Statistical Methods for Forecasting

BOVAS ABRAHAM
JOHANNES LEDOLTER

WILEY-INTERSCIENCE

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To our families

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Preface

Forecasting is an important part of decision making, and many of our decisions are based on predictions of future unknown events. Many books on forecasting and time series analysis have been published recently. Some of them are introductory and just describe the various methods heuristically. Certain others are very theoretical and focus on only a few selected topics.

This book is about the statistical methods and models that can be used to produce short-term forecasts. Our objective is to provide an intermediate-level discussion of a variety of statistical forecasting methods and models, to explain their interconnections, and to bridge the gap between theory and practice.

Forecast systems are introduced in Chapter 1. Various aspects of regression models are discussed in Chapter 2, and special problems that occur when fitting regression models to time series data are considered. Chapters 3 and 4 apply the regression and smoothing approach to predict a single time series. A brief introduction to seasonal adjustment methods is also given. Parametric models for nonseasonal and seasonal time series are explained in Chapters 5 and 6. Procedures for building such models and generating forecasts are discussed. Chapter 7 describes the relationships between the forecasts produced from exponential smoothing and those produced from parametric time series models. Several advanced topics, such as transfer function modeling, state space models, Kalman filtering, Bayesian forecasting, and methods for forecast evaluation, comparison, and control are given in Chapter 8. Exercises are provided in the back of the book for each chapter.

This book evolved from lecture notes for an MBA forecasting course and from notes for advanced undergraduate and beginning graduate statistics courses we have taught at the University of Waterloo and at the University of Iowa. It is oriented toward advanced undergraduate and beginning graduate students in statistics, business, engineering, and the social sciences.

A calculus background, some familiarity with matrix algebra, and an intermediate course in mathematical statistics are sufficient prerequisites.

Most business schools require their doctoral students to take courses in regression, forecasting, and time series analysis, and most offer courses in forecasting as an elective for MBA students. Courses in regression and in applied time series at the advanced undergraduate and beginning graduate level are also part of most statistics programs. This book can be used in several ways. It can serve as a text for a two-semester sequence in regression, forecasting, and time series analysis for Ph.D. business students, for MBA students with an area of concentration in quantitative methods, and for advanced undergraduate or beginning graduate students in applied statistics. It can also be used as a text for a one-semester course in forecasting (emphasis on Chapters 3 to 7), for a one-semester course in applied time series analysis (Chapters 5 to 8), or for a one-semester course in regression analysis (Chapter 2, and parts of Chapters 3 and 4). In addition, the book should be useful for the professional forecast practitioner.

We are grateful to a number of friends who helped in the preparation of this book. We are glad to record our thanks to Steve Brier, Bob Hogg, Paul Horn, and K. Vijayan, who commented on various parts of the manuscript. Any errors and omissions in this book are, of course, ours. We appreciate the patience and careful typing of the secretarial staff at the College of Business Administration, University of Iowa and of Marion Kaufman and Lynda Hohner at the Department of Statistics, University of Waterloo. We are thankful for the many suggestions we received from our students in forecasting, regression, and time series courses. We are also grateful to the *Biometrika* trustees for permission to reprint condensed and adapted versions of Tables 8, 12 and 18 from *Biometrika Tables for Statisticians*, edited by E. S. Pearson and H. O. Hartley.

We are greatly indebted to George Box who taught us time series analysis while we were graduate students at the University of Wisconsin. We wish to thank him for his guidance and for the wisdom which he shared so freely. It is also a pleasure to acknowledge George Tiao for his warm encouragement. His enthusiasm and enlightenment has been a constant source of inspiration.

We could not possibly discuss every issue in statistical forecasting. However, we hope that this volume provides the background that will allow the reader to adapt the methods included here to his or her particular needs.

B. ABRAHAM
J. LEDOLTER

Waterloo, Ontario
Iowa City, Iowa
June 1983

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CHAPTER 1

Introduction and Summary

Webster's dictionary defines *forecasting* as an activity "to calculate or predict some future event or condition, usually as a result of rational study or analysis of pertinent data."

