STATISTICAL TOLERANCE REGIONS

Theory, Applications, and Computation

K. KRISHNAMOORTHY

THOMAS MATHEW

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Published by John Wiley & Sons, Inc., Hoboken, New Jersey.
Published simultaneously in Canada.

Library of Congress Cataloging-in-Publication Data:

Krishnamoorthy, K. (Kalimuthu)
Statistical tolerance regions : theory, applications, and computation / K. Krishnamoorthy, Thomas Mathew.
p. cm. — (Wiley series in probability and statistics)
Includes bibliographical references and index.
ISBN 978-0-470-38026-0 (cloth)
1. Statistical tolerance regions. I. Mathew, Thomas, 1955– II. Title.
QA276.74.K75 2009
519.5—dc22 2008047069

Printed in the United States of America.

10 9 8 7 6 5 4 3 2 1

To the memory of my parents
KK

To my father K. T. Mathew, and to the memory of my mother Aleyamma Mathew
TM

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Preface

The theory of statistical tolerance intervals and tolerance regions has undergone vigorous development during the last three decades. In particular, the derivation of satisfactory tolerance intervals in the context of random effects models, and satisfactory simultaneous tolerance intervals for regression models, has been carried out only during the 1980s and 1990s. Furthermore, the construction of satisfactory tolerance regions for multivariate normal populations and multivariate regression models was accomplished only within the last ten years. The bibliography collected by Jilek (1981) lists around 270 articles on the topic, and the one by Jilek and Ackerman (1989) lists an additional 130 articles. The literature on the topic has grown considerably since the publication of the latter bibliography. However, no book-length treatment of the topic has been available since Guttman's (1970) monograph. The present book was conceived based on the perceived need to have a single source that brings together the recent developments as well as the earlier results on the topic of tolerance intervals and tolerance regions.

As opposed to a confidence interval that provides information concerning an unknown population parameter, a tolerance interval provides information on the entire population; to be specific, a tolerance interval is expected to capture a certain proportion or more of the population, with a given confidence level. For example, an upper tolerance limit for a univariate population is such that with a given confidence level, a specified proportion or more of the population will fall below the limit. This proportion is referred to as the *content* of a tolerance interval. A lower tolerance limit, or a tolerance interval having both lower and upper limits, satisfies similar conditions. For multivariate populations, we analogously have tolerance regions. The applications of tolerance intervals and tolerance regions are varied. They include clinical and industrial applications, including quality control, applications to environmental monitoring, to the assessment of agreement between two methods or devices, and applications in industrial hygiene. As suggested by the title, this book discusses the theoretical derivation of tolerance intervals and tolerance regions in a wide variety of scenarios, along with applications and examples, and also illustrates the computational procedures. Analytic formulas for the tolerance limits are available in only simple cases, for example, for the upper or lower tolerance limit for a univariate normal population. Thus it becomes necessary to use numerical methods or approximations in order to derive tolerance intervals for many populations. The book discusses the various approximations available for different tolerance interval problems, and also discusses comparisons among the different approximations, making recommendations regarding the choice of the approximation for practical use. When it comes to random or mixed effects models, the book provides the available procedures for the balanced as well as the unbalanced data situations. Furthermore, for situations where the tolerance intervals have to be numerically obtained, the book includes extensive tables providing the necessary tolerance factors for various combinations of the sample size, content and confidence level.

