch of probability theory started in the 1950s, even though its basic ideas were used during the first half of the 20th century. Its application to problems is connected with this fundamental question: "Is the proposed stochastic model a satisfactory approximation to the real model and, if so, within what limits?" In finance, we assume a stochastic model for asset return distributions and, in order to estimate portfolio risk, we sample from the fitted distribution. Then we use the generated simulations to evaluate the portfolio positions and, finally, to calculate portfolio risk. In this context, there are two issues arising on two different levels. First, the assumed stochastic model should be close to the empirical data. That is, we need a realistic model in the first place. Second, the generated scenarios should be sufficiently many in order to represent a good *approximation* model to the assumed stochastic model. In this way, we are sure that the computed portfolio risk numbers are close to what they would be had the problem been analytically tractable.

This book provides a gentle introduction into the theory of probability metrics and the problem of optimal portfolio selection, which is considered in the general context of risk and reward measures. We illustrate in numerous examples the basic concepts and where more technical knowledge is needed, an appendix is provided.

The book is organized in the following way. Chapters 1 and 2 contain introductory material from the fields of probability and optimization theory. Chapter 1 is necessary for understanding the general ideas behind probability metrics covered in Chapter 3 and ideal probability metrics in particular described in Chapter 4. The material in Chapter 2 is used when discussing optimal portfolio selection problems in Chapters 8, 9, and 10. We demonstrate how probability metrics can be applied to certain areas in finance in the following chapters:

- Chapter 5—stochastic dominance orders.
- Chapter 6—the construction of risk and dispersion measures.
- Chapter 7—problems involving average value-at-risk and spectral risk measures in particular.
- Chapter 8—reward-risk analysis generalizing mean-variance analysis.
- Chapter 9—the problem of benchmark tracking.
- Chapter 10—the construction of performance measures.

Chapters 5, 6, and 7 are also a prerequisite for the material in the last three chapters. Chapter 5 describes expected utility theory and stochastic dominance orders. The focus in Chapter 6 is on general dispersion measures and risk measures. Finally, in Chapter 7 we discuss the average value-at-risk and spectral risk measures, which are two particular families of coherent risk measures considered in Chapter 6.

The classical mean-variance analysis and the more general mean-risk analysis are explored in Chapter 8. We consider the structure of the efficient portfolios when average value-at-risk is selected as a risk measure. Chapter 9 is focused on the benchmark tracking problem. We generalize significantly the problem applying the methods of probability metrics. In Chapter 10, we discuss performance measures in the general framework of reward-risk analysis. We consider classes of performance measures that lead to practical optimal portfolio problems.

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CHAPTER 1

Concepts of Probability

1.1 INTRODUCTION

Will Microsoft's stock return over the next year exceed 10%? Will the one-month London Interbank Offered Rate (LIBOR) three months from now exceed 4%? Will Ford Motor Company default on its debt obligations sometime over the next five years? Microsoft's stock return over the next year, one-month LIBOR three months from now, and the default of Ford Motor Company on its debt obligations are each variables that exhibit randomness. Hence these variables are referred to as random variables.[1] In this chapter, we see how probability distributions are used to describe the potential outcomes of a random variable, the general properties of probability distributions, and the different types of probability distributions.[2] Random variables can be classified as either discrete or continuous. We begin with discrete probability distributions and then proceed to continuous probability distributions.

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[1] The precise mathematical definition is that a random variable is a measurable function from a probability space into the set of real numbers. In this chapter, the reader will repeatedly be confronted with imprecise definitions. The authors have intentionally chosen this way for a better general understandability and for the sake of an intuitive and illustrative description of the main concepts of probability theory. In order to inform about every occurrence of looseness and lack of mathematical rigor, we have furnished most imprecise definitions with a footnote giving a reference to the exact definition.

[2] For more detailed and/or complementary information, the reader is referred to the textbooks of Larsen and Marx (1986), Shiryaev (1996), and Billingsley (1995).