1.1. IMPORTANCE OF GOOD FORECASTS

The ability to form good forecasts has been highly valued throughout history. Even today various types of fortune-tellers claim to have the power to predict future events. Frequently their predictions turn out to be false. However, occasionally their predictions come true; apparently often enough to secure a living for these forecasters.

We all make forecasts, although we may not recognize them as forecasts. For example, a person waiting for a bus or parents expecting a telephone call from their children may not consider themselves forecasters. However, from past experience and from reading the bus schedule, the person waiting for the bus expects it to arrive at a certain time or within a certain time interval. Parents who have usually received calls from their children every weekend expect to receive one during the coming weekend also.

These people form expectations, and they make forecasts. So does a bank manager who predicts the cash flow for the next quarter, or a control engineer who adjusts certain input variables to maintain the future value of some output variable as close as possible to a specified target, or a company manager who predicts sales or estimates the number of man-hours required to meet a given production schedule. All make statements about future events, patterning the forecasts closely on previous occurrences and assuming that the future will be similar to the past.

Since future events involve uncertainty, the forecasts are usually not perfect. The objective of forecasting is to reduce the forecast error: to produce forecasts that are seldom incorrect and that have small forecast errors. In business, industry, and government, policymakers must anticipate the future behavior of many critical variables before they make decisions. Their decisions depend on forecasts, and they expect these forecasts to be accurate; a forecast system is needed to make such predictions. Each situation that requires a forecast comes with its own unique set of problems, and the solutions to one are by no means the solutions in another situation. However, certain general principles are common to most forecasting problems and should be incorporated into any forecast system.

1.2. CLASSIFICATION OF FORECAST METHODS

Forecast methods may be broadly classified into *qualitative* and *quantitative* techniques. *Qualitative* or *subjective* forecast methods are intuitive, largely educated guesses that may or may not depend on past data. Usually these forecasts cannot be reproduced by someone else, since the forecaster does not specify explicitly how the available information is incorporated into the forecast. Even though subjective forecasting is a nonrigorous approach, it may be quite appropriate and the only reasonable method in certain situations.

Forecasts that are based on mathematical or statistical models are called *quantitative*. Once the underlying model or technique has been chosen, the corresponding forecasts are determined automatically; they are fully reproducible by any forecaster. Quantitative methods or models can be further classified as deterministic or probabilistic (also known as stochastic or statistical).

In *deterministic* models the relationship between the variable of interest, Y, and the explanatory or predictor variables $X_1, \ldots, X_p$ is determined exactly:

$$Y = f(X_1, \ldots, X_p; \beta_1, \ldots, \beta_m) \tag{1.1}$$

The function f and the coefficients $\beta_1, \ldots, \beta_m$ are known with certainty. The traditional "laws" in the physical sciences are examples of such deterministic relationships.

In the social sciences, however, the relationships are usually *stochastic*. Measurement errors and variability from other uncontrolled variables introduce random (stochastic) components. This leads to *probabilistic* or *stochastic models* of the form

$$Y = f(X_1, \ldots, X_p; \beta_1, \ldots, \beta_m) + \text{noise} \tag{1.2}$$

where the noise or error component is a realization from a certain probability distribution.

Frequently the functional form f and the coefficients are not known and have to be determined from past data. Usually the data occur in time-ordered sequences referred to as *time series*. Statistical models in which the available observations are used to determine the model form are also called *empirical* and are the main subject of this book. In particular, we discuss regression and single-variable prediction methods. In *single-variable forecasting*, we use the past history of the series, let's say z_t, where t is the time index, and extrapolate it into the future. For example, we may study the features in a series of monthly Canadian consumer price indices and extrapolate the pattern over the coming months. Smoothing methods or parametric time series models may be used for this purpose. In *regression forecasting*, we make use of the relationships between the variable to be forecast and the other variables that explain its variation. For example, we may forecast monthly beer sales from the price of beer, consumers' disposable income, and seasonal temperature; or predict the sales of a cereal product by its price (relative to the industry), its advertising, and the availability of its coupons. The standard regression models measure instantaneous effects. However, there are often lag effects, where the variable of interest depends on present and past values of the independent (i.e., predictor) variables. Such relationships can be studied by combining regression and time series models.

