An Introduction to Nonlinear Partial Differential Equations

Second Edition

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An Introduction to Nonlinear Partial Differential Equations

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To Tess, for all her affection and support

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Contents

Preface

Nonlinear partial differential equations (PDEs) is a vast area, and practitioners include applied mathematicians, analysts, and others in the pure and applied sciences. This introductory text on nonlinear partial differential equations evolved from a graduate course I have taught for many years at the University of Nebraska at Lincoln. It emerged as a pedagogical effort to introduce, at a fairly elementary level, nonlinear PDEs in a format and style that is accessible to students with diverse backgrounds and interests. The audience has been a mixture of graduate students from mathematics, physics, and engineering. The prerequisites include an elementary course in PDEs emphasizing Fourier series and separation of variables, and an elementary course in ordinary differential equations.

There is enough independence among the chapters to allow the instructor considerable flexibility in choosing topics for a course. The text may be used for a second course in partial differential equations, a first course in nonlinear PDEs, a course in PDEs in the biological sciences, or an advanced course in applied mathematics or mathematical modeling. The range of applications include biology, chemistry, gas dynamics, porous media, combustion, traffic flow, water waves, plug flow reactors, heat transfer, and other topics of interest in applied mathematics.

There are three major changes from the first edition, which appeared in 1994. Because the original chapter on chemically reacting fluids was highly specialized for an introductory text, it has been removed from the new edition. Additionally, because of the surge of interest in mathematical biology, considerable material on that topic has been added; this includes linear and nonlinear age structure, spatial effects, and pattern formation. Finally, the text has been reorganized with the chapters on hyperbolic equations separated from

the chapters on diffusion processes, rather than intermixing them.

The references have been updated and, as in the previous edition, are selected to suit the needs of an introductory text, pointing the reader to parallel treatments and resources for further study. Finally, many new exercises have been added. The exercises are intermediate-level and are designed to build the students' problem solving techniques beyond what is experienced in a beginning course.

Chapter 1 develops a perspective on how to understand problems involving PDEs and how the subject interrelates with physical phenomena. The subject is developed from the basic conservation law, which, when appended to constitutive relations, gives rise to the fundamental models of diffusion, advection, and reaction. There is emphasis on understanding that nonlinear hyperbolic and parabolic PDEs describe evolutionary processes; a solution is a signal that is propagated into a spacetime domain from the boundaries of that domain. Also, there is focus on the structure of the various equations and what the terms describe physically. Chapters 2–4 deal with wave propagation and hyperbolic problems. In *Chapter 2* we assume that the equations have smooth solutions and we develop algorithms to solve the equations analytically. In *Chapter 3* we study discontinuous solutions and shock formation, and we introduce the concept of a weak solution. In keeping with our strategy of thinking about initial waveforms evolving in time, we focus on the initial value problem rather than the general Cauchy problem. The idea of characteristics is central and forms the thread that weaves through these two chapters. Next, *Chapter 4* introduces the shallow-water equations as the prototype of a hyperbolic system, and those equations are taken to illustrate basic concepts associated with hyperbolic systems: characteristics, Riemann's method, the hodograph transformation, and asymptotic behavior. Also, the general classification of systems of first-order PDEs is developed, and weakly nonlinear methods of analysis are described; the latter are illustrated by a derivation of Burgers' equation.

Chapters 1–4 can form the basis of a one-semester course focusing on wave propagation, characteristics, and hyperbolic equations.

Chapter 5 introduces diffusion processes. After establishing a probabilistic basis for diffusion, we examine methods that are useful in studying the solution structure of diffusion problems, including phase plane analysis, similarity methods, and asymptotic expansions. The prototype equations for reaction–diffusion and advection–diffusion, Fisher's equation and Burgers' equation, respectively, are studied in detail with emphasis on traveling wave solutions, the stability of those solutions, and the asymptotic behavior of solutions. The Appendix to Chapter 5 reviews phase plane analysis. In *Chapter 6* we discuss systems of reaction–diffusion equations, emphasizing applications and model building, especially in the biological sciences. We expend some effort addressing theoret-

ical concepts such as existence, uniqueness, comparison and maximum principles, energy estimates, blowup, and invariant sets; a key application includes pattern formation. Finally, elliptic equations are introduced in *Chapter 7* as a asymptotic limit of reaction–diffusion equations; nonlinear eigenvalue problems, stability, and bifurcation phenomena form the core of this chapter.

Chapter 1, along with Chapters 5–8, can form the basis of a one-semester course in diffusion and reaction–diffusion processes, with emphasis on PDEs in mathematical biology.