The book has twelve chapters and gives a rather broad coverage of its topic. Chapter 1 gives the basic concepts and definitions, and also gives some of the technical results used throughout the book. The ideas of generalized p-values and generalized confidence intervals are extensively used in some of the later chapters, and these are also described in Chapter 1. Chapter 2 gives a thorough discussion of the various tolerance intervals that have been constructed in the context of the univariate normal distribution. Chapter 3 is on the univariate linear regression model, where we describe the construction of tolerance intervals and simultaneous tolerance intervals. Chapters 4–6 are on the construction of tolerance intervals in mixed effects and random effects models. The one-way random model is given special emphasis, and is the topic covered in both Chapter 4 (the case of balanced data) and Chapter 5 (the case of unbalanced data). Other mixed and random effects models are taken up in Chapter 6. The computation of tolerance intervals for some continuous distributions other than the normal is the topic of Chapter 7. The lognormal, gamma, exponential and Weibull distributions are considered in this chapter. Non-parametric tolerance intervals form the topic of Chapter 8. Chapter 9 and Chapter 10 deal with multivariate populations; Chapter 9 is on the computation of tolerance regions for a multivariate normal distribution, and Chapter 10 addresses the problem in the context of a multivariate linear regression model. Bayesian approaches are described in Chapter 11. Some special topics not covered in the previous chapters are discussed in Chapter 12. The topics covered in this chapter include the derivation of β–expectation tolerance intervals, sample size determination, tolerance intervals for the ratio of normal random variables, tolerance intervals for binomial and Poisson distributions, and tolerance intervals based on censored data. In Chapter 3 and Chapter 10, the calibration problem is also included, since the computation of a multiple use confidence interval or region in the calibration problem can be accomplished using appropriate tolerance intervals and regions.

In each chapter of the book, the theoretical derivations are described in detail, along with the computational procedures. In fact computational algorithms are given throughout the book. However, we have not emphasized any particular software. The computational algorithms can be easily coded in any machine language (Fortran, C, SAS®, etc.). In each chapter, the results are all illustrated with data analysis based on real examples. Most of the data sets used are included in the relevant chapter. Some data sets are also given in Appendix A at the end of the book. Appendix B gives table values of tolerance factors.

The book is appropriate for a graduate level course on tolerance intervals, the prerequisite being a basic knowledge of ANOVA, mixed models, regression and multivariate analysis. In fact each chapter includes a set of exercises. For a researcher interested in the topic, the book provides the state of the art in the field. For an applied statistician or a consultant who encounter problems that call for the use of tolerance intervals, the book is expected to be a valuable resource. In a Technometrics article, Carroll and Ruppert (1991, p. 199) mention that "It appears to us that tolerance intervals should be more widely understood and used." It is hoped that this book will serve this purpose.

The authors acknowledge the support and the facilities received from the Department of Mathematics, University of Louisiana at Lafayette, and the Department of Mathematics and Statistics, University of Maryland Baltimore County. The authors are grateful to Dr. Ionut Bebu for his assistance with some of the numerical computations on Bayesian tolerance intervals. Krishnamoorthy is thankful to his wife Usha and sons Prathap and Tharany for their enduring love and moral support. Mathew wishes to express his appreciation to his wife Ruby, and daughters Stacy and Betsy for their continued affection and support.

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Chapter 1

Preliminaries

1.1 Introduction

Statistical intervals computed based on a random sample have wide applicability, for the purpose of quantifying the uncertainty about a scalar quantity associated with a sampled population. The type of interval to be computed obviously depends on the underlying problem and application. A confidence interval based on a random sample is used to provide bounds for an unknown scalar population parameter such as the population mean, standard deviation, percentile, tail probability, etc. A prediction interval based on a random sample is used to provide bounds for one or more future observations from a univariate sampled population. For multivariate populations, we have correspondingly confidence regions and prediction regions. The topic of this book is a third type of interval and region, namely tolerance intervals and tolerance regions. For a univariate population, a tolerance interval is an interval, based on a random sample, that is expected to contain a specified proportion or more of the sampled population. A tolerance region is similarly defined for a multivariate population.

