The Finite Element Method for Electromagnetic Modeling

Edited by
Gérard Meunier

iSTE

WILEY

This page intentionally left blank

The Finite Element Method for Electromagnetic Modeling

The Finite Element Method for Electromagnetic Modeling

Edited by
Gérard Meunier

iSTE

WILEY

First published in France in three volumes by Hermes Science/Lavoisier entitled "Electromagnétisme et éléments finis Vol. 1, 2 et 3"
First published in Great Britain and the United States in 2008 by ISTE Ltd and John Wiley & Sons, Inc.

ISTE Ltd
6 Fitzroy Square
London W1T 5DX
UK
www.iste.co.uk

John Wiley & Sons, Inc.
111 River Street
Hoboken, NJ 07030
USA
www.wiley.com

Library of Congress Cataloging-in-Publication Data

Electromagnétisme et éléments finis. English
The finite element method for electromagnetic modeling / edited by Gérard Meunier.
p. cm.
Includes bibliographical references and index.
ISBN: 978-1-84821-030-1
1. Electromagnetic devices--Mathematical models. 2. Electromagnetism--Mathematical models. 3. Engineering mathematics. 4. Finite element method. I. Meunier, Gérard.
TK7872.M25E4284 2008
621.301'51825--dc22

2007046086

British Library Cataloguing-in-Publication Data
A CIP record for this book is available from the British Library
ISBN: 978-1-84821-030-1

Printed and bound in Great Britain by Antony Rowe Ltd, Chippenham, Wiltshire.

FSC
Mixed Sources
Product group from well-managed forests and other controlled sources

Cert no. SGS-COC-2953
www.fsc.org
© 1996 Forest Stewardship Council

Table of Contents

This page intentionally left blank

Chapter 1

Introduction to Nodal Finite Elements

1.1. Introduction

1.1.1. *The finite element method*

The finite element method, resulting from the matrix techniques of calculation of the discrete or semi-discrete mechanical structures (assembly of beams), is a tool for resolving problems with partial differential equations involved in physics problems. We will thus tackle this method accordingly because it is useful in modeling mechanical, thermal, neutron and electromagnetic problems [ZIE 79], [SIL 83], [DHA 84], [SAB 86], [HOO 89].

The aim of this chapter is to present the principles of this method which have become essential in the panoply of the engineer. For this presentation, we will only deal with electrostatics. Indeed, this field has a familiar formulation in scalar potential, particularly suitable for the presentation of nodal finite elements, which will be the only ones discussed here.

We will develop two examples of increasing complexity which are manageable "by hand", 1D in a first part and 2D in a second. As it is very close to physical considerations, the variational approach will most of the time be favored. However, the more general method of weighted residues will also be presented. In our examples, we will see how to solve the problems at issue, but also how, using the obtained fields, to extract more relevant information.

Chapter written by Jean-Louis COULOMB.

In the third and last part, we will present the concept of a reference element and the principles that make it possible to pass from the local coordinates to the domain coordinates. We will see that beyond the possibility of handling curvilinear elements, which is quite convenient for the discretization of manufactured objects, this technique leads to a general tool for working with geometric deformations.

1.2. The 1D finite element method

1.2.1. *A simple electrostatics problem*

In order to present the finite element method, we propose, initially, to implement it on a simple 1D electrostatics example, borrowed from [HOO 89]. We will first formulate this problem in its differential form, then in its variation form. This form of integral will enable us to introduce the concept of first-order finite elements and then second-order finite elements.

We thus consider the problem of Figure 1.1 where two long distant parallel plates of *10 m* are: one with the electric potential of *0 V* and the other with the potential of *100 V*. Between the two plates, the density of electric charges and the dielectric permittivity are assumed to be constant. This problem could represent a hydrocarbon storage tank in which we wish to know the distribution of the electric potential. The lower plate corresponds to the free surface of the liquid, the upper plate to the ceiling of the tank and the intermediate part to the electrically charged vapors.