1.2 BASIC CONCEPTS

An *outcome* for a random variable is the mutually exclusive potential result that can occur. The accepted notation for an outcome is the Greek letter ω. A *sample space* is a set of all possible outcomes. The sample space is denoted by Ω. The fact that a given outcome ω_i belongs to the sample space is expressed by $\omega_i \in \Omega$. An *event* is a subset of the sample space and can be represented as a collection of some of the outcomes.[3] For example, consider Microsoft's stock return over the next year. The sample space contains outcomes ranging from 100% (all the funds invested in Microsoft's stock will be lost) to an extremely high positive return. The sample space can be partitioned into two subsets: outcomes where the return is less than or equal to 10% and a subset where the return exceeds 10%. Consequently, a return greater than 10% is an event since it is a subset of the sample space. Similarly, a one-month LIBOR three months from now that exceeds 4% is an event. The collection of all events is usually denoted by $\mathfrak{A}$. In the theory of probability, we consider the sample space Ω together with the set of events $\mathfrak{A}$, usually written as $(\Omega, \mathfrak{A})$, because the notion of probability is associated with an event.[4]

1.3 DISCRETE PROBABILITY DISTRIBUTIONS

As the name indicates, a *discrete random variable* limits the outcomes where the variable can only take on discrete values. For example, consider the default of a corporation on its debt obligations over the next five years. This random variable has only two possible outcomes: default or nondefault. Hence, it is a discrete random variable. Consider an option contract where for an upfront payment (i.e., the option price) of \$50,000, the buyer of the contract receives the payment given in Table 1.1 from the seller of the option depending on the return on the S&P 500 index. In this case, the random variable is a discrete random variable but on the limited number of outcomes.

---

[3] Precisely, only certain subsets of the sample space are called events. In the case that the sample space is represented by a subinterval of the real numbers, the events consist of the so-called "Borel sets." For all practical applications, we can think of Borel sets as containing all subsets of the sample space. In this case, the sample space together with the set of events is denoted by $(\mathbb{R}, \mathfrak{B})$. Shiryaev (1996) provides a precise definition.

[4] Probability is viewed as a function endowed with certain properties, taking events as an argument and providing their probabilities as a result. Thus, according to the mathematical construction, probability is defined on the elements of the set $\mathfrak{A}$ (called *sigma-field* or *sigma-algebra*) taking values in the interval [0, 1], $P : \mathfrak{A} \to [0, 1]$.

TABLE 1.1 Option Payments Depending on the Value of the S&P 500 Index.

If S&P 500 Return Is:	Payment Received By Option Buyer:
Less than or equal to zero	\$0
Greater than zero but less than 5%	\$10,000
Greater than 5% but less than 10%	\$20,000
Greater than or equal to 10%	\$100,000

The probabilistic treatment of discrete random variables is comparatively easy: Once a probability is assigned to all different outcomes, the probability of an arbitrary event can be calculated by simply adding the single probabilities. Imagine that in the above example on the S&P 500 every different payment occurs with the same probability of 25%. Then the probability of losing money by having invested \$50,000 to purchase the option is 75%, which is the sum of the probabilities of getting either \$0, \$10,000, or \$20,000 back. In the following sections we provide a short introduction to the most important discrete probability distributions: Bernoulli distribution, binomial distribution, and Poisson distribution. A detailed description together with an introduction to several other discrete probability distributions can be found, for example, in the textbook by Johnson et al. (1993).

1.3.1 Bernoulli Distribution

We will start the exposition with the *Bernoulli distribution*. A random variable X is *Bernoulli-distributed* with parameter p if it has only two possible outcomes, usually encoded as 1 (which might represent success or default) or 0 (which might represent failure or survival).

One classical example for a Bernoulli-distributed random variable occurring in the field of finance is the default event of a company. We observe a company C in a specified time interval I, January 1, 2007, until December 31, 2007. We define

$$X = \begin{cases} 1 & \text{if } C \text{ defaults in } I \\ 0 & \text{else.} \end{cases}$$

The parameter p in this case would be the annualized probability of default of company C.

1.3.2 Binomial Distribution

In practical applications, we usually do not consider a single company but a whole basket, $C_1, \ldots, C_n$, of companies. Assuming that all these n companies have the same annualized probability of default p, this leads to a natural generalization of the Bernoulli distribution called *binomial distribution*. A binomial distributed random variable Y with parameters n and p is obtained as the sum of n independent[5] and identically Bernoulli-distributed random variables $X_1, \dots, X_n$. In our example, Y represents the total number of defaults occurring in the year 2007 observed for companies $C_1, \dots, C_n$. Given the two parameters, the probability of observing k, $0 \leq k \leq n$ defaults can be explicitly calculated as follows:

$$P(Y = k) = \binom{n}{k} p^k (1 - p)^{n-k},$$

where

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

Recall that the factorial of a positive integer n is denoted by $n!$ and is equal to $n(n-1)(n-2) \cdot \ldots \cdot 2 \cdot 1$.