I want to acknowledge many users of the first edition who suggested improvements, corrections, and new topics. Their excitement for a second edition, along with the unwavering encouragement of my editor Susanne Steitz-Filler at Wiley, provided the stimulus to actually complete it. My own interest in nonlinear PDEs was spawned over many years by collaboration with those with whom I have had the privilege of working: Kane Yee at Kansas State, John Bdzil at Los Alamos, Ash Kapila at Rensselaer Polytechnic Institute, and several of my colleagues at Nebraska (Professors Steve Cohn, Steve Dunbar, Tony Joern in biology, Glenn Ledder, Tom Shores, Vitaly Zlotnik in geology, and my former student Bill Wolesensky, now at the College of Saint Mary). Readers of this text will see the influence of the classic books of G. B. Whitham (*Linear and Nonlinear Waves*) and J. Smoller (*Shock Waves and Reaction–Diffusion Equations*), R. Courant and K. O. Friedrichs (*Supersonic Flow and Shock Waves*), and the text on mathematical biology by J. D. Murray (*Mathematical Biology*). Finally, I express my gratitude to the National Science Foundation and to the Department of Energy for supporting my research efforts over the last several years.

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Lincoln, Nebraska

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1
Introduction to Partial Differential Equations

Partial differential equations (PDEs) is one of the basic areas of applied analysis, and it is difficult to imagine any area of applications where its impact is not felt. In recent decades there has been tremendous emphasis on understanding and modeling nonlinear processes; such processes are often governed by nonlinear PDEs, and the subject has become one of the most active areas in applied mathematics and central in modern-day mathematical research. Part of the impetus for this surge has been the advent of high-speed, powerful computers, where computational advances have been a major driving force.

This initial chapter focuses on developing a perspective on understanding problems involving PDEs and how the subject interrelates with physical phenomena. It also provides a transition from an elementary course, emphasizing eigenfunction expansions and linear problems, to a more sophisticated way of thinking about problems that is suggestive of and consistent with the methods in nonlinear analysis.

Section 1.1 summarizes some of the basic terminology of elementary PDEs, including ideas of classification. In Section 1.2 we begin the study of the origins of PDEs in physical problems. This interdependence is developed from the basic, one-dimensional conservation law. In Section 1.3 we show how constitutive relations can be appended to the conservation law to obtain equations that model the fundamental processes of diffusion, advection or transport, and reaction. Some of the common equations, such as the diffusion equation, Burgers' equation, Fisher's equation, and the porous media equation, are obtained

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as models of these processes. In Section 1.4 we introduce initial and boundary value problems to see how auxiliary data specialize the problems. Finally, in Section 1.5 we discuss wave propagation in order to fix the notion of how evolution equations carry boundary and initial signals into the domain of interest. We also introduce some common techniques for determining solutions of a certain form (e.g., traveling wave solutions). The ideas presented in this chapter are intended to build an understanding of evolutionary processes so that the fundamental concepts of hyperbolic problems and characteristics, as well as diffusion problems, can be examined in later chapters with a firmer base.

1.1 Partial Differential Equations

1.1.1 Equations and Solutions

A *partial differential equation* is an equation involving an unknown function of several variables and its partial derivatives. To fix the notion, a *second-order PDE in two independent variables* is an equation of the form

$$G(x, t, u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}) = 0, \quad (x, t) \in D, \tag{1.1.1}$$

where, as indicated, the independent variables x and t lie in some given domain D in $\mathbb{R}^2$. By a *solution* to (1.1.1) we mean a twice continuously differentiable function $u = u(x, t)$ defined on D that, when substituted into (1.1.1), reduces it to an identity on D. The function $u(x, t)$ is assumed to be twice continuously differentiable, so that it makes sense to calculate its first and second derivatives and substitute them into the equation; a smooth solution like this is called a *classical solution* or *genuine solution*. Later we extend the notion of solution to include functions that may have discontinuities, or discontinuities in their derivatives; such functions are called *weak solutions*. The xt domain D where the problem is defined is referred to as a *spacetime domain*, and PDEs that include time t as one of the independent variables are called *evolution* equations. When the two independent variables are both spatial variables, say, x and y rather than x and t, the PDE is an *equilibrium* or *steady-state* equation. Evolution equations govern time-dependent processes, and equilibrium equations often govern physical processes after the transients caused by initial or boundary conditions die away.