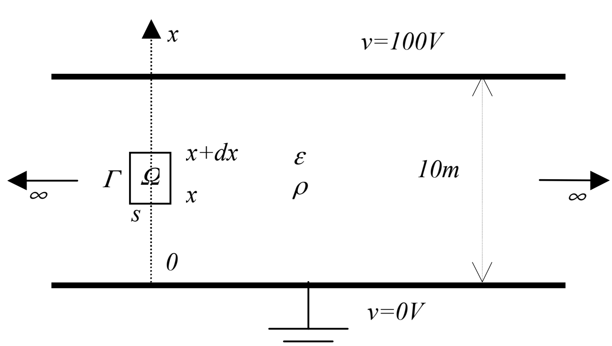

Figure 1.1. *The cloud of electric charges between the two plates*

1.2.2. *Differential approach*

The physical and geometric quantities varying only according to one direction, this problem is 1D in the interval x∈ [0, 10] and the electric field E and electric flux density $D = \varepsilon E$ vectors have only one non-zero component E_x and D_x.

Let us consider a parallelepipedic elementary volume of constant section s in the direction perpendicular to x and of length dx. The flux of the electric density vector, leaving its border Γ, and the internal electric charge to its volume Ω are respectively:

$$\oiint_{\Gamma} D.d\Gamma = \left[D_x(x+dx) - D_x(x)\right].s \tag{1.1}$$

$$\iiint_{\Omega} \rho d\Omega = \rho.s.dx \tag{1.2}$$

The Gaussian electric law implies the equality of these two integrals, which gives, for the electric flux density, the following differential equation:

$$\frac{dD_x}{dx} = \rho \tag{1.3}$$

This equation is specifically one of Maxwell's equations:

$$divD = \rho \tag{1.4}$$

applied to a 1D problem in which the variations in the orthogonal directions to the x axis are zero.

On the terminals of the domain, the boundary conditions are expressed in terms of electric potential $v(0) = 0\ V$ and $v(10) = 100\ V$. It is thus judicious to specify the problem entirely in terms of v which is connected to the electric field by the relation $E_x = -grad\ v$, which, in our 1D case, gives $E_x = -\frac{dv}{dx}$. The equation and the boundary conditions governing the distribution of the electric potential are thus

$$\frac{d}{dx}\left[-\varepsilon\frac{dv}{dx}\right] = \rho \qquad \text{for } x \in [0, 10] \tag{1.5}$$

$$v = 0 \qquad \text{for } x = 0$$

$$v = 100 \qquad \text{for } x = 10$$

In our case, the electric permittivity is constant, which simplifies the equation and becomes

$$\frac{d^2v}{dx^2} = -\frac{\rho}{\varepsilon}, \qquad v(0) = 0, \qquad v(10) = 100 \tag{1.6}$$

This problem has the following analytical solution

$$v(x) = -\frac{\rho}{2\varepsilon}x^2 + \left[1 + \frac{\rho}{2\varepsilon}\right]10x \tag{1.7}$$

the knowledge of which will be useful for us when evaluating the quality of the solution given by the finite element method, which we will present below.

1.2.3. *Variational approach*

In fact, the finite element method does not directly use the previous differential form, but is based on an equivalent integral form. For this reason we will develop the *variational* approach which here is connected to the internal energy of the device. This approach is based on a functional (i.e. a function of the unknown function *v(x)*) which is extremal when *v(x)* is the solution. The functional, called coenergy for reasons which will be explained later, corresponding to electrostatics problem [1.5] is

$$W_c(v) = \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{dv}{dx}\right]^2 dx - \int_0^{10} \rho v \, dx \tag{1.8}$$

We will show that, if it exists, a continuous and derivable function $v_m(x)$ which fulfills the boundary conditions $v_m(0) = 0$ and $v_m(10) = 100$ and which makes functional [1.8] extremal is also the solution of problem [1.5].

For that, let us consider a function *v(x)* built on the basis of $v_m(x)$ as follows

$$v(x) = v_m(x) + \alpha\varphi(x) \tag{1.9}$$

where α is an unspecified real number and $\varphi(x)$ is an arbitrary continuous and derivable function which becomes zero at the boundary of the domain ($\varphi(0) = 0$ and $\varphi(10) = 0$). By construction, function *v(x)* automatically verifies the boundary conditions $v(0) = 0$ and $v(10) = 100$.