1.3. CONCEPTUAL FRAMEWORK OF A FORECAST SYSTEM

In this book we focus our attention exclusively on quantitative forecast methods. In general, a quantitative forecast system consists of two major components, as illustrated in Figure 1.1. At the first stage, the *model-building stage*, a forecasting model is constructed from pertinent data and available theory. In some instances, theory (for example, economic theory) may suggest particular models; in other cases, such theory may not exist or may be incomplete, and historical data must be used to specify an appropriate model. The tentatively entertained model usually contains unknown parameters; an estimation approach, such as least squares, can be used to determine these constants. Finally, the forecaster must check the adequacy of the fitted model. It could be inadequate for a number of reasons; for example, it could include inappropriate variables or it could have misspecified the functional relationship. If the model is unsatisfactory, it has to be respecified, and the iterative cycle of model specification–estimation–diagnostic checking must be repeated until a satisfactory model is found.

At the second stage, the *forecasting stage*, the final model is used to obtain the forecasts. Since these forecasts depend on the specified model,

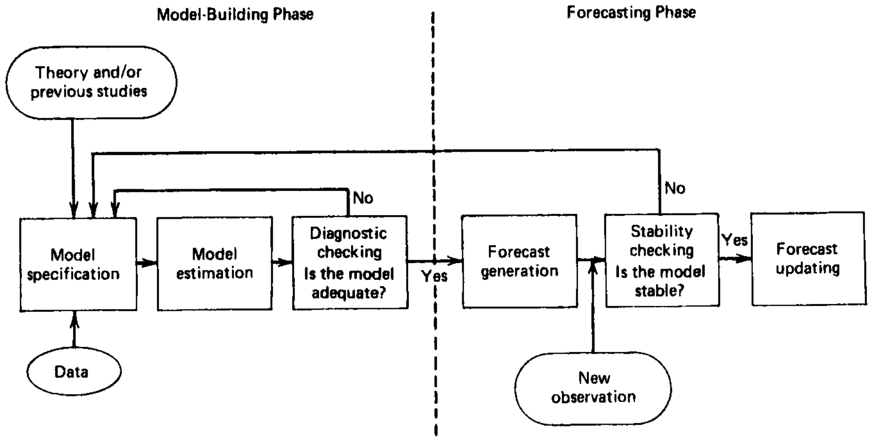

Figure 1.1. Conceptual framework of a forecasting system.

one has to make sure that the model and its parameters stay constant during the forecast period. The stability of the forecast model can be assessed by checking the forecasts against the new observations. Forecast errors can be calculated, and possible changes in the model can be detected. For example, particular functions of these forecast errors can indicate a bias in the forecasts (i.e., consistent over- or underpredictions). The most recent observation can also be used to update the forecasts. Since observations are recorded sequentially in time, updating procedures that can be applied routinely and that avoid the computation of each forecast from first principles are very desirable.

1.4. CHOICE OF A PARTICULAR FORECAST MODEL

Among many other forecast criteria, the choice of the forecast model or technique depends on (1) what degree of accuracy is required, (2) what the forecast horizon is, (3) how high a cost for producing the forecasts can be tolerated, (4) what degree of complexity is required, and (5) what data are available.

Sometimes only crude forecasts are needed; in other instances great accuracy is essential. In some applications, inaccuracy can be very costly; for example, inaccurate forecasts of an economic indicator could force the Federal Reserve Board to boost its lending rate, thus creating a chain of undesirable events. However, increasing the accuracy usually raises substantially the costs of data acquisition, computer time, and personnel. If a small loss in accuracy is not too critical, and if it lowers costs substantially, the simpler but less accurate model may be preferable to the more accurate but more complex one.

The forecast horizon is also essential, since the methods that produce short-term and long-term forecasts differ. For example, a manufacturer may wish to predict the sales of a product for the next 3 months, while an electric utility may wish to predict the demand for electricity over the next 25 years.