Here is a simple example to illustrate the differences among a confidence interval, prediction interval and tolerance interval. The application deals with the assessment of air lead levels in a laboratory. The data are given in Chapter 2 (see Table 2.1) and represent air lead levels collected by the National Institute of Occupational Safety and Health (NIOSH) at a laboratory for health hazard evaluation purpose. The air lead levels were collected from $n = 15$ different areas within the facility. It was noted that the log-transformed lead levels fitted a normal distribution well (that is, the data are from a lognormal distribution). Let μ and σ^2, respectively, denote the population mean and variance for the log-transformed data. If X denotes the corresponding random variable, we thus have $X \sim N(\mu, \sigma^2)$. We note that $\exp(\mu)$ is the median air lead level. A confidence interval for μ can be constructed the usual way, based on the t-distribution; this in turn will provide a confidence interval for the median air lead level. If $\bar{X}$ and S denote the sample mean and standard deviation of the log-transformed data for a sample of size n, a 95% confidence interval for μ is given by $\bar{X} \pm t_{n-1;.975}\frac{S}{\sqrt{n}}$, where $t_{m;1-\alpha}$ denotes the $1-\alpha$ quantile of a t-distribution with m degrees of freedom. It may also be of interest to derive a 95% upper confidence bound for the median air lead level. Such a bound for μ is given by $\bar{X} + t_{n-1;.95}\frac{S}{\sqrt{n}}$. Consequently, a 95% upper confidence bound for the median air lead level is given by $\exp(\bar{X} + t_{n-1;.95}\frac{S}{\sqrt{n}})$. Now suppose we want to predict the air lead level at a particular area within the laboratory. A 95% upper prediction limit for the log-transformed lead level is given by $\bar{X} + t_{n-1;.95}S\sqrt{1+\frac{1}{n}}$. A two-sided prediction interval can be similarly computed. The meaning and interpretation of these intervals are well known. For example, if the confidence interval $\bar{X} \pm t_{n-1;.975}\frac{S}{\sqrt{n}}$ is computed repeatedly from independent samples, 95% of the intervals so computed will include the true value of μ, in the long run. In other words, the interval is meant to provide information concerning the parameter μ only. A prediction interval has a similar interpretation, and is meant to provide information concerning a single lead level only. Now suppose we want to use the sample to conclude whether or not at least 95% of the population lead levels are below a threshold. The confidence interval and prediction interval cannot answer this question, since the confidence interval is only for the median lead level, and the prediction interval is only for a single lead level. What is required is a tolerance interval; more specifically, an upper tolerance limit. The upper tolerance limit is to be computed subject to the condition that at least 95% of the population lead levels is below the limit, with a certain confidence level, say 99%. Once such an upper tolerance limit is computed, we can verify if it is less than the threshold value.

We shall now give the precise definitions of tolerance intervals. This will be followed by a summary of several preliminary concepts and results, to be used for the derivation of tolerance intervals in some of the later chapters.

1.1.1 One-Sided Tolerance Intervals

Let X be a continuous random variable with cumulative distribution function (cdf) $F_X(x) = P(X \le x)$. For a given p $(0 < p < 1)$, the inverse cdf is defined by

$$F_X^{-1}(p) = \inf\{x : F_X(x) \ge p\}. \tag{1.1.1}$$

The quantity $F_X^{-1}(p)$ is obviously the p quantile or $100p$ percentile of the distribution F_X. We shall also denote the p quantile by q_p. Notice that a proportion p of the population (with distribution function F_X) is less than or equal to q_p. If $F_X(x)$ is a strictly increasing function of x (this is the case for many commonly used distributions), then $F_X^{-1}(p)$ is the value of x for which $F_X(x) = P(X \leq x) = p$.

Let X_1, X_2, ..., X_n be a random sample from $F_X(x)$, and write $\boldsymbol{X} = (X_1, X_2, ..., X_n)$. In order to define a tolerance interval, we need to specify its *content* and *confidence level*. These will be denoted by p and $1-\alpha$, respectively, and the tolerance interval will be referred to as a p content and $(1-\alpha)$ coverage (or p content and $(1-\alpha)$ confidence) tolerance interval or simply a $(p, 1-\alpha)$ tolerance interval $(0 < p < 1,\ 0 < \alpha < 1)$. In practical applications, p and $1-\alpha$ usually take values from the set $\{0.90, 0.95, 0.99\}$. The interval will be constructed using the random sample $\boldsymbol{X}$, and is required to contain a proportion p or more of the sampled population, with confidence level $1-\alpha$. Formally, a $(p, 1-\alpha)$ one-sided tolerance interval of the form $(-\infty, U(\boldsymbol{X})]$ is required to satisfy the condition