The introduction into [1.8] of this function *v(x)* defines a simple function of α

$$W_c(\alpha) = \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d}{dx}[v_m + \alpha\varphi]\right]^2 dx - \int_0^{10} \rho[v_m + \alpha\varphi]dx \qquad [1.10]$$

Note that, by assumption, for $\alpha = 0$ this function is extremal. Let us now express the increase of W_c with respect to its extremum,

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d\varphi}{dx}\right]^2 dx + \alpha \int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx}dx - \alpha \int_0^{10} \rho\varphi dx \qquad [1.11]$$

The integration by parts of the second integral gives

$$\int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx}dx = \left[\varepsilon \frac{dv_m}{dx}\varphi\right]_0^{10} - \int_0^{10} \frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right]\varphi dx$$

$$\int_0^{10} \varepsilon \frac{dv_m}{dx}\frac{d\varphi}{dx}dx = -\int_0^{10} \frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right]\varphi dx \qquad [1.12]$$

because the arbitrary function $\varphi(x)$ is zero on the boundaries of the domain.

We thus obtain for the increase of the functional

$$W_c(\alpha) - W_c(0) = \alpha^2 \frac{1}{2}\int_0^{10} \varepsilon \left[\frac{d\varphi}{dx}\right]^2 dx - \alpha \int_0^{10} \left\{\frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right] + \rho\right\}\varphi dx \qquad [1.13]$$

This polynomial of the second-degree is extremum for $\alpha = 0$, therefore the coefficient of α must be zero. This coefficient is an integral, to be zero whatever the arbitrary function $\varphi(x)$, and it is necessary that the weighting coefficient of this function becomes zero for any X

$$\frac{d}{dx}\left[\varepsilon \frac{dv_m}{dx}\right] + \rho = 0 \qquad \forall x \in [0, 10] \qquad [1.14]$$

which corresponds precisely to equation [1.5], which we want to solve. Therefore, if function $v_m(x)$ exists, it is indeed the solution of the specified problem. Moreover, the coefficient of α^2 being positive, the extremum is a minimum.

The result that we have just obtained is a particular case of a proof that is much more general of the calculus of variations. Equation [1.14] is in fact the *Euler* *equation* of functional [1.8], and could thus have been obtained directly by application of a traditional theorem.

1.2.4. *First-order finite elements*

In order to present the finite element method, we introduce several concepts shown in Figure 1.2. First of all, in the field of study, we define *nodes* at the positions $x_1 = 0$, $x_2 = 10/3$, $x_3 = 20/3$ and $x_4 = 10$. The electric potentials v_1, v_2, v_3 and v_4 at these nodes are called *nodal values*. Two of these nodal values, $v_1 = 0$ and $v_4 = 100$, are already known thanks to the boundary conditions, while two others, v_2 and v_3, will have to be determined by application of the finite element method.

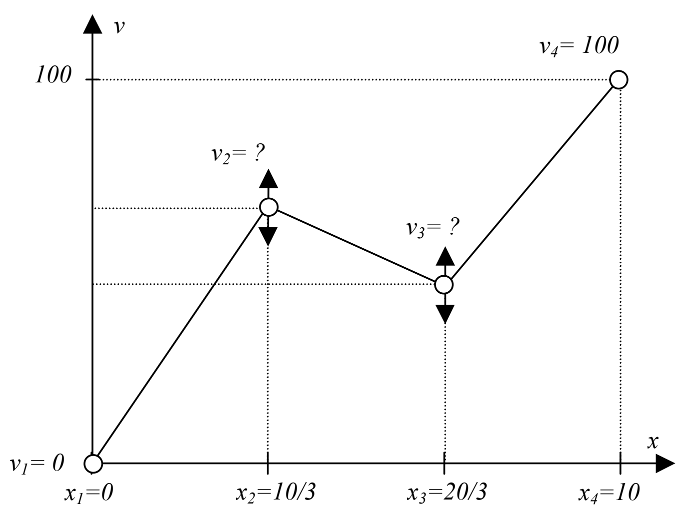

Figure 1.2. *Subdivision of the domain into three first-order finite elements*

We thus define a subdivision of the domain into *finite elements* $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_4]$ on which we apply an *interpolation* for the electric potential. We choose the linear interpolation (order 1) which is the simplest of the interpolations ensuring the continuity of the potential and its derivability per piece, as that is required by the variational approach. On the element $[x_i, x_{i+1}]$, this gives for the potential

$$v(x) = v_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + v_{i+1} \frac{x_i - x}{x_i - x_{i+1}} \qquad [1.15]$$

and for its gradient

$$\frac{dv}{dx} = \frac{v_{i+1} - v_i}{x_{i+1} - x_i} \qquad [1.16]$$

In order to determine the unknown nodal values v_2 and v_3, we will use functional [1.8], into which we will introduce the function $v(x)$ defined in [1.15] per piece on each finite element. We will then obtain a function of the only two unknown factors. The extremality conditions of this function will be the equations defining these unknown factors.