A forecaster should try building simple models, which are easy to understand, use, and explain. An elaborate model may lead to more accurate forecasts but may be more costly and difficult to implement. Ockham's razor, also known as the principle of parsimony, says that in a choice among competing models, other things being equal, the simplest is preferable.

Another important consideration in the choice of an appropriate forecast method is the availability of suitable data; one cannot expect to construct accurate empirical forecast models from a limited and incomplete data base.

1.5. FORECAST CRITERIA

The most important criterion for choosing a forecast method is its accuracy, or how closely the forecast predicts the actual event. Let us denote the actual observation at time t with z_t and its forecast, which uses the information up to and including time $t-1$, with $z_{t-1}(1)$. Then the objective is to find a forecast such that the future forecast error $z_t - z_{t-1}(1)$ is as small as possible. However, note that this is a future forecast error and, since z_t has not yet been observed, its value is unknown; we can talk only about its expected value, conditional on the observed history up to and including time $t-1$. If both negative (overprediction) and positive (underprediction) forecast errors are equally undesirable, it would make sense to choose the forecast such that the *mean absolute error* $E|z_t - z_{t-1}(1)|$, or the *mean square error* $E[z_t - z_{t-1}(1)]^2$ is minimized. The forecasts that minimize the mean square error are called *minimum mean square error* (*MMSE*) *forecasts*. The mean square error criterion is used here since it leads to simpler mathematical solutions.

1.6. OUTLINE OF THE BOOK

This book is about the statistical methods and models that can be used to produce short-term forecasts. It consists of four major parts: regression, smoothing methods, time series models, and selected special topics.

In Chapter 2 we discuss the *regression model*, which describes the relationship between a dependent or response variable and a set of independent or predictor variables. We discuss how regression models are built from historical data, how their parameters are estimated, and how they can be used for forecasting. We describe the matrix representation of regression models and cover such topics as transformations, multicollinearity, and the special problems that occur in fitting regression models to time series data. We discuss how to detect serial correlation in the errors, the consequences of such correlation, and generalizations of regression models that take account of this correlation explicitly.

In Chapters 3 through 7 we discuss how to forecast a single time series, without information from other, possibly related, series. In Chapter 3 we review regression and smoothing as methods of forecasting nonseasonal time series. Nonseasonal series are characterized by time trends and uncorrelated error or noise components. The trend component is usually a polynomial in time; constant, linear, and quadratic trends are special cases.

In models with stable, nonchanging trend components and uncorrelated errors, the parameters can be estimated by least squares. If the trend components change with time, *discounted least squares*, also known as *general exponential smoothing*, can be used to estimate the parameters and derive future forecasts. There the influence of the observations on the parameter estimates diminishes with the age of the observations. Special cases lead to simple, double, and triple exponential smoothing and are discussed in detail.

In Chapter 4 we apply the regression and smoothing methods to forecast seasonal series. Seasonal time series are decomposed into trend, seasonal, and error components. The seasonal component is expressed as a sum of either trigonometric functions or seasonal indicators. We describe the regression approach for series with stable trend and seasonal components, and discuss general exponential smoothing and Winters' additive and multiplicative methods for series with time-changing components. In addition, seasonal adjustment is introduced, with emphasis on the Census X-11 method.

A stochastic modeling or time series analysis approach to forecast a single time series is given in Chapters 5 and 6. In Chapter 5 we discuss the class of *autoregressive integrated moving average (ARIMA) models*, which can represent many stationary and nonstationary stochastic processes. A stationary stochastic process is characterized by its mean, its variance, and its autocorrelation function. Transformations, in particular successive differences, transform nonstationary series with changing means into stationary series. The patterns in the autocorrelation functions implied by specific ARIMA models are analyzed in detail; to simplify the model-specification we also introduce partial autocorrelations and describe their patterns. We explain Box and Jenkins' (1976) three-stage iterative model-building strategy, which consists of model specification, parameter estimation, and diagnostic checking. We show how forecasts from ARIMA models can be derived; how minimum mean square error forecasts and corresponding prediction intervals can be easily calculated; and how the implied forecast functions, which consist of exponential and polynomial functions of the forecast horizon, adapt as new observations are observed.