Graphically, a solution $u = u(x, t)$ of (1.1.1) is a smooth surface in three-dimensional xtu space lying over the domain D in the xt plane, as shown in Figure 1.1. An alternative representation is a plot in the xu-plane of the function $u = u(x, t_0)$ for some fixed time $t = t_0$ (see Figures 1.1 and 1.2). Such

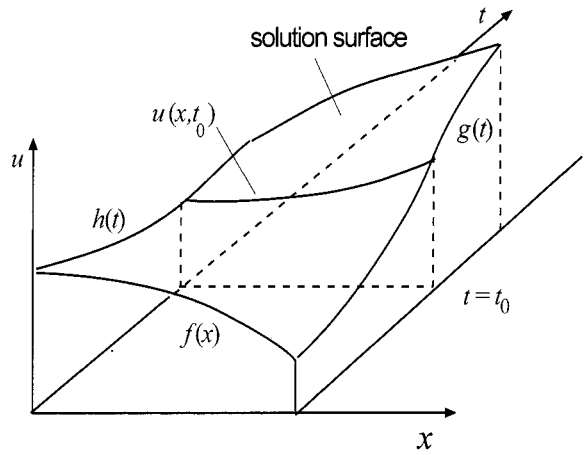

Figure 1.1 Solution surface $u = u(x,t)$ in xtu space, also showing a time snapshot or wave profile $u(x,t_0)$ at time t_0. The functions f, g, and h represent values of u on the boundary of the domain, which are often prescribed as initial and boundary conditions.

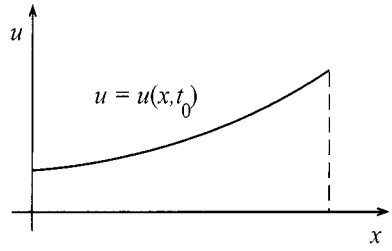

Figure 1.2 Time snapshot $u(x,t_0)$ at $t = t_0$ graphed in xu space. Often several snapshots for different times t are graphed on the same set of xu coordinates to indicate how the wave profiles are evolving in time.

representations are called *time snapshots* or *wave profiles* of the solution; time snapshots are profiles in space of the solution $u = u(x,t)$ frozen at a fixed time t_0, or, stated differently, slices of the solution surface at a fixed time t_0. Occasionally, several time snapshots are plotted simultaneously on the same set of xu axes to indicate how profiles change. It is also helpful on occasion to think of a solution in abstract terms. For example, suppose that $u = u(x,t)$ is a solution of a PDE for $x \in \mathbb{R}$ and $0 \leq t \leq T$. Then for each t, $u(x,t)$ is a function of x (a profile), and it generally belongs to some space of functions $\mathbf{X}$. To fix the idea, suppose that $\mathbf{X}$ is the set of all twice continuously differentiable

functions on $\mathbb{R}$ that approach zero at infinity. Then the solution can be regarded as a mapping from the time interval $[0,T]$ into the function space $\mathbf{X}$; that is, to each t in $[0,T]$ we associate a function $u(\cdot,t)$, which is the wave profile at time t.

A PDE has infinitely many solutions, depending on arbitrary functions. For example, the *wave equation*

$$u_{tt} - c^2 u_{xx} = 0 \tag{1.1.2}$$

has a general solution that is the superposition (sum) of a right traveling wave $F(x-ct)$ of speed c and a left traveling wave $G(x+ct)$ of speed c; that is,

$$u(x,t) = F(x-ct) + G(x+ct) \tag{1.1.3}$$

for any twice continuously differentiable functions F and G. (See the Exercises at the end of this section.) We contrast the situation in ordinary differential equations, where solutions depend on arbitrary constants; there, initial or boundary conditions fix the arbitrary constants and select a unique solution. For PDEs this occurs as well; initial and boundary conditions are usually imposed and select one of the infinitude of solutions. These auxiliary or subsidiary conditions are suggested by the underlying physical problem from which the PDE arises, or by the type of PDE. A condition on u or its derivatives given at $t=0$ along some segment of the x axis is called an *initial condition*, while a condition along any other curve in the xt plane is called a *boundary condition*. PDEs with auxiliary conditions are called *initial value problems, boundary value problems,* or *initial–boundary value problems*, depending on the type of subsidiary conditions that are specified.