The subdivision of the domain allows the integral giving the functional to be expressed in a sum of integrals on the finite elements

$$W_c = \int_0^{10} = \int_{x_1}^{x_2} + \int_{x_2}^{x_3} + \int_{x_3}^{x_4} = W_{c\,1} + W_{c\,2} + W_{c\,3} \qquad [1.17]$$

The elementary contribution $W_{c\,i}$ of the element $[x_i, x_{i+1}]$ is written

$$W_{c\,i} = \frac{1}{2}\int_{x_i}^{x_{i+1}} \varepsilon \left[\frac{v_{i+1} - v_i}{x_{i+1} - x_i}\right]^2 dx - \int_{x_i}^{x_{i+1}} \rho \left[v_i \frac{x_{i+1} - x}{x_{i+1} - x_i} + v_{i+1} \frac{x_i - x}{x_i - x_{i+1}} \right] dx$$

$$W_{c\,i} = \frac{1}{2}\varepsilon \frac{[v_{i+1} - v_i]^2}{x_{i+1} - x_i} - \frac{1}{2}\rho [v_{i+1} + v_i][x_{i+1} - x_i] \qquad [1.18]$$

The integral thus becomes

$$\begin{aligned} W_c = \; & \frac{1}{2}\varepsilon \frac{[v_2 - v_1]^2}{x_2 - x_1} - \frac{1}{2}\rho [v_2 + v_1][x_2 - x_1] \\ & + \frac{1}{2}\varepsilon \frac{[v_3 - v_2]^2}{x_3 - x_2} - \frac{1}{2}\rho [v_3 + v_2][x_3 - x_2] \\ & + \frac{1}{2}\varepsilon \frac{[v_4 - v_3]^2}{x_4 - x_3} - \frac{1}{2}\rho [v_4 + v_3][x_4 - x_3] \end{aligned} \qquad [1.19]$$

The stationarity conditions of W_c, with respect to the two unknown variables v_2 and v_3, lead to the following two equations

$$\begin{aligned}\frac{\partial W_c}{\partial v_2} &= \varepsilon\frac{v_2 - v_1}{x_2 - x_1} - \frac{1}{2}\rho[x_2 - x_1] - \varepsilon\frac{v_3 - v_2}{x_3 - x_2} - \frac{1}{2}\rho[x_3 - x_2] = 0 \\ \frac{\partial W_c}{\partial v_3} &= \varepsilon\frac{v_3 - v_2}{x_3 - x_2} - \frac{1}{2}\rho[x_3 - x_2] - \varepsilon\frac{v_4 - v_3}{x_4 - x_3} - \frac{1}{2}\rho[x_4 - x_3] = 0\end{aligned} \qquad [1.20]$$

To go numerically further, we arbitrarily fix the ratio between the electric permittivity and the density of electric charges

$$\frac{\rho}{\varepsilon} = 1 \qquad [1.21]$$

We obtain the system of two equations with two unknown variables according to

$$\begin{aligned}\frac{3v_2}{5} - \frac{3v_3}{10} &= \frac{10}{3} \\ -\frac{3v_2}{10} + \frac{3v_3}{5} &= \frac{100}{3}\end{aligned} \qquad [1.22]$$

which has the solution

$$\begin{aligned}v_2 &= \frac{400}{9} \\ v_3 &= \frac{700}{9}\end{aligned} \qquad [1.23]$$

In Figure 1.3, we can evaluate the quality of the approximation obtained. The interpolation by first-order finite elements is not very far away from the reference solution. It is even exact at the nodes of the grid. In fact, this coincidence is related to the simplicity of the problem taken as an illustration and will not be found in more realistic applications. Here, the exact solution is a second-degree polynomial, whose average behavior is perfectly represented on each piece by linear interpolations.

In order to improve the solution, we have two strategies. The first consists of decreasing the size of the finite elements; it is called the *h method* by reference to the diameter of the elements which is often denoted h. The second consists of increasing the order of the finite elements; it is denoted the *p method* because p is often used to represent the order of the approximation. It is this second strategy which we will implement below.