Bernoulli distribution and binomial distribution are revisited in Chapter 4 in connection with a fundamental result in the theory of probability called the *Central Limit Theorem*. Shiryaev (1996) provides a formal discussion of this important result.

1.3.3 Poisson Distribution

The last discrete distribution that we consider is the *Poisson distribution*. The Poisson distribution depends on only one parameter, λ, and can be interpreted as an approximation to the binomial distribution when the parameter p is a small number.[6] A Poisson-distributed random variable is usually used to describe the random number of events occurring over a certain time interval. We used this previously in terms of the number of defaults. One main difference compared to the binomial distribution is that the number of events that might occur is unbounded, at least theoretically. The parameter λ indicates the rate of occurrence of the random events, that is, it tells us how many events occur on average per unit of time.

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[5] A definition of what independence means is provided in Section 1.6.4. The reader might think of independence as no interference between the random variables.

[6] The approximation of Poisson to the binomial distribution concerns the so-called *rare events*. An event is called *rare* if the probability of its occurrence is close to zero. The probability of a rare event occurring in a sequence of independent trials can be approximately calculated with the formula of the Poisson distribution.

The probability distribution of a Poisson-distributed random variable N is described by the following equation:

$$P(N = k) = \frac{\lambda^k}{k!}e^{-\lambda},\ k = 0, 1, 2, \ldots$$

1.4 CONTINUOUS PROBABILITY DISTRIBUTIONS

If the random variable can take on any possible value within the range of outcomes, then the probability distribution is said to be a *continuous random variable*.[7] When a random variable is either the price of or the return on a financial asset or an interest rate, the random variable is assumed to be continuous. This means that it is possible to obtain, for example, a price of 95.43231 or 109.34872 and any value in between. In practice, we know that financial assets are not quoted in such a way. Nevertheless, there is no loss in describing the random variable as continuous and in many times treating the return as a continuous random variable means substantial gain in mathematical tractability and convenience. For a continuous random variable, the calculation of probabilities is substantially different from the discrete case. The reason is that if we want to derive the probability that the realization of the random variable lays within some range (i.e., over a subset or subinterval of the sample space), then we cannot proceed in a similar way as in the discrete case: The number of values in an interval is so large, that we cannot just add the probabilities of the single outcomes. The new concept needed is explained in the next section.

1.4.1 Probability Distribution Function, Probability Density Function, and Cumulative Distribution Function

A *probability distribution function* P assigns a probability $P(A)$ for every event A, that is, of realizing a value for the random value in any specified subset A of the sample space. For example, a probability distribution function can assign a probability of realizing a monthly return that is negative or the probability of realizing a monthly return that is greater than 0.5% or the probability of realizing a monthly return that is between 0.4% and 1.0%.

---

[7] Precisely, not every random variable taking its values in a subinterval of the real numbers is continuous. The exact definition requires the existence of a density function such as the one that we use later in this chapter to calculate probabilities.

To compute the probability, a mathematical function is needed to represent the probability distribution function. There are several possibilities of representing a probability distribution by means of a mathematical function. In the case of a continuous probability distribution, the most popular way is to provide the so-called *probability density function* or simply *density function*.

In general, we denote the density function for the random variable X as $f_X(x)$. Note that the letter x is used for the function argument and the index denotes that the density function corresponds to the random variable X. The letter x is the convention adopted to denote a particular value for the random variable. The density function of a probability distribution is always nonnegative and as its name indicates: Large values for $f_X(x)$ of the density function at some point x imply a relatively high probability of realizing a value in the neighborhood of x, whereas $f_X(x) = 0$ for all x in some interval (a, b) implies that the probability for observing a realization in (a, b) is zero.