$$P_{\boldsymbol{X}}\left\{P_X\left(X \leq U(\boldsymbol{X}) \middle| \boldsymbol{X}\right) \geq p\right\} = 1-\alpha, \tag{1.1.2}$$

where X also follows F_X, independently of $\boldsymbol{X}$. That is, $U(\boldsymbol{X})$ is to be determined such that at least a proportion p of the population is less than or equal to $U(\boldsymbol{X})$ with confidence $1-\alpha$. The interval $(-\infty, U(\boldsymbol{X})]$ is called a one-sided tolerance interval, and $U(\boldsymbol{X})$ is called a one-sided upper tolerance limit. Note that based on the definition of the p quantile q_p, we can write (1.1.2) as

$$P_{\boldsymbol{X}}\left\{q_p \leq U(\boldsymbol{X})\right\} = 1-\alpha. \tag{1.1.3}$$

It is clear from (1.1.3) that $U(\boldsymbol{X})$ is a $1-\alpha$ upper confidence limit for the p quantile q_p.

A $(p, 1-\alpha)$ one-sided lower tolerance limit $L(\boldsymbol{X})$ is defined similarly. Specifically, $L(\boldsymbol{X})$ is determined so that

$$P_{\boldsymbol{X}}\left\{P_X\left(X \geq L(\boldsymbol{X}) \middle| \boldsymbol{X}\right) \geq p\right\} = 1-\alpha,$$

or equivalently,

$$P_{\boldsymbol{X}}\left\{L(\boldsymbol{X}) \leq q_{1-p}\right\} = 1-\alpha.$$

Thus, $L(\boldsymbol{X})$ is a $1-\alpha$ lower confidence limit for q_{1-p}.

1.1.2 Tolerance Intervals

There are two types of two-sided tolerance intervals. One is constructed so that it would contain at least a proportion p of the population with confidence $1-\alpha$, and is simply referred to as the tolerance interval. A second type of tolerance interval is constructed so that it would contain at least a proportion p of the *center* of the population with confidence $1-\alpha$, and is usually referred to as an equal-tailed tolerance interval.

A $(p, 1-\alpha)$ two-sided tolerance interval $(L(\boldsymbol{X}), U(\boldsymbol{X}))$ satisfies the condition

$$P_{\boldsymbol{X}}\left\{P_X\left(L(\boldsymbol{X}) \le X \le U(\boldsymbol{X})\middle|\boldsymbol{X}\right) \ge p\right\} = 1-\alpha, \tag{1.1.4}$$

or equivalently,

$$P_{\boldsymbol{X}}\left\{F_X(U(\boldsymbol{X})) - F_X(L(\boldsymbol{X})) \ge p\right\} = 1-\alpha. \tag{1.1.5}$$

In other words, the interval $(L(\boldsymbol{X}), U(\boldsymbol{X}))$ is constructed so that it would contain at least a proportion p of the population with confidence $1-\alpha$. The quantities $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$ are referred to as the tolerance limits. It is important to note that the computation of $L(\boldsymbol{X})$ and $U(\boldsymbol{X})$ *does not* reduce to the computation of confidence limits for certain percentiles.