The class of ARIMA models is extended to include seasonal time series models. Multiplicative seasonal ARIMA models are described in Chapter 6, and their implied autocorrelation and partial autocorrelation functions are discussed. The minimum mean square error forecasts from such models are illustrated with several examples. The forecast functions include polynomial trends and trigonometric seasonal components that, unlike those in the seasonal regression model described in Chapter 4, adapt as new observations become available.

In Chapter 7 we discuss the relationships between the forecasts from exponential smoothing and the forecasts derived from ARIMA time series models. We show that the time series approach actually includes general exponential smoothing as a special case; exponential smoothing forecast procedures are implied by certain restricted ARIMA models. Implications of these relationships for the forecast practitioner are discussed.

In Chapter 8 we introduce several more advanced forecast techniques. We describe *transfer function models*, which relate an output series to present and past values of an input series; *intervention time series modeling*, which can be used to assess the effect of an exogenous intervention; *Bayesian forecasting* and *Kalman filtering*; *time series models with time-varying coefficients*; *adaptive filtering*; and *post-sample forecast evaluation* and *tracking signals*.

Throughout the book we emphasize a model-based approach to forecasting. We discuss how models are built to generate forecasts, and match commonly used forecast procedures to models within which they generate optimal forecasts. We stress the importance of checking the adequacy of a model before using it for forecasting. An appropriate forecast system has to produce uncorrelated one-step-ahead forecast errors, since correlations among these forecast errors would indicate that there is information in the data that has not yet been used.

Many actual examples are presented in the book. The data sets are real and have been obtained from published articles and consulting projects. Exercises are given in the back of the book for each chapter.

CHAPTER 2

The Regression Model and Its Application in Forecasting

Regression analysis is concerned with modeling the relationships among variables. It quantifies how a response (or dependent) variable is related to a set of explanatory (independent, predictor) variables. For example, a manager might be interested in knowing how the sales of a particular product are related to its price, the prices of competitive products, and the amount spent for advertising. Or an economist might be interested in knowing how a change in per capita income affects consumption, how a price change in gasoline affects gasoline demand, or how the gross national product is related to government spending. Engineers might be interested in knowing how the yield of a particular chemical process depends on reaction time, temperature, and the type of catalyst used.

If the true relationships among the variables were known exactly, the investigator (manager, economist, engineer) would be in a position to understand, predict, and control the response. For example, the economist could predict gasoline sales for any fixed price (forecasting) or could choose the price to keep the gasoline sales at a fixed level (control).

The true relationships among the studied variables, however, will rarely be known, and one must rely on empirical evidence to develop approximations. In addition, the responses will vary, even if the experiment is repeated under apparently identical conditions. This variation that occurs from one repetition to the next is called *noise*, *experimental variation*, *experimental error*, or merely *error*. The variation can come from many sources; it is usually due to measurement error and variation in other, uncontrollable variables. To take explicit account of it, we have to consider probabilistic (or statistical) models.

In this chapter we introduce one such probabilistic model, namely the regression model. We discuss how to construct such a model from empirical data and how to use it in forecasting. In our discussion, we emphasize general principles of statistical model building, such as the specification of models, the estimation of unknown coefficients (parameters), and diagnostic procedures to check the adequacy of the considered model.

2.1. THE REGRESSION MODEL

Let us study the relationship between a dependent variable Y and p independent variables $X_1, X_2, \ldots, X_p$. An index t is introduced to denote the dependent variable at time t (or for subject t) by y_t and the p independent variables by $x_{t1}, x_{t2}, \ldots, x_{tp}$. For observations that occur in natural time order, the index t stands for time. In situations where there is no such ordering, t is just an arbitrary index. For example, in cross-sectional data, where we get observations on different subjects (companies, counties, etc.), the particular ordering has no meaning.