Figure 1.1 aids in understanding a continuous probability distribution. The shaded area is the probability of realizing a return less than b and greater than a. As probabilities are represented by areas under the density function, it follows that the probability for every single outcome of a continuous random variable always equals zero. While the shaded area

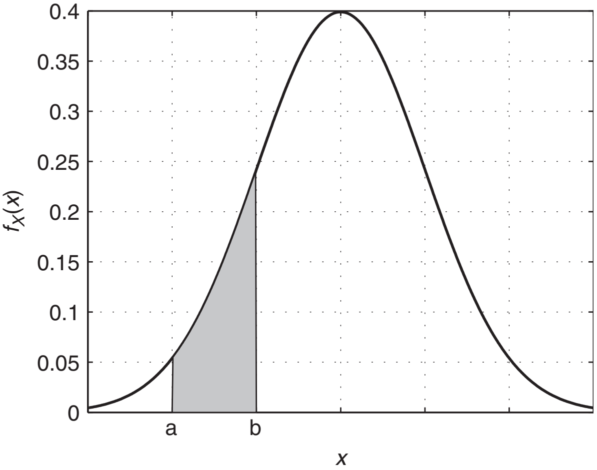

FIGURE 1.1 The probability of the event that a given random variable, X, is between two real numbers, a and b, which is equal to the shaded area under the density function, $f_X(x)$.

in Figure 1.1 represents the probability associated with realizing a return within the specified range, how does one compute the probability? This is where the tools of calculus are applied. Calculus involves differentiation and integration of a mathematical function. The latter tool is called *integral calculus* and involves computing the area under a curve. Thus the probability that a realization from a random variable is between two real numbers a and b is calculated according to the formula,

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx.$$

The mathematical function that provides the cumulative probability of a probability distribution, that is, the function that assigns to every real value x the probability of getting an outcome less than or equal to x, is called the *cumulative distribution function* or *cumulative probability function* or simply *distribution function* and is denoted mathematically by $F_X(x)$. A cumulative distribution function is always nonnegative, nondecreasing, and as it represents probabilities it takes only values between zero and one.[8] An example of a distribution function is given in Figure 1.2.

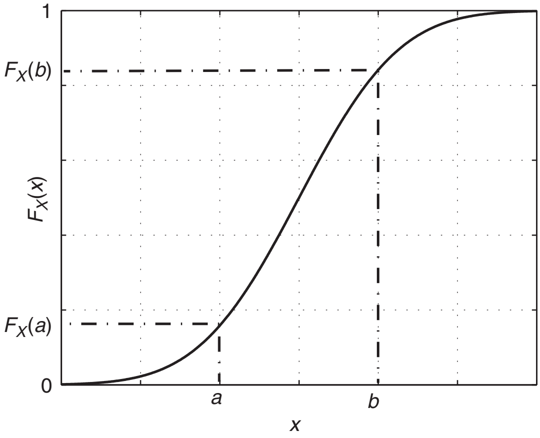

FIGURE 1.2 The probability of the event that a given random variable X is between two real numbers a and b is equal to the difference $F_X(b) - F_X(a)$.

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[8] Negative values would imply negative probabilities. If F decreased, that is, for some $x < y$ we have $F_X(x) > F_X(y)$, it would create a contradiction because the probability

The mathematical connection between a probability density function f, a probability distribution P, and a cumulative distribution function F of some random variable X is given by the following formula:

$$P(X \leq t) = F_X(t) = \int_{-\infty}^{t} f_X(x)dx.$$

Conversely, the density equals the first derivative of the distribution function,

$$f_X(x) = \frac{dF_X(x)}{dx}.$$

The cumulative distribution function is another way to uniquely characterize an arbitrary probability distribution on the set of real numbers. In terms of the distribution function, the probability that the random variable is between two real numbers a and b is given by

$$P(a < X \leq b) = F_X(b) - F_X(a).$$

Not all distribution functions are continuous and differentiable, such as the example plotted in Figure 1.2. Sometimes, a distribution function may have a jump for some value of the argument, or it can be composed of only jumps and flat sections. Such are the distribution functions of a discrete random variable for example. Figure 1.3 illustrates a more general case in which $F_X(x)$ is differentiable except for the point $x = a$ where there is a jump. It is often said that the distribution function has a point mass at $x = a$ because the value a happens with nonzero probability in contrast to the other outcomes, $x \neq a$. In fact, the probability that a occurs is equal to the size of the jump of the distribution function. We consider distribution functions with jumps in Chapter 7 in the discussion about the calculation of the average value-at-risk risk measure.