In order to define an equal-tailed tolerance interval, assume that $p > 0.5$. A $(p, 1-\alpha)$ equal-tailed tolerance interval $(L(\boldsymbol{X}), U(\boldsymbol{X}))$ is such that, with confidence $1-\alpha$, no more than a proportion $\frac{1-p}{2}$ of the population is less than $L(\boldsymbol{X})$ and no more than a proportion $\frac{1-p}{2}$ of the population is greater than $U(\boldsymbol{X})$. This requirement can be stated in terms of percentiles. Note that the condition $L(\boldsymbol{X}) \le q_{\frac{1-p}{2}}$ is equivalent to no more than a proportion $\frac{1-p}{2}$ of the population being less than $L(\boldsymbol{X})$, and the condition $q_{\frac{1+p}{2}} \le U(\boldsymbol{X})$ is equivalent to no more than a proportion $1-\frac{1+p}{2} = \frac{1-p}{2}$ of the population being greater than $U(\boldsymbol{X})$. Consequently, for $(L(\boldsymbol{X}), U(\boldsymbol{X}))$ to be a $(p, 1-\alpha)$ equal-tailed tolerance interval, the condition to be satisfied is

$$P_{\boldsymbol{X}}\left(L(\boldsymbol{X}) \le q_{\frac{1-p}{2}} \text{ and } q_{\frac{1+p}{2}} \le U(\boldsymbol{X})\right) = 1-\alpha. \tag{1.1.6}$$

Apart from the one-sided and two-sided $(p, 1-\alpha)$ tolerance intervals introduced above, intervals have also been constructed so as to contain a proportion β of the population, on the average. Such intervals are referred to as β–expectation tolerance intervals. It has been noted that these are simply $100\beta\%$ prediction intervals for a future observation from the population, constructed using a random sample from the population.

Early works on the tolerance interval problem are due to Wilks (1941, 1942), Wald (1943) and Wald and Wolfowitz (1946). The book by Guttman (1970) gives a concise treatment of tolerance intervals and regions, both $(p, 1-\alpha)$ tolerance intervals as well as $\beta-$expectation tolerance intervals; see also the book by Aitchison and Dunsmore (1975, Chapters 5 and 6). Extensive bibliographies of the literature on tolerance intervals and regions are given in the articles by Jilek (1981) and Jilek and Ackerman (1989). Reviews of the literature on the topic are provided in Patel (1986, 1989), and in the book by Hahn and Meeker (1991). For brief introductions and review, we refer to Guttman (1988) and Vangel (2008a, b). Several articles on tolerance intervals and regions also provide tables of tolerance factors that facilitate easy computation of the required intervals and regions, and the book by Odeh and Owen (1980) gives the required factors in the context of the normal distribution. The PC calculator *StatCalc* (Krishnamoorthy, 2006) can also be conveniently used to compute tolerance factors for univariate and multivariate normal populations.

1.1.3 Survival Probability and Stress-Strength Reliability

Estimation of Survival Probability

In many applications it is desired to estimate the probability that a random variable exceeds a specified value. For example, in lifetime data analysis, it is of interest to assess the probability that the lifetime of an item exceeds a value; this probability is commonly referred to as the survival probability. In industrial hygiene, it is of interest to estimate the probability that the exposure level (level of exposure to a contaminant in a workplace) of a worker exceeds the occupational exposure limit (OEL; usually set by the Occupational Safety and Health Administration). This is referred to as an *exceedance probability.* To assess the lifetime of an item, a lower confidence limit for the survival probability is warranted. Such a lower confidence limit can be easily deduced from a suitable lower tolerance limit, as shown below.

Let X be a continuous random variable with the distribution function $F_X(x)$. For a given t, define the survival probability $S_t = P(X > t) = 1 - F_X(t)$. Let $\boldsymbol{X}$ be a sample from F_X, and $L(\boldsymbol{X}) = L(\boldsymbol{X}; p)$ be a $(p, 1-\alpha)$ lower tolerance limit for the distribution of X. Being a $(p, 1-\alpha)$ lower tolerance limit, we have

$$P_{\boldsymbol{X}}\left\{P_X\left(X \geq L(\boldsymbol{X}; p) | \boldsymbol{X}\right) \geq p\right\} = 1 - \alpha.$$

That is,

$$P_{\boldsymbol{X}}\left\{S_{L(\boldsymbol{X};p)} \geq p\right\} = 1 - \alpha.$$