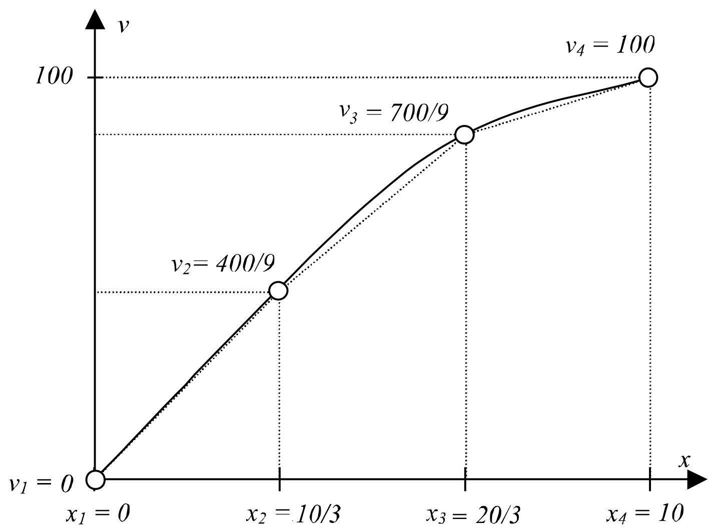

Figure 1.3. *Exact solution in continuous line and solution by first-order finite elements in dotted lines*

1.2.5. *Second-order finite elements*

We now decide to implement the second-order elements. In order to simplify our work to the maximum, we define a minimal subdivision of the domain, i.e. three nodes at the positions $x_1 = 0$, $x_2 = 5$, $x_3 = 10$ having the three nodal values v_1, v_2, and v_3 and defining only one second-order finite element $[x_1, x_2, x_3]$. The nodal values on the limits are $v_1 = 0\ V$ and $v_3 = 100\ V$. Only the internal nodal value v_2 is to be determined by the finite element method. On the single finite element, the electric potential is interpolated by

$$v(x) = v_1 \frac{[x_2 - x][x_3 - x]}{[x_2 - x_1][x_3 - x_1]} + v_2 \frac{[x_3 - x][x_1 - x]}{[x_3 - x_2][x_1 - x_2]} + v_3 \frac{[x_1 - x][x_2 - x]}{[x_1 - x_3][x_2 - x_3]} \quad [1.24]$$

and its gradient by

$$\frac{dv}{dx} = v_1 \frac{2x - x_2 - x_3}{[x_2 - x_1][x_3 - x_1]} + v_2 \frac{2x - x_3 - x_1}{[x_3 - x_2][x_1 - x_2]} + v_3 \frac{2x - x_1 - x_2}{[x_1 - x_3][x_2 - x_3]} \quad [1.25]$$

The introduction of these approximations into functional [1.8], the integration then the application of the stationarity condition with respect to v_2, led to the equation

$$2\varepsilon v_2 = \varepsilon v_1 + \varepsilon v_3 + 25\rho \qquad [1.26]$$

which, for the numerical values selected previously $v_1 = 0$, $v_3 = 100$ and $\rho/\varepsilon = 1$ results, for the unknown nodal value, in $v_2 = 125/2$, which is the good value. Figure 1.4 shows the exact solution and the second-order finite elements solution. These are exactly superimposed. Indeed, the exact solution [1.7] is a second-degree polynomial, which is precisely the type of approximation implemented in the second-order finite element method. Here again, this coincidence is only related to the simplicity of the concerned problem. In more complex applications, we will no longer find such perfect solutions.

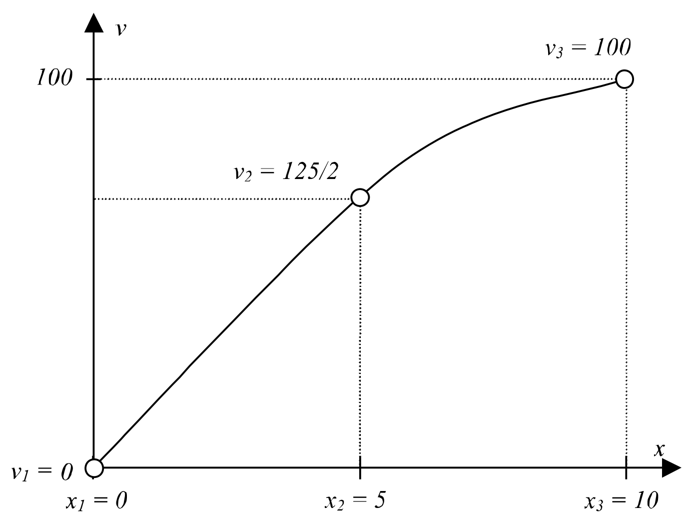

Figure 1.4. *The exact solution and the second-order finite elements are exactly superimposed*