Example. The initial value problem for the wave equation is

$$u_{tt} - c^2 u_{xx} = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{1.1.4}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x \in \mathbb{R}, \tag{1.1.5}$$

where f and g are given twice continuously differentiable functions on $\mathbb{R}$. The unique solution is given by (see Exercise 2)

$$u(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, \tag{1.1.6}$$

which is *D' Alembert's formula*. So, in this example we think of the auxiliary data (1.1.5) as selecting one of the infinitude of solutions given by (1.1.3). Note that the solution at (x,t) depends only on the initial data (1.1.5) in the interval $[x-ct, x+ct]$. □

Statements regarding the single second-order PDE (1.1.1) can be generalized in various directions. Higher-order equations (as well as first-order equations), several independent variables, and several unknown functions (governed by systems of PDEs) are all possibilities.

1.1.2 Classification

PDEs are classified into different types, depending on either the type of physical phenomena from which they arise or a mathematical basis. As the reader has learned from previous experience, there are three fundamental types of equations: those that govern diffusion processes, those that govern wave propagation, and those that govern equilibrium phenomena. Equations of mixed type also occur. We consider a single, second order PDE of the for

$$a(x,t)u_{xx} + 2b(x,t)u_{xt} + c(x,t)u_{tt} = d(x,t,u,u_x,u_t), \quad (x,t) \in D, \quad (1.1.7)$$

where a, b, and c are continuous functions on D, and not all of a, b, and c vanish simultaneously at some point of D. The function d on the right side is assumed to be continuous as well. Classification is based on the combination of the second-order derivatives in the equation. If we define the *discriminant* Δ by $\Delta = b^2 - ac$, then (1.1.7) is *hyperbolic* if $\Delta > 0$, *parabolic* if $\Delta = 0$, and *elliptic* if $\Delta < 0$.

Hyperbolic and parabolic equations are evolution equations that govern wave propagation and diffusion processes, respectively, and elliptic equations are associated with equilibrium or steady-state processes. In the latter case, we use x and y as independent variables rather than x and t. There is also a close relationship between the classification and the kinds of initial and boundary conditions that may be imposed on a PDE to obtain a well-posed mathematical problem, or one that is physically relevant. Because classification is based on the highest-order derivatives in (1.1.7), or the *principal part* of the equation, and because Δ depends on x and t, equations may change type as x and t vary throughout the domain.

Now we demonstrate that equation (1.1.7) can be transformed to certain simpler, or *canonical*, forms, depending on the classification, by a change of independent variables

$$\xi = \xi(x,t), \quad \eta = \eta(x,t). \quad (1.1.8)$$

We now perform this calculation, with the view of actually trying to determine (1.1.8) such that (1.1.7) reduces to a simpler form in the $\xi\eta$ coordinate system. The transformation (1.1.8) is assumed to be invertible, which requires that the Jacobian $J = \xi_x\eta_t - \xi_t\eta_x$ be nonzero in any region where the transformation is applied. A straightforward application of the chain rule, which the reader

can verify, shows that the left side of (1.1.7) becomes, under the change of independent variables (1.1.8)

$$au_{xx} + 2bu_{xt} + cu_{tt} + \cdots = Au_{\xi\xi} + 2Bu_{\xi\eta} + Cu_{\eta\eta} + \cdots, \tag{1.1.9}$$

where the three dots denote terms with lower-order derivatives, and where

$$\begin{aligned} A &= a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2, \\ B &= a\xi_x\eta_x + b(\xi_x\eta_t + \xi_t\eta_x) + c\xi_t\eta_t, \\ C &= a\eta_x^2 + 2b\eta_x\eta_t + c\eta_t^2. \end{aligned}$$

Notice that the expressions for A and C have the same form, namely

$$a\phi_x^2 + 2b\phi_x\phi_t + c\phi_t^2,$$

and are independent.

In the *hyperbolic case* we can choose ξ and η such that $A = C = 0$. To this end, set

$$a\phi_x^2 + 2b\phi_x\phi_t + c\phi_t^2 = 0. \tag{1.1.10}$$

Because the discriminant Δ is positive, we can write (1.1.10) as (assume that a is not zero)

$$\frac{\phi_x}{\phi_t} = -\frac{b \pm \sqrt{b^2 - ac}}{a}.$$

To determine ϕ, we regard it as defining loci (curves) in the xt plane via the equation $\phi(x,t) = \text{const}$. The differentials dx and dt along one of these curves satisfy the relation $\phi_x dx + \phi_t dt = 0$ or $dt/dx = -\phi_x/\phi_t$. Therefore

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a} \tag{1.1.11}$$

is a differential equation whose solutions determine the curves $\phi(x,t) = \text{const}$. On choosing the $+$ and $-$ signs in (1.1.11), respectively, we obtain $\xi(x,t)$ and $\eta(x,t)$ as integral curves of (1.1.11), making $A = C = 0$. Consequently, if (1.1.7) is hyperbolic, it can be reduced to the *canonical hyperbolic form*

$$u_{\xi\eta} + \cdots = 0,$$

where the three dots denote terms involving lower-order derivatives (we leave it as an exercise to show that B is nonzero in this case).