If $L(\boldsymbol{X};p) \geq t$, then we obviously have $S_t \geq S_{L(\boldsymbol{X};p)}$. Furthermore, if $S_{L(\boldsymbol{X};p)} \geq p$, we can conclude that $S_t \geq p$ whenever $L(\boldsymbol{X};p) \geq t$. Consequently the maximum value of p for which $L(\boldsymbol{X};p) \geq t$ gives a $1-\alpha$ lower bound, say p_l, for S_t. That is,

$$p_l = \max\{p : L(\boldsymbol{X};p) \geq t\}. \tag{1.1.7}$$

In general, $L(\boldsymbol{X};p)$ is a decreasing function of p, and so the maximum in (1.1.7) is attained when $L(\boldsymbol{X};p) = t$. That is, p_l is the solution of $L(\boldsymbol{X};p) = t$. A lower tolerance limit can also be used to test one-sided hypotheses concerning S_t. If it is desired to test

$$H_0 : S_t \leq p_0 \quad \text{vs.} \quad H_a : S_t > p_0$$

at a level α, then H_0 will be rejected if a $(p_0, 1-\alpha)$ lower tolerance limit is greater than t.

An upper confidence limit for an exceedance probability is often used to assess the exposure level (exposure to pollution or contaminant) in a workplace. For instance, if t denotes the occupational exposure limit and X denotes the exposure measurement for a worker, then the exceedance probability is defined by $P(X > t)$. If $U(\boldsymbol{X};p)$ is a $(p, 1-\alpha)$ upper tolerance limit, and is less than or equal to t, then we can conclude that $P(X > t)$ is less than $1-p$. Arguing as in the case of (1.1.7), we conclude that if $p_u = \max\{p : U(\boldsymbol{X};p) \leq t\}$, then $1-p_u$ is a $1-\alpha$ upper confidence limit for the exceedance probability. In general, $U(\boldsymbol{X};p)$ is a nondecreasing function of p, and so p_u is the solution of the equation $U(\boldsymbol{X};p) = t$.

Stress-Strength Reliability

The classical stress-strength reliability problem concerns the proportion of the time the strength X of a component exceeds the stress Y to which it is subjected. If $X \leq Y$, then either the component fails or the system that uses the component may malfunction. If both X and Y are random, then the reliability R of the component can be expressed as $R = P(X > Y)$. A lower limit for R is commonly used to assess the reliability of the component. Writing $R = P(X - Y > 0)$, we see that R can be considered as a survival probability. Therefore, the procedures given for estimating survival probability can be applied to find a lower confidence limit for R. More specifically, if it is desired to test

$$H_0 : R \leq R_0 \quad \text{vs.} \quad H_a : R > R_0$$

at a level α, then the null hypothesis will be rejected if a $(R_0, 1-\alpha)$ lower tolerance limit for the distribution of $X - Y$ is greater than zero.

1.2 Some Technical Results

In this section, we shall give a number of technical results to be used in later chapters. In particular, the first result has important applications in the derivation of tolerance intervals for a univariate normal distribution and other normal based models.

Result 1.2.1 Let $X \sim N(0, c)$ independently of $Q \sim \frac{\chi_m^2}{m}$, where χ_ν^2 denotes a chi-square random variable with degrees of freedom (df) m. Let $0 < p < 1$, $0 < \gamma < 1$, and let Φ denote the standard normal distribution function.

(i) The factor k_1 that satisfies

$$P_{X,Q}\left(\Phi\left(X + k_1\sqrt{Q}\right) \geq p\right) = \gamma \tag{1.2.1}$$

is given by

$$k_1 = \sqrt{c} \times t_{m;\gamma}\left(\frac{z_p}{\sqrt{c}}\right), \tag{1.2.2}$$

where z_p denotes the p quantile of a standard normal distribution, and $t_{\nu;\eta}(\delta)$ denotes the η quantile of a noncentral t distribution with df ν and noncentrality parameter δ.