1.3. The finite element method in two dimensions

1.3.1. *The problem of the condenser with square section*

We will again be interested in a problem of electrostatics, but this time of a 2D nature, in order to handle a more realistic example of implementation of the finite element method. We will find the differential then the variational forms of this type of problem, with the associated boundary conditions. We will present the general concepts of domain meshing and finite element interpolation. We will explain the Ritz method and we will implement it to find an approximate solution to the problem. Lastly, we will see how to take advantage of this solution to obtain local and global information that is more explicit than a simple set of nodal values.

The studied device is a condenser whose cross-section is represented in Figure 1.5 and whose depth *h* is very large in front of the section dimensions.

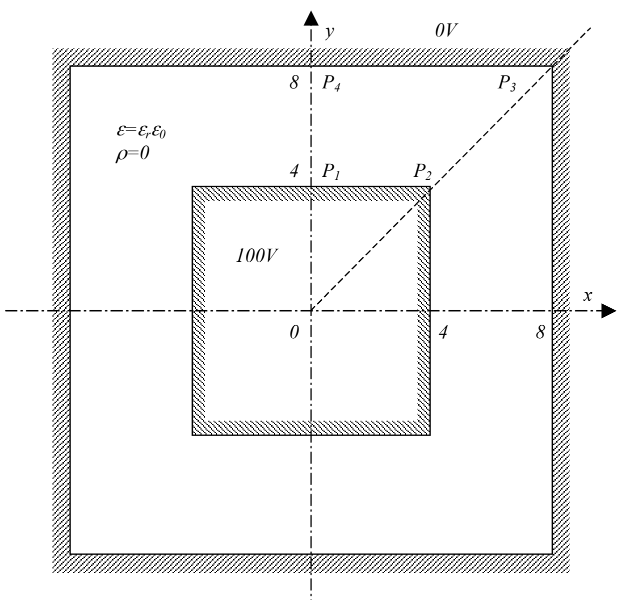

Figure 1.5. *Cross-section of the long condenser*

This condenser is composed of two overlapped conductors of square sections, one with the electric potential of *100 V* and the other with the potential of *0 V*. Taking into account the high dimension of the condenser in the direction perpendicular to the xOy plane, the 2D study of the device in its cross-section will give a very good idea of its global behavior. In fact, we are interested here in the capacitor of this condenser, which we will obtain by using the finite element method. For this purpose, we will initially determine the distribution of the electric potential within the dielectric, assumed to be perfect, placed between the two electrodes.

1.3.2. *Differential approach*

The Maxwell's equations, representative of the distribution of the electrostatic field in the dielectric, are

$$divD = \rho \qquad \text{(Gauss law)} \tag{1.27}$$

$$curlE = 0 \qquad \text{(Faraday law in static mode)} \tag{1.28}$$

$$D = \varepsilon E \qquad \text{(constitutive law of the dielectric material)} \tag{1.29}$$

where D is the electric flux density vector, E the electric field vector, ρ the density of electric charges and ε the permittivity of the dielectric.

The introduction of v, the electric scalar potential, such that

$$E = -grad\ v \tag{1.30}$$

automatically solves the second Maxwell's equation since the rotational of a gradient is systematically zero. By combining the first and third equations, we obtain the partial differential equation of the electric potential

$$div[\varepsilon\ grad\ v] = -\rho \tag{1.31}$$

which, in the reference frame xOy, is written

$$\frac{\partial}{\partial x}\left[\varepsilon \frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[\varepsilon \frac{\partial v}{\partial y}\right] = -\rho \tag{1.32}$$

and in the particular case of a constant electric permittivity and of a density of electric charges equal to zero

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \Delta v = 0 \tag{1.33}$$

However, for the sought generality, we will use expression [1.31] in the rest of this presentation.