The differential equations (1.1.11) are called the *characteristic equations* associated with (1.1.7), and the two sets of solution curves $\xi(x,t) = \text{const}$ and $\eta(x,t) = \text{const}$ are called the *characteristic curves*, or just the *characteristics*; ξ and η are called *characteristic coordinates*. In summary, in the hyperbolic case there are two real families of characteristics that provide a coordinate system

where the equation reduces to a simpler form. Characteristics are the fundamental concept in the analysis of hyperbolic problems because characteristic coordinates form a natural curvilinear coordinate system in which to examine these problems. In some cases, PDEs simplify to ODEs along the characteristic curves.

In the *parabolic case* ($b^2 - ac = 0$) there is just one family of characteristic curves, defined by

$$\frac{dt}{dx} = \frac{b}{a}.$$

Thus we may choose $\xi = \xi(x,t)$ as an integral curve of this equation to make $A = 0$. Then, if $\eta = \eta(x,t)$ is chosen as any smooth function independent of ξ (i.e., so that the Jacobian is nonzero), one can easily determine that $B = 0$ automatically, giving the *parabolic canonical form*

$$u_{\xi\xi} + \cdots = 0.$$

Characteristics rarely play a role in parabolic problems.

In the *elliptic case* ($b^2 - ac < 0$) there are no real characteristics and, as in the parabolic case, characteristics play no role in elliptic problems. However, it is still possible to eliminate the mixed derivative term in (1.1.7) to obtain an elliptic canonical form. The procedure is to determine complex characteristics by solving (1.1.11), and then take real and imaginary parts to determine a transformation (1.1.8) that makes $A = C$ and $B = 0$ in (1.1.9). We leave it as an exercise to show that the transformation is given by

$$\alpha = \frac{1}{2}(\xi + \eta), \quad \beta = \frac{1}{2i}(\xi - \eta).$$

Then the *elliptic canonical form* is

$$u_{\alpha\alpha} + u_{\beta\beta} + \cdots = 0,$$

where the Laplacian operator becomes the principal part.

Example. It is easy to see that the characteristic curves for the wave equation (1.1.2), which is hyperbolic, are the straight lines $x - ct = \text{const}$ and $x + ct = \text{const}$. These are shown in Figure 1.3. In this case the characteristic coordinates are given by $\xi = x - ct$ and $\eta = x + ct$. In these coordinates the wave equation transforms to $u_{\xi\eta} = 0$. We regard characteristics as curves in spacetime moving with speeds c and $-c$, and from the general solution (1.1.3) we observe that signals are propagated along these curves. In hyperbolic problems, in general, the characteristics are curves in spacetime along which signals are transmitted. □

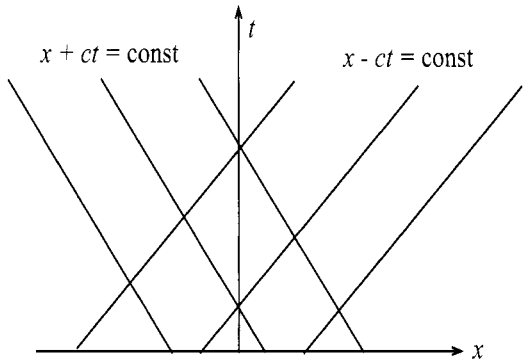

Figure 1.3 Characteristic diagram for the wave equation showing the forward and backward characteristics $x - ct = \text{const}$ and $x + ct = \text{const}$.

If the coefficients a, b, and c of the second-order derivatives in equation (1.1.7) depend on x, t, and u, then (1.1.7) is called a *quasilinear* equation. In this case we make the same classification as above, depending on the sign of the discriminant Δ; now the type of the equation depends not only on the spacetime domain but also on the solution u itself. The canonical forms listed above are no longer valid in this case, and the characteristics defined by (1.1.11) cannot be determined a priori since a, b, and c depend on u, the unknown solution itself. Therefore, there is a significant increase in difficulty when the principal part of the equation is nonlinear.