(ii) The factor k_2 that satisfies

$$P_{X,Q}\left(\Phi(X + k_2\sqrt{Q}) - \Phi(X - k_2\sqrt{Q}) \geq p\right) = \gamma \tag{1.2.3}$$

is the solution of the integral equation

$$\sqrt{\frac{2}{\pi c}}\int_0^\infty P_Q\left(Q \geq \frac{\chi_{1;p}^2(x^2)}{k_2^2}\right) e^{-\frac{x^2}{2c}}dx = \gamma, \tag{1.2.4}$$

where $\chi_{\nu;\eta}^2(\delta)$ denotes the η quantile of a noncentral chi-square distribution with df ν and noncentrality parameter δ.

(iii) An approximation to k_2 that satisfies (1.2.4) is given by

$$k_2 \simeq \left(\frac{m\chi_{1;p}^2(c)}{\chi_{m;1-\gamma}^2}\right)^{\frac{1}{2}}, \tag{1.2.5}$$

where $\chi_{\nu;\eta}^2$ denotes the η quantile of a chi-square distribution with df ν.

Proof.

(i) Note that the inner probability inequality in (1.2.1) holds if and only if $X + k_1\sqrt{Q} \geq z_p$. So we can write (1.2.1) as

$$\begin{aligned} P_{X,Q}\left(X + k_1\sqrt{Q} \geq z_p\right) &= P_{X,Q}\left(\frac{X - z_p}{\sqrt{Q}} \geq -k_1\right) \\ &= P_{X,Q}\left(\sqrt{c}\frac{X/\sqrt{c} + z_p/\sqrt{c}}{\sqrt{Q}} \leq k_1\right) \\ &= \gamma. \end{aligned} \tag{1.2.6}$$

To get the second step of (1.2.6), we used the fact that X and $-X$ are identically distributed. Because $X/\sqrt{c} \sim N(0,1)$ independently of $Q \sim \frac{\chi^2_m}{m}$, we have

$$\frac{X/\sqrt{c} + z_p/\sqrt{c}}{\sqrt{Q}} \sim t_m\left(\frac{z_p}{\sqrt{c}}\right),$$

where $t_\nu(\delta)$ denotes a noncentral t random variable with degrees of freedom ν and noncentrality parameter δ. Therefore, k_1 that satisfies (1.2.6) is given by (1.2.2).

(ii) Note that, for a fixed X, $\Phi(X + r) - \Phi(X - r)$ is an increasing function of r. Therefore, for a fixed X, $\Phi(X + k_2\sqrt{Q}) - \Phi(X - k_2\sqrt{Q}) \geq p$ if and only if $k_2\sqrt{Q} > r$ or $Q > \frac{r^2}{k_2^2}$, where r is the solution of the equation

$$\Phi(X + r) - \Phi(X - r) = p, \tag{1.2.7}$$

or equivalently,

$$P_Z\left((Z - X)^2 \leq r^2|X\right) = p, \tag{1.2.8}$$

Z being a standard normal random variable (see Exercise 1.5.3). For a fixed X, $(Z - X)^2 \sim \chi^2_1(X^2)$, where $\chi^2_m(\delta)$ denotes a noncentral chi-square random variable with noncentrality parameter δ. Therefore, conditionally given X^2, r^2 that satisfies (1.2.8) is the p quantile of $\chi^2_1(X^2)$, which we denote by $\chi^2_{1;p}(X^2)$. Using these results, and noticing that r is a function of X^2 and p, we have

$$\begin{aligned} P_Q\left(\Phi(X + k_2\sqrt{Q}) - \Phi(X - k_2\sqrt{Q}) > p\Big|X\right) &= P_Q\left(Q > \frac{r^2}{k_2^2}\Big|X\right) \\ &= P_Q\left(Q > \frac{\chi^2_{1;p}(X^2)}{k_2^2}\Big|X\right). \end{aligned} \tag{1.2.9}$$