To go further in the definition of the problem, we should specify the field of study and the boundary conditions. We could take the whole cross-section of the dielectric of Figure 1.5 as field of study, with $v = 0\ V$ on the external edge and *v = 100 V* on the internal edge. However, the presence of several symmetries allows the zone of study to be considerably reduced, and thus the efforts of calculation. Indeed, we have just to calculate the solution in the eighth $[P_1, P_2, P_3, P_4]$ of the domain (see Figure 1.6), then to reconstitute, thanks to symmetries, the distribution of the electric potential in all the dielectric.

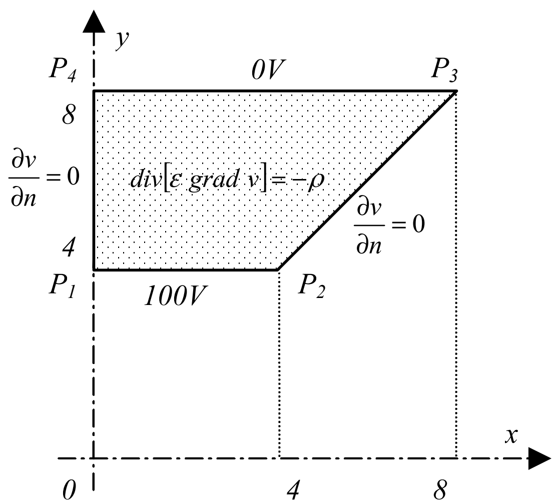

Figure 1.6. *Reduction of the field of study thanks to symmetries*

With a partial differential equation such as [1.31], of elliptic type, and in order to specify the problem clearly, it is necessary to impose conditions on all the limits of the field of studies, either on the state variable *v*, called the Dirichlet condition, or on its normal derivative $\frac{\partial v}{\partial n}$, called the Neumann condition. We already know that $v = 100\ V$ on the edge P_1P_2 and that $v = 0\ V$ on the edge P_3P_4. It remains to define the conditions on the rest of the border. On the axes of symmetry P_2P_3 and P_4P_1, the field has a particular direction: it is tangential. In fact, no electric flux crosses these parts of the border. Mathematically it means that the normal component of the induction is zero $D_n = 0$, i.e. a zero normal component of the field $E_n = 0$ and thus that the homogenous Neumann condition $\frac{\partial v}{\partial n} = 0$ on the electric potential, which we will take as conditions on these limits.

1.3.3. *Variational approach*

The functional of coenergy of the previous differential equation which generalizes that given in [1.8] to the 2D case is

$$\iint \left[\frac{\varepsilon \left[grad\ v \right]^2}{2} - \rho v \right] dxdy \qquad [1.34]$$

This first functional would be well adapted to the specified problem; however, we would rather use the following expression

$$W_c(v) = \iint \left[\int_0^{-grad\ v} D\, dE - \rho v \right] hdxdy \qquad [1.35]$$

This second functional is more general because it is able to handle a possible nonlinearity in the constitutive law *D(E)*, and the presence of the depth *h* of the device makes it homogenous to an electrical energy.

Let us check that the continuous and derivable function $v_m(x,y)$ which satisfies the boundary conditions $v_m = 100\ V$ on P_1P_2 and $v_m = 0\ V$ on P_3P_4 and which makes functional [1.35] stationary, is a solution of equation [1.31] and also satisfies $\frac{\partial v_m}{\partial n} = 0$ on P_2P_3 and $P_4P_1P_2P_3$.

For this purpose, starting from $v_m(x,y)$, we build the function

$$v(x,y) = v_m(x,y) + \delta v(x,y) \qquad [1.36]$$

where $\delta v(x,y)$ is a continuous and derivable function, zero on the Dirichlet type boundaries which play the role of an unspecified infinitesimal variation around the balance function $v_m(x,y)$. By construction, $v(x,y)$ always verifies the boundary conditions of the problem on P_1P_2 and P_3P_4.

Let us introduce $v(x,y)$ into functional [1.35] and express the variation

$$\delta W_c = \iint \left[-D\ grad\ \delta v - \rho\, \delta v \right] hdxdy \qquad [1.37]$$

By using the vector relation

$$div\left[D\, \delta v \right] = divD\, \delta v + D\, grad\, \delta V \qquad [1.38]$$