1.4.2 The Normal Distribution

The class of *normal distributions*, or *Gaussian distributions*, is certainly one of the most important probability distributions in statistics and due to some of its appealing properties also the class which is used in most applications in finance. Here we introduce some of its basic properties.

The random variable X is said to be normally distributed with parameters μ and σ, abbreviated by $X \in N(\mu, \sigma^2)$, if the density of the random

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of getting a value less than or equal to x must be smaller or equal to the probability of getting a value less than or equal to y.

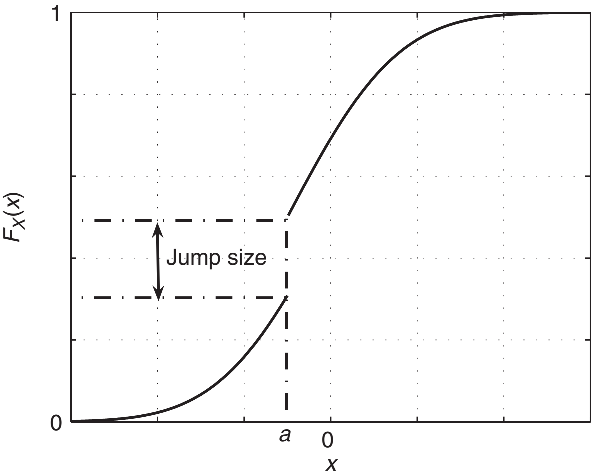

FIGURE 1.3 A distribution function $F_X(x)$ with a jump at $x = a$.

variable is given by the formula,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, x \in \mathbb{R}.$$

The parameter μ is called a *location parameter* because the middle of the distribution equals μ and σ is called a *shape parameter* or a *scale parameter*. If $\mu = 0$ and $\sigma = 1$, then X is said to have a *standard normal distribution*.

An important property of the normal distribution is the *location-scale invariance* of the normal distribution. What does this mean? Imagine you have random variable X, which is normally distributed with the parameters μ and σ. Now we consider the random variable Y, which is obtained as $Y = aX + b$. In general, the distribution of Y might substantially differ from the distribution of X but in the case where X is normally distributed, the random variable Y is again normally distributed with parameters and $\tilde{\mu} = a\mu + b$ and $\tilde{\sigma} = a\sigma$. Thus we do not leave the class of normal distributions if we multiply the random variable by a factor or shift the random variable. This fact can be used if we change the scale where a random variable is measured: Imagine that X measures the temperature at the top of the Empire State Building on January 1, 2008, at 6 A.M. in degrees Celsius. Then $Y = \frac{9}{5}X + 32$ will give the temperature in degrees Fahrenheit, and if X is normally distributed, then Y will be too.

Another interesting and important property of normal distributions is their summation stability. If you take the sum of several independent[9] random variables that are all normally distributed with location parameters μ_i and scale parameters σ_i, then the sum again will be normally distributed. The two parameters of the resulting distribution are obtained as

$$\mu = \mu_1 + \mu_2 + \cdots + \mu_n$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.$$

The last important property that is often misinterpreted to justify the nearly exclusive use of normal distributions in financial modeling is the fact that the normal distribution possesses a *domain of attraction*. A mathematical result called the *central limit theorem* states that under certain technical conditions the distribution of a large sum of random variables behaves necessarily like a normal distribution. In the eyes of many, the normal distribution is the unique class of probability distributions having this property. This is wrong and actually it is the class of stable distributions (containing the normal distributions) that is unique in the sense that a large sum of random variables can only converge to a stable distribution. We discuss the stable distribution in Chapter 4.

1.4.3 Exponential Distribution

The exponential distribution is popular, for example, in queuing theory when we want to model the time we have to wait until a certain event takes place. Examples include the time until the next client enters the store, the time until a certain company defaults or the time until some machine has a defect.

As it is used to model waiting times, the exponential distribution is concentrated on the positive real numbers and the density function f and the cumulative distribution function F of an exponentially distributed random variable τ possess the following form:

$$f_\tau(x) = \frac{1}{\beta} e^{-\frac{x}{\beta}}, \ x > 0$$

and

$$F_\tau(x) = 1 - e^{-\frac{x}{\beta}}, \ x > 0.$$

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[9] A definition of what independent means is provided in section 1.6.4. The reader might think of independence as nointerference between the random variables.