There are other ways to approach the classification problem. In the preceding discussion the focus was on determining transformations under which a simplification occurs. In Section 6.1 we take a different perspective and ask whether it is possible to determine the solution u near a curve where the values of u and its first derivatives are known. That discussion is accessible to the reader at the present juncture, if desired. Yet another view of classification is presented in Chapter 4, where hyperbolic systems are discussed. Finally, from a physical perspective, we observe later in this chapter that hyperbolic problems are associated with wave propagation; parabolic problems, with diffusion; and elliptic problems, with equilibria.

1.1.3 Linear versus Nonlinear

The most important classification criterion is to distinguish PDEs as *linear* or *nonlinear*. Roughly, a homogeneous PDE is linear if the sum of two solutions is a solution, and a constant multiple of a solutions is a solution. Otherwise, it is

nonlinear. The division of PDEs into these two categories is a significant one. The mathematical methods devised to deal with these two classes of equations are often entirely different, and the behavior of solutions differs substantially. One underlying cause is the fact that the solution space to a linear, homogeneous PDE is a vector space, and the linear structure of that space can be used with advantage in constructing solutions with desired properties that can meet diverse boundary and initial conditions. Such is not the case for nonlinear equations.

It is easy to find examples where nonlinear PDEs exhibit behavior with no linear counterpart. One is the breakdown of solutions and the formation of singularities, such as shock waves. A second is the existence of solitions, which are solutions to nonlinear dispersion equations. These solitary wave solutions maintain their shapes through collisions, in much the same was as linear equations do, even though the interactions are not linear. Nonlinear equations have come to the forefront because, basically, the world is nonlinear!

More formally, linearity and nonlinearity are usually defined in terms of the properties of the operator that defines the PDE itself. Let us assume that the PDE (1.1.1) can be written in the form

$$Lu = F, \tag{1.1.12}$$

where $F = F(x,t)$ and L is an operator that contains all the operations (differentiation, multiplication, composition, etc.) that act on $u = u(x,t)$. For example, the wave equation $u_{tt} - u_{xx} = 0$ can be written $Lu = 0$, where L is the partial differential operator $\partial_t^2 - \partial_x^2$. In (1.1.12) we reiterate that all terms involving the unknown function u are on the left side of the equation and are contained in the expression Lu; the right side of (1.1.12) contains in F only expressions involving the independent variables x and t. If $F = 0$, then (1.1.12) is said to be *homogeneous*; otherwise, it is *nonhomogeneous*. We say that an operator L is *linear* if it is additive and if constants factor out of the operator, that is, (1) $L(u+v) = Lu + Lv$, and (2) $L(cu) = cLu$, where u and v are functions (in the domain of the operator) and c is any constant. The PDE (1.1.12) is *linear* if L is a linear operator; otherwise, the PDE is *nonlinear*.

Example. The equation $Lu = u_t + uu_x = 0$ is nonlinear because, for example, $L(cu) = cu_t + c^2uu_x$, which does not equal $cLu = c(u_t + uu_x)$. □

Conditions (1) and (2) stated above imply that a linear homogeneous equation $Lu = 0$ has the property that if $u_1, u_2, \ldots, u_n$ are n solutions, the linear combination

$$u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

is also a solution for any choice of the constants $c_1, c_2, \ldots, c_n$. This fact is called the *superposition principle* for linear equations. For nonlinear equations we cannot superimpose solutions in this manner. The superposition principle can often be extended to infinite sums for linear problems, provided that convergence requirements are met. Superposition for linear equations allows one to construct, from a given set of solutions, another solution that meets initial or boundary requirements by choosing the constants $c_1, c_2, \ldots$ judiciously. This observation is the basis for the Fourier method, or eigenfunction expansion method, for linear, homogeneous boundary value problems, and we review this procedure at the end of the section. Moreover, superposition can often be extended to a family of solutions depending on a continuum of values of a parameter. More precisely, if $u = u(x, t; k)$ is a family of solutions of a linear homogeneous PDE for all values of k in some interval of real numbers I, one can superimpose these solutions formally using integration by defining

$$u(x,t) = \int_I c(k)u(x,t;k)\,dk,$$

where $c = c(k)$ is a function of the parameter k. Under certain conditions that must be established, the superposition $u(x, t)$ may again be a solution. As in the finite case, there is flexibility in selecting $c(k)$ to meet boundary or initial conditions. In fact, this procedure is the vehicle for transform methods for solving linear PDEs (Laplace transforms, Fourier transforms, etc.). We review this technique below. Finally, for a homogeneous, linear PDE the real and imaginary parts of a complex solution are both solutions. This is easily seen from the calculation

$$L(v + iw) = Lv + iLw = 0 + 0 = 0,$$

where the real-valued functions v and w satisfy $Lv = 0$ and $Lw = 0$. None of these methods based on superposition are applicable to nonlinear problems, and other methods must be sought. In summary, there is a profound difference between properties and solution methods for linear and nonlinear problems.