In its most general form, the regression model can be written as

$$y_t = f(\mathbf{x}_t; \boldsymbol{\beta}) + \varepsilon_t \tag{2.1}$$

where $f(\mathbf{x}_t; \boldsymbol{\beta})$ is a mathematical function of the p independent variables $\mathbf{x}_t = (x_{t1}, \ldots, x_{tp})'$ and unknown parameters $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_m)'$. In the following discussion we assume that, apart from the unknown parameters, the functional form of the model is known.

The model in (2.1) is probabilistic, since the error term ε_t is a random variable. It is assumed that:

1. Its mean, $E(\varepsilon_t) = 0$, and its variance, $V(\varepsilon_t) = \sigma^2$, are constant and do not depend on t.
2. The errors ε_t are uncorrelated; that is, $\text{Cov}(\varepsilon_t, \varepsilon_{t-k}) = E(\varepsilon_t \varepsilon_{t-k}) = 0$ for all t and $k \neq 0$.
3. The errors come from a normal distribution. Then assumption 2 implies independence among the errors.

Due to the random nature of the error terms ε_t, the dependent variable y_t itself is a random variable. The model in Equation (2.1) can therefore also be expressed in terms of the conditional distribution of y_t given $\mathbf{x}_t =$

$(x_{t1}, \ldots, x_{tp})'$. In this context the assumptions can be rewritten:

1. The conditional mean, $E(y_t|\mathbf{x}_t) = f(\mathbf{x}_t; \boldsymbol{\beta})$, depends on the independent variables $\mathbf{x}_t$ and the parameters $\boldsymbol{\beta}$, and the variance $V(y_t|\mathbf{x}_t) = \sigma^2$ is independent of $\mathbf{x}_t$ and time.
2. The dependent variables y_t and y_{t-k} for different time periods (or subjects) are uncorrelated:
$$\mathrm{Cov}(y_t, y_{t-k}) = E[y_t - f(\mathbf{x}_t; \boldsymbol{\beta})][y_{t-k} - f(\mathbf{x}_{t-k}; \boldsymbol{\beta})] = 0.$$
3. Conditional on $\mathbf{x}_t$, y_t follows a normal distribution with mean $f(\mathbf{x}_t; \boldsymbol{\beta})$ and variance σ^2; this is denoted by $N(f(\mathbf{x}_t; \boldsymbol{\beta}), \sigma^2)$.

These assumptions imply that the mean of the conditional distribution of y_t is a function of the independent variables $\mathbf{x}_t$. This relationship, however, is not deterministic, as for each fixed $\mathbf{x}_t$ the corresponding y_t will scatter around its mean. The variation of this scatter does not depend on t or on the levels of the independent variables. Furthermore, the error ε_t cannot be predicted from other errors.

In our discussion we assume that the independent variables are fixed and nonstochastic. This simplifies the derivations in this chapter. However, most results also hold if the predictor variables are random.

Several special cases of the general regression model are given below:

1. $y_t = \beta_0 + \varepsilon_t$ (constant mean model)
2. $y_t = \beta_0 + \beta_1 x_t + \varepsilon_t$ (simple linear regression model)
3. $y_t = \beta_0 \exp(\beta_1 x_t) + \varepsilon_t$ (exponential growth model)
4. $y_t = \beta_0 + \beta_1 x_t + \beta_2 x_t^2 + \varepsilon_t$ (quadratic model)
5. $y_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \varepsilon_t$ (linear model with two independent variables)
6. $y_t = \beta_0 + \beta_1 x_{t1} + \beta_2 x_{t2} + \beta_{11} x_{t1}^2 + \beta_{22} x_{t2}^2 + \beta_{12} x_{t1} x_{t2} + \varepsilon_t$ (quadratic model with two independent variables)

In Figure 2.1 we have plotted $E(y_t|\mathbf{x}_t) = f(\mathbf{x}_t; \boldsymbol{\beta})$ for the models given above. An individual y_t will vary around its mean function according to a normal distribution with constant variance σ^2. This is illustrated in Figure 2.1 for model 2.

2.1.1. Linear and Nonlinear Models

All models except the third are *linear in the parameters*, which means that the derivatives of $f(\mathbf{x}_t; \boldsymbol{\beta})$ with respect to the parameters in $\boldsymbol{\beta}$ do not depend