If most solution methods for linear problems are inapplicable to nonlinear equations, what methods can be developed? We mention a few.

1. *Perturbation Methods.* Perturbation methods are applicable to problems where a small or large parameter can be identified. In this case an approximate solution is sought as a series expansion in the parameter.

2. *Similarity Methods.* The similarity method is based on the PDE and its auxiliary conditions being invariant under a family of transformations depending on a small parameter. The invariance transformation allows one to identify a canonical change of variables that reduces the PDE to an ordinary differential equation (ODE), or reduces the order of the PDE.

3. *Characteristic Methods.* Nonlinear hyperbolic equations, which are associated with wave propagation, can be analyzed with success in characteristic coordinates (i.e., coordinates in spacetime along which the waves or signals propagate).

4. *Transformations.* Sometimes it is possible to identify transformations that change a given nonlinear equation into a simpler equation that can be solved.

5. *Numerical Methods.* Fast, large-scale computers have given tremendous impetus to the development and analysis of numerical algorithms to solve nonlinear problems and, in fact, have been a stimulus to to the analysis of nonlinear equations.

6. *Traveling Wave Solutions.* Seeking solutions with special properties is a key technique. For example, traveling waves are solutions to evolution problems that represent fixed waveforms moving in time. The assumption of a traveling wave profile to a PDE sometimes reduces it to an ODE, often facilitating the analysis and solution. Traveling wave solutions form one type of similarity solution.

7. *Steady State Solutions and Their Stability.* Many PDEs have steady-state, or time-independent, solutions. Studying these equilibrium solutions and their stability is an important activity in many areas of application.

8. *Ad Hoc Methods.* The mathematical and applied science literature is replete with articles illustrating special methods that analyze a certain type of nonlinear PDE, or restricted classes of nonlinear PDEs.

These methods are primarily solution methods, which represent one aspect of the subject of nonlinear PDEs. Other basic issues are questions of existence and uniqueness of solutions, the regularity (smoothness) of solutions, and the investigation of stability properties of solutions. These and other theoretical questions have spawned investigations based on modern topological and algebraic concepts, and the subject of nonlinear PDEs has evolved into one of the most diverse, active areas of applied analysis.

1.1.4 Linear Equations

In this subsection we review, through examples, two techniques from elementary PDEs that illustrate the use of the superposition principles mentioned above. These calculations arise later in analyzing the local stability of equilibrium solutions to nonlinear problems.

Example. (*Separation of Variables*) Consider the following problem for $u = u(x,t)$ on the bounded interval $I: 0 \le x \le 1$ with $t > 0$, that is

$$u_t = Au, \quad 0 < x < 1, \; t > 0, \tag{1.1.13}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{1.1.14}$$

$$u(x,0) = f(x), \quad 0 \le x \le 1, \tag{1.1.15}$$

where A is a linear, spatial differential operator of the form

$$Au = -(pu_x)_x + qu.$$

The functions $p = p(x)$ and $q = q(x)$ are given, with p of one sign on I, and p, p', and q continuous on I. Problems of this type are solved by Fourier's method, or the method of eigenfunction expansions. The idea is to construct infinitely many solutions that satisfy the PDE and the boundary conditions, equations (1.1.13) and (1.1.14), and then superimpose them, rigging up the constants so that the initial condition (1.1.15) is satisfied. This technique is called *separation of variables*, based on an assumption that the solution has the form $u(x,t) = g(t)y(x)$, where g and y are to be determined. When we substitute this form into the PDE and rearrange terms we obtain

$$\frac{g'}{g} = \frac{Ay}{y},$$

where the left side depends only on t and the right side depends only on x. A function of t can equal a function of x for all x and t only if both are equal to a constant, say, $-\lambda$, called the *separation constant*. Therefore

$$\frac{g'}{g} = \frac{Ay}{y} = -\lambda,$$

and we obtain two ODEs, one for g and one for y:

$$g' = -\lambda g, \quad -Ay = \lambda y.$$

We say that the equation separates. If we substitute the assumed form of u into the boundary conditions (1.1.14), then we obtain

$$y(0) = y(1) = 0.$$

The temporal equation is easily solved to get $g(t) = ce^{-\lambda t}$, where c is an arbitrary constant. The spatial equation along with its homogeneous (zero) boundary conditions give a boundary value problem (BVP) for y:

$$-Ay = \lambda y, \quad 0 < x < 1, \tag{1.1.16}$$

$$y(0) = y(1) = 0. \tag{1.1.17}$$

This BVP for y, which is differential eigenvalue problem called a *Sturm–Liouville problem*, has the property there are infinitely many real, discrete values of the separation constant λ, say, $\lambda = \lambda_n$, $n = 1, 2, ...$, for which there are corresponding solutions $y = y_n(x)$, $n = 1, 2, ...$ The λ_n are called the *eigenvalues* for the problem and the corresponding solutions $y = y_n(x)$ are called the *eigenfunctions*. The eigenvalues have the property that they are ordered and $|\lambda_n| \to \infty$ as $n \to \infty$. Therefore we have obtained a countably infinite number of solutions to the PDE that satisfy the boundary conditions:

$$u_n(x,t) = c_n e^{-\lambda_n t} y_n(x), \quad n = 1, 2, ...$$

Now, here is where superposition is used. We add up these solutions and pick the constants c_n so that the initial condition (1.1.15) is satisfied, thus obtaining the solution to the problem; that is, we form

$$u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n t} y_n(x).$$

Formally applying the initial condition gives

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} c_n y_n(x). \tag{1.1.18}$$

The right side is an expansion of the initial condition f in terms of the eigenfunctions y_n, and we can use it to determine the coefficients c_n. This calculation is enabled by a very important property of the eigenfunctions, namely, orthogonality. If we define the inner product of two functions ϕ and ψ by

$$(\phi, \psi) = \int_0^1 \phi(x)\psi(x)dx,$$

then we say ϕ and ψ are *orthogonal* if $(\phi, \psi) = 0$. The set of eigenfunctions y_n of the Sturm–Liouville problem (1.1.16)–(1.1.17) are mutually orthogonal, or

$$(y_n, y_m) \equiv \int_0^1 y_n(x)y_m(x)dx = 0, \quad n \neq m.$$

Therefore, if we multiply (1.1.18) by a fixed but arbitrary y_m and formally integrate over the interval I, we then obtain

$$(f, y_m) = \sum_{n=1}^{\infty} c_n(y_n, y_m).$$

Because of orthogonality, the infinite series on the right side collapses to the single term $c_m(y_m, y_m)$. Therefore the coefficient c_m is given by

$$c_m = \frac{(f, y_m)}{(y_m, y_m)}.$$

This relation is true for any m, and so the coefficients c_n are

$$c_n = \frac{(f, y_n)}{(y_n, y_n)}, \quad n = 1, 2, ... \tag{1.1.19}$$

Therefore, we have obtained the solution of (1.1.13)–(1.1.15) in the form of a series representation, or eigenfunction expansion,

$$u(x,t) = \sum_{n=1}^{\infty} \frac{(f, y_n)}{(y_n, y_n)} e^{-\lambda_n t} y_n(x).$$

The preceding calculation took a lot for granted, but it can be shown rigorously that the steps are valid. □

An expansion of a function $f(x)$ in terms of the eigenfunctions $y_n(x)$, as in (1.1.18), is called the generalized *Fourier series* for f, and the coefficients c_n, given by (1.1.19), are the *Fourier coefficients*. It can be shown that that the series converges in the mean-square sense:

$$\int_0^1 \left(f(x) - \sum_{n=1}^{N} c_n y_n(x) \right)^2 dx \to 0 \quad \text{as} \quad N \to \infty.$$

Pointwise and uniform convergence theorems require suitable smoothness conditions on the function f.

The method of separation of variables is successful under general boundary conditions of the form

$$\alpha u(0,t) + \beta u_x(0,t) = 0, \quad \gamma u(1,t) + \delta u_x(1,t) = 0,$$

where α, β, γ, and δ are given constants. Of course, the interval over which the problem is defined may be any *bounded* interval $a \le x \le b$; we chose $a = 0$ and $b = 1$ for simplicity of illustration. The method may be extended to problems over higher-dimensional, bounded, spatial domains, as well as to nonhomogeneous problems. For example, if the PDE in (1.1.13)–(1.1.15) is replaced by the nonhomogeneous equation

$$u_t = Au + F(x,t), \quad 0 < x < 1, \ t > 0,$$

we can expand the nonhomogeneous term F as a Fourier series of the eigenfunctions for the homogenous problem, or

$$F(x,t) = \sum_{n=1}^{\infty} \gamma_n(t) y_n(x),$$